

# Józef Marcinkiewicz—his life and work

on the occasion of his 110<sup>th</sup> birthday anniversary  
and the 80<sup>th</sup> anniversary of his death

by

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\*\* Picture of Marcinkiewicz from the *Collected Papers* (Pic #1)

## 1 Introduction

In this survey article we outline the life and mathematics of Józef Marcinkiewicz, one of the most brilliant analysts of the pre-WWII era. We present his important results which started new directions in analysis in the late 1930–ties and in the period after the WWII.

The main source of information on J. Marcinkiewicz’s mathematical achievements is the volume, *Józef Marcinkiewicz. Collected Papers* [48], edited by Professor Antoni Zygmund who also wrote an informative Introduction section, and a recent exhaustive monograph [46] by Lech Maligranda. The latter author highlighted Marcinkiewicz’s connections to the contemporary analysis to the greatest extent and provided the up-to-date list of references. We placed the complete bibliography of Marcinkiewicz’s work (between items [49] and [105]) in this paper. These articles also bear the same pagination as in [48]; the idea of double numbering was borrowed from [46].

In recent times various articles appeared on Marcinkiewicz. Among them [13], [27] and, very recently, [41].

## 2 The life of Józef Marcinkiewicz

Józef Marcinkiewicz was a mathematician of phenomenal talent, one of the best ever born in Poland. The influence of his research activities on mathematics, and most of all—on mathematical analysis—

is difficult to overestimate. During his very short life (at the time of his death he was barely thirty), within the period of merely 6 years of working in academia, including one-year period of military training, he authored and co-authored 55 seminal research papers. There were only few mathematicians who could match his productivity, originality and depth of work done in such a short period of time.

Marcinkiewicz was born on April 12, 1910 in Cimoszka, a small village nearby Janów in the sokolski county (nowadays, podlaski province) in Poland, in an affluent farmer's family of Klemens Marcinkiewicz and Aleksandra, née Chodakiewicz. Józef had four siblings: the oldest sister Stanisława, two older brothers Mieczysław and Edward, and a younger one—Kazimierz. The war fate of his family was very tragic. Both parents died of hunger in 1941, half a year after being deported to Buchara in Uzbekistan (then the Soviet republic). The youngest brother Kazimierz was shot in 1946 by the Polish communist secret police (UB) for 'an attempt to escape the prison,' and Józef was murdered in Kharkov in Spring of 1940; his name appears on the list of the Katyń massacre victims. His untimely death was a huge loss to Polish and world mathematics.

As a child, Marcinkiewicz was sickly and, at the beginning, was home schooled. Subsequently, he attended King Sigismund August State Gymnasium in Białystok, where he graduated in 1930. Two of his math teachers the most influenced scientific interests of young Józef. The first was Zenon Krassowski, who shared his private math library and whose son befriended Józef. The second was Konstanty Kosiński, a person of wide intellectual horizons, community activist and a journalist; he quickly recognized enormous talent of his young student.

The next three years following high school graduation, Marcinkiewicz spent as a student in the Stefan Batory University (USB) in Wilno (now, Vilnius—the capital of Lithuania). He was of a very good luck to be under the tutelage of wonderful people and outstanding researchers, like professors Stefan Kempisty (1892–1940), Juliusz Rudnicki (1881–1948), and most of all, Antoni Zygmund (1900–1992) who became his academic advisor, teacher and friend.

As a sophomore, Marcinkiewicz obtained permission from his mentor to attend his lecture *Introduction to theory of Lebesgue integral*. This course preceded another class, on orthogonal series, offered by Zygmund. After merely three years of study, Marcinkiewicz obtained his Master of Philosophy in Mathematics degree. During his university years, Marcinkiewicz was very engaged in the activities of Math and Physics Circle, whose president he became during 1932–1933 academic year.

\*\* The participants of the XI Congress of Scientific Mathematical–Physical and Astronomical Circles (25–28 May 1933) in Wilno. J. Marcinkiewicz is sitting in the middle of the front row and A. Zygmund is second from the right. (Pic #2)

Among colleagues and friends of Marcinkiewicz were: Stanisław Kolankowski (physics major), Wanda Onoszko, Danuta Grzesikowska-Sadowska and Leon Jeśmianowicz, a future math professor at the Nicolaus Copernicus University in Toruń, Poland. In [29], Professor Jeśmianowicz made some recollections on the atmosphere and academic life of math students and faculty in the pre-war Wilno,

before it became incorporated by the Lithuanian Soviet Socialist Republic (1940–41) and (1944–1990/91). After 1991, it became the capital of the Republic of Lithuania. Those mathematicians who survived WWII emigrated from Wilno westward and settled in various cities of the post-war Poland, including Toruń (where the traditions of USB are still being continued), Gdańsk, and Łódź. Jeśmianowicz also notes that A. Zygmund was known for his sarcastic and pointed remarks and 'in this regard he yielded only to Marcinkiewicz.' In 1987, the author had a chance to have a longer conversation with L. Jeśmianowicz where, among other issues, he shared some information on Zygmund's activities as an adviser of the Math Circle of Polish–Jewish students at USB.

Marcinkiewicz graduated from USB in June of 1933. His Master's thesis contained original results on trigonometric interpolation. Its title was *Convergence of the Fourier-Lebesgue series* and was supervised by A. Zygmund. Among other things, he proved a theorem on existence of a continuous periodic function, whose sequence of interpolation trigonometric polynomials corresponding to uniformly distributed nodes is divergent almost everywhere (more on this later).

In 1933–35, Marcinkiewicz was working as an assistant to the chair of A. Zygmund. An extended version of his Master's thesis became a basis for his doctorate [52] (archived in [106], see the references in [46]). Marcinkiewicz defended his doctoral dissertation in June of 1935; its title was *Interpolation polynomials of absolutely continuous functions*, again under the supervision of A. Zygmund.

\*\* Marcinkiewicz's Alma Mater (Pic # 3)

As a recipient of a fellowship from the Fund for National Culture, he spent the academic year 1935–1936 at the Jan Kazimierz University in Lwów at the chair of Stefan Banach (December 1935–August 1936). He collaborated with Juliusz P. Schauder (1899–1943), who had returned to Lwów in 1934 (after spending time working with J. Hadamard and J. Leray in Paris), Stefan Kaczmarz (1895–1939) and Władysław Orlicz (1903–1990).

Thus far, Marcinkiewicz's research focused on problems related to trigonometric series. It is clear that studying with Kaczmarz and Schauder, and being influenced by Orlicz and (indirectly) by Banach, expanded his interest to general orthogonal series. Seven of Marcinkiewicz's papers published after 1935 deal with it. The paper [75], on multipliers of orthogonal systems, is the only work co-authored with Kaczmarz. It is worth mentioning that just in 1935, Kaczmarz and Steinhaus wrote the first book on theory of general orthogonal series [30].

\*\* Marcinkiewicz & Zygmund in Wilno, 1936 (Pic # 4)

Marcinkiewicz was appointed a senior assistant to the chair of mathematics at USB for the period 1 September 1936–31 August 1937 and, in April 1937, he filled in an application to initiate the habilitation process. One month later, Zygmund wrote the following opinion about the achievements of his pupil:

*From the above discussion the work of Dr. Marcinkiewicz shows that it contains a number of interesting and important results. Some of them, due to their final form, will certainly appear in textbooks in mathematics. It should be mentioned that in some of the early papers we can already see strong and subtle arithmetic*

*techniques; things of rare quality. The entire collection is extremely favorable and testifies to the multilateral and original mathematical talent of the author.*

In June 1937 Marcinkiewicz submitted his habilitation (senior doctorate), *On summability of orthogonal series*, and was appointed a *docent* (associate professor) at USB. The title of his habilitation lecture was *Arithmetization of notion of eventual variable*. At the age of 27, Marcinkiewicz became the youngest PhD who had ever obtained habilitation at USB.

In 1938 he received, this time one-year, scholarship from the Fund for National Culture for traveling to Paris, London and Stockholm to master his probability theory and mathematical statistics skills. He stayed in Paris for six months (October 1938–March 1939), where he collaborated with Stefan Bergman (1895–1977), Raphaël Salem (1898–1963) and also stayed in touch with a high stature mathematician Paul Levy (1886–1971). After Paris, Marcinkiewicz headed to London on his scientific journey where he spent five months (April–August 1939) at the University College, London. A trip to Stockholm never materialized, since at the end of August 1939, he returned from London to Wilno. Staying in London was not an option, it meant desertion for him.

In June 1939, Marcinkiewicz was appointed Extraordinary Professor at the University of Poznań, Poland. Beginning the academic year 1939–40, he was supposed to become the chair of the Department of Mathematics at the University of Poznań. Turmoils of the WWII did not spare university records and not until recently, there was no known document indicating that Marcinkiewicz was indeed appointed as the University of Poznań Professor. Several years ago, after three-day extensive search in the University of Poznań archives, Professor Lech Maligranda was able to dig out the proof (personal communication, June 2015).

In his article [6], Bielski portrays Marcinkiewicz *as a genuine patriot, citizen, and a scholar*, and describes in a greater detail his military training and an intensive effort made to excel as a reservist. After obtaining his MSc in 1933, Marcinkiewicz was drafted to a 1-year service (09–20–1933 to 09–16–1934), along with his friend S. Kolankowski. Then, in the subsequent years 1935, 1936, and 1938, he participated in 6-week military exercises. The last one took place just before his trip to France in 1938. His superiors were of the highest opinion about officer cadet Marcinkiewicz. Here are some quotes: *mature character, outstanding personality, outstanding intelligence, very energetic and full of initiative, sense of honor and ambition—very high, very dutiful and hardworking, social skills—very high, commands with ease, memory and logical thinking—very good, easily earns respect among soldiers, deep, precise and smart mind, very liked by the soldiers, independent, immune to negative influence, possesses administrative and organizational skills, very scrupulous and accurate, general assessment: outstanding.*

As a reserve officer (second lieutenant, 2<sup>nd</sup> Battalion, 205<sup>th</sup> Infantry Regiment), Marcinkiewicz took part in the defense of Lwów (September 12–21, 1939). The 2<sup>nd</sup> Battalion was the first unit of the Polish military that reached Lwów and was fighting Germans. The night of September 20/21, the Germans left the city and Lwów surrendered to Soviets who entered the city (September 22). Marcinkiewicz, together with other Polish officers, was taken a prisoner of war (September 25) by

the Red Army. S. Kolankowski recalled, [14], that Marcinkiewicz as a fit and strong person had a real chance to escape the transportation to a camp. He refused to do so. Kolankowski decided to run away and he survived.

From the scanty correspondence to his family ([14]), one learns that Marcinkiewicz asked for sending him some of his books and a copy of his doctoral diploma. The Soviets must have known whom they were keeping in the internment camp and apparently made an offer of collaboration, which Marcinkiewicz most likely rejected. That sealed his faith.

Marcinkiewicz was kept in the Starobielsk camp since September 1939 until April or May 1940. He was subsequently murdered in Kharkov where thousands of Polish officers were executed. He was likely buried in the village Piatichatki (now in Ukraine). The exact date of Józef Marcinkiewicz's death remains unknown, as some official Soviet documents are inaccessible or have been destroyed. The only known information is that it happened between April 5 and 12 May 12 of 1940.

We refer to the article [14], which is a collection of documents and testimonies, including the one of Mrs. Stanisława Lewicka, the sister of J. Marcinkiewicz, S. Kolankowski and Zbigniew Godlewski, MD who was a military physician and befriended Marcinkiewicz. Godlewski and some other 400 POWs, in a mysterious way, found their place in the transport to Pawliszczew Bór and survived. It is the only testimony from the inside of the camp that relates to Marcinkiewicz. Among other documents in the above publication one can find two reviews, mentioned earlier, of the work of Marcinkiewicz, one for a doctoral degree, another for habilitation; both reviews were handwritten by A. Zygmund. There is also a job contract for the academic year 1935–1936 which Marcinkiewicz spent at Jan Kazimierz University in Lwów (he was appointed Junior Assistant; one reads from this contract that the teaching load for this position was 12 hrs/wk (sic!)).

Józef Marcinkiewicz will be remembered primarily as an excellent mathematician and a true Polish patriot. He was passionate about mathematics, possessed an outstanding ability to concentrate on mathematical problems, had a tremendous proving power, and an extraordinary insight into mathematics. Marcinkiewicz's premature death was a huge loss to Polish and to the world mathematics. In his article about Marcinkiewicz ([48], p. 1), A. Zygmund recollects:

*His first mathematical paper appeared in 1933; the last one he sent for publication in the Summer of 1939. This short period of mathematical activity left however, a definite imprint on Mathematics, and but for his premature death he would probably have been one of the most outstanding contemporary mathematicians. Considering what he did during his short life and what he might have done in normal circumstances one may view his early death as a great blow to Polish Mathematics, and probably its heaviest individual loss during the second world war.*

In 1940 Zygmund with his family emigrated to the United States, and since 1947 he worked at the University of Chicago, where he established the most prominent school of analysis of the second half of the twentieth century. Arguably, it was Zygmund who, among Polish mathematicians (beside S. Banach), mostly influenced the world mathematics in the 20<sup>th</sup> century, and it was mainly due to him who survived the WWII and became a leading world analyst, that Marcinkiewicz's name

became widely known among mathematicians. A. Zygmund wrote in the 2<sup>nd</sup> (1959) edition of his *Trigonometric Series* (see also [135]):

*Dedicated to the memories of  
A. Rajchman and J. Marcinkiewicz  
my teacher and my pupil.*

### 3 Research achievements of Józef Marcinkiewicz

The domain of J. Marcinkiewicz's scientific activities was widely defined function theory. That includes Fourier series, interpolation theory of functions, probability theory (law of large numbers, law of iterated logarithm), complex functions, interpolation of operators, and maximal functions.

He secured his place in Analysis for: Marcinkiewicz interpolation theorem of operators, Marcinkiewicz integral and Marcinkiewicz function, Grünwald–Marcinkiewicz interpolation theorem of functions, Marcinkiewicz multiplier theorem,  $L^p$ –Marcinkiewicz–Zygmund inequalities, Marcinkiewicz theorem on characteristic functions, Marcinkiewicz spaces, Marcinkiewicz average, Marcinkiewicz–Zygmund law of large numbers, Jessen–Marcinkiewicz–Zygmund strong differentiation theorem, and holomorphic Marcinkiewicz symbols.

There were five co-authors of Marcinkiewicz papers: A. Zygmund (15), S. Bergmann (2), B. Jessen (1), Kaczmarz (1) and R. Salem (1). Thus, he was a single author of 36 papers. Numerous of his publications do not have any bibliographic items, simply because in many cases his work was so pioneering that there were no predecessors, that is, there was no one to be cited.

#### 3.1 Interpolation of functions

For the interval  $[-1, 1]$ , consider the array of points

$$T := \{x_{k,n} = \cos((2k-1)\pi/(2n)) : k = 1, \dots, n, n \in \mathbb{N}\}.$$

Given  $n \in \mathbb{N}$ , the points  $\{x_{1,n}, x_{2,n}, \dots, x_{n,n}\}$  are the distinct zeros of the Chebyshev polynomial

$$T_n(x) = \cos(n \arccos(x)),$$

where  $x \in [-1, 1]$ . Given a function  $f : [-1, 1] \rightarrow \mathbb{R}$  and  $n \in \mathbb{N}$ , let  $L_{n-1}(f, T, x)$  be the unique polynomial of degree  $\leq n-1$ , such that  $L_{n-1}(f, T, x_{k,n}) = f(x_{k,n})$ .

**Theorem 1** (*G. Grünwald, J. Marcinkiewicz 1936, [23], [63]*) *There exists a continuous function  $f$  on  $[-1, 1]$  such that for all  $x \in [-1, 1]$ ,*

$$\limsup_{n \rightarrow \infty} |L_{n-1}(f, T, x)| = +\infty.$$

See also [25]; for more recent developments, we refer to T. M. Mills and P. Vertesi [44].

Marcinkiewicz considered the trigonometric interpolation based on the equidistant nodes  $\left\{\frac{2k\pi}{2n+1}\right\}$  but it differs only formally from the Lagrange interpolation based on the Chebyshev matrix  $T$  presented above.

The lives of Józef Marcinkiewicz and Hungarian Géza Grünwald had several interesting and surprising coincidences. They both were born in 1910 and defended their doctoral dissertations in 1935. They both proved essentially the same theorem (see above) that Lagrange interpolation based at the zeros of Chebyshev polynomials can diverge *everywhere* even for continuous functions. In the experts' opinion, this theorem is one of the strongest results obtained by J. Marcinkiewicz and the strongest one by G. Grünwald. They both perished during WWII, Marcinkiewicz in the Katyn Massacre in Spring of 1940, Grünwald in the Holocaust on September 7, 1943 in Hungary.

Initially, both proved the existence of a continuous function  $g$  for which the sequence  $(L_{n-1}(T, g, x))$  diverges *almost everywhere*. On p. 16 of [48] A. Zygmund writes:

It is curious that a year later both authors could, independently of each other, strengthen their examples by constructing continuous functions whose Chebyshev interpolating polynomials diverge *everywhere*.

Grünwald-Marcinkiewicz theorem aside, instead of the array  $T$  presented at the beginning of this section, one may consider a general infinite triangular array

$$\begin{bmatrix} x_{1,1} & & & & \\ x_{1,2} & x_{2,2} & & & \\ x_{1,3} & x_{2,3} & x_{4,3} & & \\ \dots & \dots & \dots & \dots & \\ x_{1,n} & x_{2,n} & \dots & x_{n,n} & \\ \dots & \dots & \dots & \dots & \end{bmatrix}$$

with  $\{x_{i,j}\}$  representing nodes of an interpolation process on some interval  $[a, b]$  (assuming that for all  $i, j, n \in \mathbb{N}$ , if  $1 \leq i \neq j \leq n$ , we have  $x_{i,n} \neq x_{j,n}$ ). Let, as before, for a given continuous function  $f : [a, b] \rightarrow \mathbb{R}$ ,  $L_{n-1}(f, x)$  denote the unique (Lagrange) polynomial of degree not exceeding  $(n-1)$  interpolating  $f$  at the nodes  $x_{1,n}, x_{2,n}, \dots, x_{n,n}$  (i.e.,  $L_{n-1}(x_{i,n}) = f(x_{i,n})$ ), located at the  $n^{th}$  row of the above array. We say that the process  $(L_{n-1}(f))$  converges at  $x \in [a, b]$  to  $f$  if  $L_{n-1}(f, x) \rightarrow f(x)$  and similarly,  $(L_{n-1}(f))$  converges uniformly to  $f$ .

Already in 1914, G. Faber, [18], showed that there is no array of (non-equidistant) system of nodes for which the Lagrange interpolation polynomials would converge uniformly for any continuous function  $f$  on  $[-1, 1]$ . In turn, the following holds:

**Theorem 2** (Marcinkiewicz 1936, [62]) *For every continuous function  $f$  on  $[a, b]$  there exists a triangular matrix of nodes such that the corresponding interpolation process converges to  $f$  uniformly.*

A simple proof of this version of Marcinkiewicz's theorem was given in [15], Chapter 5, Theorem 1; see also a comment on p. 220 in [46].

### 3.2 Marcinkiewicz average

Let  $\tau_v f$  be the *shift* (translation) operator of a function  $f$  by an element  $v \in \mathbb{R}^n$ , i.e.,  $\tau_v f(x) = f(x - v)$ . Operator  $\mathcal{M}(R)$  is called the *Marcinkiewicz's average* of an operator  $R$  if

$$[\mathcal{M}(R)](f)(x) = \int_{[0,1]^n} \tau_{-v} R(\tau_v f)(x) dv.$$

In other words,  $\int_{[0,1]^n} \tau_{-v} R(\tau_v f)(x) dv = \int_{[0,1]^n} R(f(\cdot - v))(x + v) dv$ . The shift operators have been omnipresent in analysis for the past 80 years. Marcinkiewicz was using his transformation  $\mathcal{M}(R)$  in the paper [62] in relation to the interpolation theory of functions.

As an example of newer applications, the following spline characterization of the Hardy space of analytic functions  $H^1(\mathbb{T})$  on the torus  $\mathbb{T}$  is known:

**Proposition 1** (*Ciesielski 1983, [12]*) *For each  $r > 2$  there is a positive constant  $C_r$  such that for any  $f \in L^1(\mathbb{T})$  we have*

$$\frac{1}{C_r} \|f\|_{H^1(\mathbb{T})} \leq \left\| \sup_{n \geq 1} \left| \int_{\mathbb{T}} \tau_{-z} (Q_n^r \tau_z f) dz \right| \right\|_{L^1(\mathbb{T})} \leq C_r \|f\|_{H^1(\mathbb{T})}.$$

Here,  $Q_n^r$  is the orthogonal projection of  $L^2(\mathbb{T})$  onto the subspace  $S_n^r(\mathbb{T})$  of periodic splines of order  $r$  associated with the uniform mesh of cardinality  $n$ . The author also observed that the Marcinkiewicz average of the above orthogonal projection can be expressed as the convolution integral with the kernel representing the fundamental cardinal spline of order  $2r$ . Later, Beška and Dziedziul [5], by applying the ideas developed in [12], proved that the Marcinkiewicz average is also a useful tool in theory of Hardy spaces  $H^p(\mathbb{R}^n)$ .

### 3.3 Strong differentiability of functions

One of the remarkable results whose author or co-author was J. Marcinkiewicz, is the so-called Jessen–Marcinkiewicz–Zygmund theorem [56], which imposes a mild condition on a measurable function  $f$  so that its integral is strongly almost everywhere differentiable. Some terminology is necessary here; an excellent introduction to this subject is (still) a book [118] by S. Saks (1897–1942). Its Chapter IV, *Derivation of additive functions of a set and of an interval*, serves well this purpose.

An *interval* (rectangular box)  $\mathcal{I}$  in  $\mathbb{R}^n$ ,  $n > 1$ , is a Cartesian product of nonempty intervals,  $\delta(\mathcal{I})$  is its diameter. Given an interval  $\mathcal{I}_0 \subseteq \mathbb{R}^n$  and a function  $f$  integrable on  $\mathcal{I}_0$ , the integral of the function  $f$  is *strongly differentiable* at the point  $x_0 \in \mathcal{I}_0$  if the limit

$$\lim_{\delta(\mathcal{I}) \rightarrow 0} \frac{1}{|\mathcal{I}|} \int_{\mathcal{I}} f(x) dx$$

exists and is finite, where  $\mathcal{I} \subseteq \mathcal{I}_0$  is any interval containing  $x_0$ . The above limit is called the *strong derivative* of the integral of the function  $f$  at  $x_0$ .

**Theorem 3** (*Jessen–Marcinkiewicz–Zygmund 1935, [56]*) *Let  $\mathcal{I}_0$  be a fixed interval in  $\mathbb{R}^n$ . If a function  $f$  is measurable and  $|f(x)|(\log^+ |f(x)|)^{n-1}$  is integrable on the interval  $\mathcal{I}_0$ , then the strong derivative of the integral of the function  $f$  exists for almost all elements in  $\mathcal{I}_0$  and is equal to  $f$ .*

The authors pointed at two results:

- (a) There is a function  $f \in L^1(\mathcal{I}_0)$  such that its integral is nowhere strongly differentiable (proved by Saks in [118]), and
- (b) If  $f \in L^p(\mathcal{I}_0)$ ,  $p > 1$ , the strong derivative of the integral of  $f$  exists and is equal to  $f$  (proved by A. Zygmund, *Fund. Math.* 23 (1934), 143–149).

Their aim was to generalize and complete these results and to apply the generalizations to the theory of multiple Fourier series. It is interesting to note that the sufficient condition given in the above JMZ theorem is in a sense the optimal one. Let  $\phi : [0, \infty) \rightarrow \mathbb{R}$  be an increasing function satisfying the conditions

$$\phi(0) = 0, \quad \liminf_{t \rightarrow \infty} \frac{\phi(t)}{t} > 0,$$

and let  $L^\phi$  be the class of functions  $f$  such that  $\phi(|f|)$  is integrable over  $[0, 1]^n$ . The following result holds:

**Proposition 2** *If for every  $f \in L^\phi$  the integral of  $f$  is strongly differentiable almost everywhere, then  $\phi(t) > ct(\log^+ |f|)^{n-1}$  for some constant  $c > 0$ . In other words,  $f \cdot (\log^+ |f|)^{n-1}$  is integrable over  $[0, 1]^n$ .*

The above result is due to Saks and to Busemann and Feller. On a further discussion on matters related to JMZ theorem and to the strong maximal function, we refer to [46], pp. 197–198. On a separate note, especially valuable is a comment on p. 196, related to a maximal function associated with the mean average summation of double trigonometric series and investigated in [96].

### 3.4 Interpolation of operators

In his *Comptes Rendus* note ([94], 1939), Marcinkiewicz published three theorems on interpolation of linear/quasi-linear transformations on  $L^p$  spaces without offering any proofs. The announcements in this journal are very brief and detailed proofs appear elsewhere. Evidently, Marcinkiewicz must have had all the proofs beforehand, but due to the sequence of tragic events in his life, no one else knew these proofs. A. Zygmund reconstructed a proof of the general Marcinkiewicz interpolation theorem at the beginning of 1950-ies and the paper [133] was published in 1956. As Zygmund indicated in the footnote <sup>(2)</sup>, in June of 1939 he got a letter from Marcinkiewicz with the ideas of the proof (the diagonal case only, and more specifically, for weak type  $(1, 1)$  and  $(2, 2)$  transformations). A contemporary form of the general interpolation theorem, elaboration on the remaining two theorems in [94] (second one was actually on the interpolation in Orlicz spaces), and some new material, were presented in the work [133], which long time ago became a classic in the subject matter literature and, by all means, is a must-read/study by any aspiring student of mathematical analysis. Even

though that article is single-authored, Zygmund was always underscoring the fact that he merely reconstructed the proofs from the ideas shared with him by Marcinkiewicz, and by no means was he the author of the Marcinkiewicz interpolation theorem.

Around the mid-1950-ies, some of the Zygmund's students, including M. Cotlar and W. J. Riordan gave new proofs of Marcinkiewicz theorem or proposed some extensions. While the Riesz–Thorin theorem (RT) was called the *convexity theorem*, Marcinkiewicz seems to be the one who coined the term *interpolation of operators*. More on this later.

There are many very good/superb sources containing various versions and generalizations of the original Marcinkiewicz interpolation theorem. We mention the classical ones, like Bergh–Löfström [4], Krein, Petunin and Semenov [40], Stein [124] (Chap.1, Sect.4–diagonal case and Appendix B–general case, both cases in  $L^p(\mathbb{R}^n)$ ), Stein and Weiss [126] or a more recent one, by Lunardi [43], with applications to elliptic PDEs.

In order to present the Marcinkiewicz theorem, we need a brief introduction. A function  $f$  defined on a measure space  $(\Omega, \mu)$  is of *weak type*  $p$  ( $0 < p < \infty$ ), if there exists a constant  $C$  such that for every  $\lambda \in \mathbb{R}_+$  we have

$$\mu(\{\omega \in \Omega : |f(\omega)| > \lambda\}) \leq \left(\frac{C}{\lambda}\right)^p$$

and write  $f \in L^{p,\infty}(\Omega, \mu)$ . Here,  $L^{p,\infty}(\Omega, \mu)$  is the space of all functions of weak type  $p$  and  $\|f\|_{p,\infty}$  denotes the infimum over all constants  $C$  in the above inequality. Even though this quantity is not a norm, it will be called a (weak  $p$ )–norm. It somewhat describes the size of an element  $f$ : indeed, we have (only)  $\|f + g\|_{p,\infty} \leq 2(\|f\|_{p,\infty} + \|g\|_{p,\infty})$ , see p. 7 in [4]. The Chebyshev inequality readily implies that any function  $f \in L^p(\Omega, \mu)$  is in  $L^{p,\infty}(\Omega, \mu)$  and  $\|f\|_{p,\infty} \leq \|f\|_p$ . For  $1 \leq p < \infty$ , we also call  $L^{p,\infty}(\Omega, \mu)$  a weak- $L^p$  space.

Given are two measure spaces  $(\Omega, \mu)$  and  $(\Lambda, \nu)$  and a linear transformation  $T$  defined on  $L^p(\Omega, \mu)$  whose values are measurable functions on  $(\Lambda, \nu)$ .  $T$  is said to be of *weak type*  $(p, q)$  if there exists a constant  $C$  such that

$$\nu(\{y \in \Lambda : |Tf(y)| > \lambda\}) \leq \left(\frac{C\|f\|_p}{\lambda}\right)^q \quad \text{for all } \lambda > 0.$$

In case of  $q = +\infty$ , the weak type  $(p, \infty)$  condition means  $\|Tf\|_\infty \leq C\|f\|_p$ , that is, strong type  $(p, \infty)$  of  $T$ .

**Theorem 4** (Marcinkiewicz 1939, [94], Zygmund 1956, [133]) *Let  $1 \leq p_0, p_1, q_0, q_1 \leq +\infty$  and for any  $\theta \in (0, 1)$  define the exponents  $p, q$  by the equalities*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

*If  $p_0 \leq q_0$  and  $p_1 \leq q_1$  with  $q_0 \neq q_1$ , the boundedness of any linear or sublinear operator  $T : L^{p_0} \rightarrow L^{q_0,\infty}$  and  $T : L^{p_1} \rightarrow L^{q_1,\infty}$  implies the boundedness of  $T : L^p \rightarrow L^q$  and*

$$\|T\|_{L^p \rightarrow L^q} \leq C \|T\|_{L^{p_0} \rightarrow L^{q_0,\infty}}^{1-\theta} \|T\|_{L^{p_1} \rightarrow L^{q_1,\infty}}^\theta,$$

where  $C = C(p_0, p_1, q_0, q_1, \theta)$  is independent of  $f$ , and  $C \rightarrow +\infty$  when  $\theta \rightarrow 0^+$  or  $\theta \rightarrow 1^-$ .

Some comments are in place: The above two conditions imposed on the operator  $T$  simply mean that  $T$  is simultaneously of weak type  $(p_i, q_i)$  for  $i = 0, 1$ , and together imply that  $T$  is of strong type  $(p, q)$  which means boundedness of  $T : L^p \rightarrow L^q$ .

The conditions  $p_0 \leq q_0$  and  $p_1 \leq q_1$  imposed on the exponents indicate that the points  $\left(\frac{1}{p_0}, \frac{1}{q_0}\right)$  and  $\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$  belong to the lower triangle with vertices  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$ . When both of the above points are in an open upper triangle, Hunt showed (in 1964) that Marcinkiewicz theorem is no longer valid. For more detailed discussion and comments, consult [47].

Zygmund in [133] actually proved validity of Marcinkiewicz theorem for a *quasi-linear* transformation  $T$ , i.e.,  $|T(f_1 + f_2)| \leq \kappa(|Tf_1| + |Tf_2|)$  if  $T(f_1 + f_2)$  is defined whenever both  $Tf_1$  and  $Tf_2$  are defined, and  $\kappa \geq 1$  is a constant independent of  $f_1$  and  $f_2$ ; in addition,  $|T(\alpha f)| = |\alpha||T(f)|$  (positive homogeneity). The inequality is understood as a pointwise inequality. When  $\kappa = 1$ ,  $T$  with the above property becomes a *sublinear* transformation.

When the exponents are on the diagonal of the above triangle (rather square), i.e., they look like  $(p_0, p_0)$  and  $(p_1, p_1)$ , Marcinkiewicz interpolation theorem takes a simple form called the *diagonal case*:

**Theorem 5** (*Marcinkiewicz theorem 1939–diagonal case*). *If  $1 \leq p_0 < p_1 \leq \infty$  and  $T$  is an arbitrary linear or sublinear operator of weak type  $(p_0, p_0)$  and weak type  $(p_1, p_1)$  (that is, for  $i = 0, 1$ ,  $T : L^{p_i} \rightarrow L^{p_i, \infty}$  is bounded) then  $T : L^p \rightarrow L^p$  is bounded for any  $p_0 < p < p_1$  (that is,  $T$  is of strong type  $(p, p)$ ). In addition, as  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$  for some  $0 < \theta < 1$ , one has*

$$\|T\|_{L^p \rightarrow L^p} \leq 2 \left( \frac{p}{p-p_0} + \frac{p}{p_1-p} \right)^{\frac{1}{p}} \|T\|_{L^{p_0} \rightarrow L^{p_0, \infty}}^{1-\theta} \|T\|_{L^{p_1} \rightarrow L^{p_1, \infty}}^{\theta}.$$

There are manifold generalizations of Marcinkiewicz theorem made by various authors, including Astashkin & Maligranda, Cotlar, Krein, Riordan, Stein, Stein & Weiss, Strichartz. The entirely new discipline, the interpolation theory, was developed (real and complex methods) based on theorems of Riesz-Thorin and Marcinkiewicz.

To show applicability of Marcinkiewicz's theorem, we invoke some background material, fundamental for the classical function theory and which had been known to analysts of the pre-WWII era. When  $F \in L^p[-\pi, \pi]$  for some  $p \geq 1$ , the function  $U(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) F(t) dt$  represents a harmonic function on the disk  $|z| < 1$ . Here,  $z = re^{i\theta}$  and

$$P_r(t) = \frac{1-r^2}{1+r^2-2r \cos t}$$

is the Poisson kernel associated with the unit disk. Thus,  $U(re^{i\theta})$  is a convolution of the function  $F$  and the Poisson kernel. A deep work of Fatou and Lebesgue makes the following claim true:

**Proposition 3** *Let  $p \in (1, \infty]$  and let  $U(z)$  be harmonic in  $|z| < 1$ , and for some constant  $C$ ,*

$$\left( \int_{-\pi}^{\pi} |U(re^{i\theta})|^p d\theta \right) \leq C$$

*uniformly in  $0 \leq r < 1$ . Then, for almost all  $\theta$ ,  $U(z)$  tends to a finite limit, say  $U(e^{i\theta})$ , as  $z \rightarrow e^{i\theta}$  non-tangentially in  $|z| < 1$ ,  $U(re^{i\theta}) \in L^p(-\pi, \pi)$ , and for  $0 \leq r < 1$ ,*

$$U(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} U(e^{it}) dt.$$

In case of  $p = 1$ , situation is more exotic as we need to handle a convolution of a Poisson kernel with a measure.

For any  $U(z)$ -harmonic in  $|z| < 1$  and having a representation given above, there exists a unique harmonic conjugate  $V(z)$ , so  $U(z) + iV(z)$  becomes analytic in  $|z| < 1$ . Recovering  $V(z)$  requires use of tools of a standard vector calculus course. It is determined up to a constant. So, assume  $V(0) = 0$ . Such a unique harmonic conjugate is denoted by  $\tilde{U}(z)$ . Given expansion of  $U$  in  $|z| < 1$

$$U(re^{i\theta}) = \sum_{-\infty}^{\infty} A_n r^{|n|} e^{in\theta}$$

one gets,

$$\tilde{U}(re^{i\theta}) = - \sum_{-\infty}^{\infty} i \operatorname{sgn} n A_n r^{|n|} e^{in\theta},$$

where  $\operatorname{sgn} 0 = 0$ . We readily check that

$$U(re^{i\theta}) + i\tilde{U}(re^{i\theta}) = A_0 + \sum_{-\infty}^{\infty} 2A_n r^n e^{i\theta}$$

is *analytic* in  $|z| < 1$ . In case of  $U(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta-t) d\mu(t)$  with a measure  $\mu$  on  $[-\pi, \pi]$  the above series expansion for  $U$  works with coefficients  $A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int} d\mu(t)$ . The series development for  $\tilde{U}$  now looks like

$$\tilde{U}(re^{i\theta}) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{-\infty}^{\infty} i \operatorname{sgn} n r^{|n|} e^{in(\theta-t)} d\mu(t).$$

The quantity

$$Q_r(t) := -\frac{1}{2\pi} \sum_{-\infty}^{\infty} i \operatorname{sgn} n r^{|n|} e^{in\theta}$$

is called the *conjugate Poisson kernel*. Direct manipulation shows

$$Q_r(\theta) = \frac{2r \sin \theta}{1 + r^2 - 2r \cos \theta}.$$

Thus, if  $U(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) d\mu(t)$ , then the harmonic conjugate is

$$\tilde{U}(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2r \sin(\theta - t)}{1 + r^2 - 2r \cos(\theta - t)} d\mu(t).$$

We are interested in an absolutely continuous measure  $\mu$ , i.e.,  $d\mu(t) = F(t)dt$ , where a function  $F$  belongs to some  $L^p(-\pi, \pi)$ , with  $p \geq 1$ . Assume that  $F$  is  $2\pi$ -periodically extended to  $\mathbb{R}$ , hence

$$\begin{aligned} \tilde{U}(re^{i\theta}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2r \sin(\theta - t)}{1 + r^2 - 2r \cos(\theta - t)} F(t) dt \\ &= \frac{1}{\pi} \int_0^{\pi} \frac{\sin s}{1 + r^2 - 2r \cos s} [F(\theta - s) - F(\theta + s)] ds. \end{aligned}$$

We have  $\frac{r \sin s}{1 + r^2 - 2r \cos s} = \frac{2r \sin \frac{s}{2} \cos \frac{s}{2}}{1 + r^2 - 4r \sin^2 \frac{s}{2}}$ , so if  $\int_0^{\pi} (|F(\theta - s) - F(\theta + s)|) < \infty$ ,

we have that for  $r \rightarrow 1$ ,

$$\tilde{U}(re^{i\theta}) \rightarrow \frac{1}{2\pi} \int_0^{\pi} \frac{F(\theta - s) - F(\theta + s)}{\tan \frac{s}{2}} ds$$

with the integral on the right being absolutely convergent. For a general function  $F$ , no differentiability is required and the convergence will be almost everywhere with respect to  $\theta$ . One should be mindful about convergence of the above integral, and it should be interpreted as

$$\tilde{F}(\theta) = \lim_{\varepsilon \rightarrow 1} \int_{\varepsilon}^{\pi} \frac{F(\theta - s) - F(\theta + s)}{\tan \frac{s}{2}} ds.$$

Whenever it exists, the limit is the same as

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left\{ \int_{-\pi}^{\theta - \varepsilon} + \int_{\theta + \varepsilon}^{\pi} \right\} \frac{F(t) dt}{2 \tan \frac{\theta - t}{2}} &=: p.v. \int_{-\pi}^{\pi} \frac{F(t) dt}{2 \tan \frac{\theta - t}{2}} \\ &= p.v. \int_{-\pi}^{\pi} \frac{F(\theta - t) dt}{2 \tan \frac{t}{2}} = -p.v. \int_{-\pi}^{\pi} \frac{F(\theta + s) ds}{2 \tan \frac{s}{2}}, \end{aligned}$$

that is,

$$\tilde{F}(\theta) = -p.v. \int_{-\pi}^{\pi} \frac{F(\theta + s) ds}{2 \tan \frac{s}{2}}.$$

So, it is the Cauchy *principal value* of the integral.

A systematic exposition of the subject outlined above can be found in the Koosis' book [37]; consult also Duren [16]. It turns out that the conjugation mapping  $F \rightarrow \tilde{F}$  is of weak type  $(1, 1)$  (Kolmogorov, [35]) and strong type  $(2, 2)$  (Parseval identity), see also [37]. Since any type  $(2, 2)$  transformation is of weak type  $(2, 2)$ , Marcinkiewicz interpolation theorem, the diagonal case here, implies that  $F \rightarrow \tilde{F}$  is of strong type  $(p, p)$  for  $1 < p \leq 2$ . By the duality argument one gets that the conjugation operator is of strong type  $(p, p)$  for all  $p > 1$ , that is,  $\|\tilde{F}\|_p \leq C_p \|f\|_p$ . This

transformation is not of strong type  $(1, 1)$  and this fact makes the problem more interesting. The norms  $C_p$  will grow indefinitely as  $p$  approaches 1 and it prohibits the case  $p = 1$  of being strong type  $(1, 1)$ . This is the case when the RT convexity theorem cannot do the job. Marcinkiewicz was fully aware of the role of the RT theorem. It was proved first by M. Riesz (1926), then extended by his student G. O. Thorin in his MSc thesis in 1938. In his article [134], A. Zygmund discusses the similarities and differences of the RT theorem and the Marcinkiewicz interpolation theorem. The important thing is that Marcinkiewicz theorem can be proved using purely real methods, while the RT requires complex methods. A. Zygmund writes:

*Really you don't know what's happening [with the proof of the Riesz–Thorin theorem]. You apply the three-circle theorem and the result comes out neatly. But what actually is going on? You do not know. Now, the proof of Marcinkiewicz has the advantage that actually you see with naked eyes what various parts of functions contribute to the value of the operation.*

At the end of this section, we indicate the paper by T. Iwaniec [28]. The author points out that both the RT theorem and Marcinkiewicz deal with the estimates of the  $L^p$  norm of an operator knowing its behavior at the endpoints of the interval of the exponents  $p$ , where the operator is still defined. While the RT theorem works well (with numerous applications, especially when dealing with sharp inequalities for singular integrals) for linear transformations, e.g., quick and elegant proof of the Hausdorff–Young inequality ([4], p. 6), Marcinkiewicz's approach can be adopted to nonlinear operators, as the author explains in his work. He highlights new advances of Marcinkiewicz interpolation theorem which arise from the study of the nonlinear  $p$ -harmonic type PDEs. One of the tools that prompts this nonlinear setting is the Hodge decomposition of differential forms.

### 3.5 Marcinkiewicz spaces

A concept of Marcinkiewicz space is closely related to interpolation. Given a function  $f$  on a measure space  $(\Omega, \mu)$ , the *distribution function* of  $f$  is

$$\lambda_f(\alpha) := \mu(\{\omega \in \Omega : |f(\omega)| > \alpha\}).$$

The function  $\lambda_f(\alpha)$  is right-continuous and non-increasing function of  $\alpha$ . Change of variables in the integral gives

$$\|f\|_p = \left( p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty$$

and

$$\|f\|_\infty = \inf\{\alpha : \lambda_f(\alpha) = 0\}.$$

We can also define  $f^*$ , the so-called *nonincreasing rearrangement* of a function  $f$ :

$$f^*(s) := \inf\{\alpha : \lambda_f(\alpha) \leq s\}.$$

It also enjoys the property that  $\lambda_{f^*}(\alpha) = \lambda_f(\alpha)$  for  $\alpha \geq 0$ . The weak  $L^p$ -space  $L^{p,\infty}(\Omega, \mu)$  (introduced by Marcinkiewicz), is called now a *Marcinkiewicz space* and is endowed with a quasi-norm  $\|f\|_{p,\infty} = \sup_{\alpha} \alpha \lambda_f(\alpha)^{\frac{1}{p}}$ . In fact, we have

$$\|f\|_{p,\infty} = \sup_{s>0} s^{\frac{1}{p}} f^*(s) = \sup_{\alpha>0} \alpha \lambda_f(\alpha)^{\frac{1}{p}}.$$

In [47], Maligranda indicates that the above definition can easily be generalized to the situation when  $I = (0, 1)$  or  $I = (0, \infty)$  and a quasi-concave function  $\phi : I \cup \{0\} \rightarrow \mathbb{R}_+$  such that  $\phi(0) = 0$ . Quasi-concave means that the inequality  $\phi(s) \leq \max(1, s/t)\phi(t)$  holds true for all  $s, t \in I$ .

The first generalization is the *Marcinkiewicz function space*  $M_\phi^*$  on  $I$ , which consists of measurable functions equipped with the quasi-norm

$$\|f\|_\phi := \sup_{t \in I} \phi(t) f^*(t) < \infty.$$

Another Marcinkiewicz function space  $M_\phi$  on  $I$  is generated by the norm

$$\|f\|_\phi := \sup_{t \in I} \phi(t) f^{**}(t), \quad \text{where} \quad f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds.$$

We have the inclusion  $M_\phi \subset M_\phi^*$  and  $\|f\|_\phi^* \leq \|f\|_\phi$  for  $f \in M_\phi$ . When  $\phi(t) = t^{\frac{1}{p}}$  and  $1 < p < \infty$ , we get  $M_\phi^* = M_\phi = L^{p,\infty}$ , while for  $\phi(t) = t$ ,  $M_\phi^* = L^{p,\infty}$  and  $M_\phi = L^1$ .

Maligranda goes further and discusses properties of the above spaces in a greater detail. In addition, he presents two other types of Marcinkiewicz sequence spaces  $m_\phi^*$  and  $m_\phi$ .

### 3.6 Wiener–Lévy theorem

Let  $A$  denote the space of all functions  $f \in L^1(\mathbb{T})$  with absolutely convergent Fourier series, i.e.,  $f(t) \sim \sum_{-\infty}^{\infty} c_n e^{int}$  and  $\sum_{-\infty}^{\infty} |c_n| < \infty$ , where

$$c_n := \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) e^{-in\theta} d\theta.$$

The Wiener–Lévy theorem says that if  $f \in A$  and  $\phi$  is analytic on a domain containing the range of  $f$ , then the composition function  $\phi \circ f(t)$  also belongs to  $A$ . Wiener's original formulation was the case of  $\phi(z) = \frac{1}{z}$ . We easily check that  $A$  equipped with the norm  $\|f\|_A := \sum_{-\infty}^{\infty} |c_n|$  is a Banach algebra (isometric to  $l^1(\mathbb{Z})$ ).

In [102], Marcinkiewicz worked on improvements of the Wiener–Lévy theorem trying to get estimates of  $\|e^{iaf}\|_A$  when  $f$  was a real-valued function and  $a \in \mathbb{R}$ . J.-P. Kahane rediscovered this important work and gave it a proper recognition, see [32].

To see the input of Marcinkiewicz, we need some notation. The class  $A_p$ ,  $0 < p < \infty$ , is a collection of all functions for which  $\sum_{-\infty}^{\infty} |c_n|^p < \infty$ . Let  $I$  be an open interval on the real axis and let  $W(t)$  be a real-valued function defined on  $I$ . We say that  $W \in G_p$ , given that for each compact subset  $J$  of  $I$  there exists a constant  $B$  depending on  $J$  such that  $|W^{(n)}| \leq B^n (n!)^{\frac{1}{p}}$  for all  $t \in J$ .

and all  $n \geq 1$ . (NB,  $W \in G_1 \equiv W$  is real-analytic). Here is Marcinkiewicz's extension of the classical Wiener–Lévy theorem in which the conditions on  $f(x)$  are strengthened while conditions on  $W(t)$  are weakened:

**Theorem 6** (Marcinkiewicz 1939, [102]) *Let  $I$  be an open interval on the real axis and let  $W(t)$  be defined on  $I$  with  $W \in G_p$  for some  $0 < p \leq 1$ . If  $(x)$  maps  $[\pi, \pi]$  into  $I$  and  $f \in A_p$ , then  $W \circ f \in A_1$ . (Zygmund noted that actually  $W \circ f \in A_p$ .)*

Riviere and Sagher [116] established the following converse to Marcinkiewicz's theorem.

**Proposition 4** (Riviere–Sagher [116], 1966) *Let  $I$  be an open interval on the real axis and let  $W(t)$  be defined on  $I$ . Let  $0 < p \leq 1$  and suppose that for each  $f \in A_p$  whose range is contained in  $I$ ,  $W \circ f \in A_p$  for some  $\rho < 2$  (depending on  $f$ ). Then  $W \in G_p$ .*

Additional information on extensions of the Wiener–Lévy theorem can be found in [122].

Marcinkiewicz's idea contributed to the solution of the following two problems in Banach algebras:

- (1) Is the function  $\phi$ , for which the conclusion of the Wiener–Levy theorem holds, necessarily analytic?
- (2) The problem of spectral synthesis for the Wiener algebra  $A$ : Is there a one-to-one correspondence between closed ideals of  $A$  and closed subsets of  $\mathbb{T} \simeq \mathbb{R}/2\pi\mathbb{Z}$ ?

Using the Wiener–Lévy theorem one can identify the set of maximal ideals of  $A$  as elements of  $\mathbb{T}$ .

As it was mentioned earlier, Kahane succeeded in the complete estimate of the norm  $\|e^{iaf}\|_A$ . Problem (1) was solved by Y. Katznelson in the positive, while Problem (2) was solved by P. Malliavin in the negative. Rudin's book [117] may come handy.

### 3.7 Marcinkiewicz integral

Given a closed subset  $F$  of an open and bounded interval  $(a, b)$ ,  $d_F(\cdot)$  is a distance function to  $F$ , and  $\lambda > 0$ . In the series of papers, [57], [82], [58] and [98], Marcinkiewicz studied the integral

$$I_\lambda(x) := \int_a^b \frac{(d_F(t))^\lambda}{|t - x|^{\lambda+1}} dt$$

and its limiting version for  $\lambda = 0$ ,

$$I_0(x) = \int_a^b \frac{|\log \text{dist}(t, F)|^{-1}}{|x - t|} dt.$$

**Theorem 7** (Marcinkiewicz, 1935–39) *For the integrals defined above, one has  $I_\lambda(x) < \infty$  almost everywhere on  $F$ ; moreover,  $I_\lambda$  is summable on  $F$  and  $\int_F I_\lambda(x) dx \leq \frac{2}{\lambda} |(a, b) \setminus F|$ .*

If  $f \in L^1(a, b)$  then

$$J_\lambda(f)(x) := \int_{\mathbb{R}} \frac{d_F(y)^\lambda f(y)}{|x - y|^{1+\lambda}} dy$$

is summable on  $F$  and

$$\int_F |J_\lambda| \leq (2/\lambda) \int_a^b |f|.$$

Here are multidimensional versions of the above theorem (see [124]): Let  $F$  be a closed subset of  $\mathbb{R}^n$ . Then the integral

$$I(x) := \int_{|y| \leq 1} \frac{d_F(x+y)}{|y|^{n+1}} dy$$

is finite for almost every  $x \in F$  and  $I(x) = \infty$  for any  $x \notin F$ .

The Marcinkiewicz integral is a critical tool in Calderón–Zygmund proof of the weak-type  $(1, 1)$  estimate for higher-dimensional singular integrals ([10] and [124], Chapter 1).

### 3.8 Marcinkiewicz function

In their three-part work [42], Littlewood and Paley introduced and studied the following function  $g$ :

$$g(f)(\theta) = \left( \int_0^1 (1-r) |\Phi'(re^{i\theta})|^2 d\theta \right)^{\frac{1}{2}},$$

where  $\Phi(z)$  is an analytic function in  $|z| < 1$  and whose real part has boundary value  $f(\theta)$ . They proved existence of two constants  $A_p, B_p$  independent of  $f$ , such that for any  $1 < p < \infty$ , one has

$$\|g(f)\|_p \leq A_p \|f\|_p \quad \text{and} \quad \|f\|_p \leq B_p \|g(f)\|_p;$$

the latter inequality is valid under the restriction  $\int_0^{2\pi} f(\theta) d\theta = 0$ .

Lusin, in his earlier studies ([45]) developed the function, called the *Lusin area function*

$$S(\Phi)(\theta) := \left( \int_{\Omega(\theta)} |\Phi'(x+iy)|^2 dx dy \right)^{\frac{1}{2}},$$

where  $\Omega(0) = \Omega$  is a standard ‘kite-shaped’ region inside the unit circle with vertex at  $z = 1$  and  $\Omega(\theta)$  is the region  $\Omega$  rotated through an angle  $\theta$  around  $z = 0$ . Marcinkiewicz and Zygmund showed in [81] that for  $0 < p < \infty$ ,

$$\|S(\Phi)\|_p \leq A_p \|\Phi\|_p.$$

As a consequence (by a celebrated M. Riesz’s theorem), one gets  $\|S(\Phi)\|_p \leq A_p \|f\|_p$ . Marcinkiewicz and Zygmund noted in [81] that  $S$  is essentially a majorant of  $g$ .

Subsequently, in [82] Marcinkiewicz introduced what we call the *Marcinkiewicz function*:

$$\mu(f)(\theta) := \left( \int_0^{2\pi} \frac{|F(\theta+t) + F(\theta-t) - 2F(\theta)|^2}{t^3} dt \right)^{\frac{1}{2}}$$

where  $F(\theta) = \int_0^\theta f(t)dt$ . His intention was to provide an analogue of the function  $g(f)$ , defined above, without reaching into the interior of the unit disk for its definition.

In [132], Zygmund proved that for any  $1 < p < \infty$ , the following inequalities (conjectured by Marcinkiewicz) hold true:

$$\|\mu(f)\|_p \leq A_p \|f\|_p \quad \text{and} \quad \|f\|_p \leq A_p \|\mu(f)\|_p,$$

again, with the restriction  $\int_0^{2\pi} f(\theta)d\theta = 0$  for the latter inequality.

In [82], Marcinkiewicz proved the following two statements:

**Theorem 8** (*Marcinkiewicz, 1938*)

(1) *Suppose  $f$  is periodic and integrable. Then*

$$\mu(f)(\theta) = \left( \int_0^{2\pi} \frac{|F(\theta+t) + F(\theta-t) - 2F(\theta)|^2}{t^3} dt \right)^{\frac{1}{2}}$$

*is finite almost everywhere.*

(2) *Let  $F$  be any function in  $L^2[0, 2\pi]$ ,  $2\pi$ -periodic, and differentiable in a set  $E$  of positive measure. Then the integral  $\mu(f)(\theta)$  is finite for almost all points of the set  $E$ .*

Obviously, statement (2) is more general than (1).

Marcinkiewicz also considered more general integrals

$$\mu_p(f)(\theta) := \left( \int_0^{2\pi} \frac{|F(\theta+t) + F(\theta-t) - 2F(\theta)|^p}{t^{p+1}} dt \right)^{\frac{1}{p}},$$

and proved the inequalities

$$\|\mu_p(f)\|_p \leq C_p \|f\|_p \quad \text{for } p \geq 2, \text{ and}$$

$$\|f\|_p \leq C_p \|\mu_p(f)\|_p \quad \text{for } 1 < p \leq 2 \quad \text{with} \quad \int_0^{2\pi} f(\theta)d\theta = 0.$$

The real-line version of the Marcinkiewicz function is

$$\mu(f)(x) := \left( \int_{\mathbb{R}^1} \frac{|F(x+t) + F(x-t) - 2F(x)|^2}{t^3} dt \right)^{\frac{1}{2}}, \quad f \in L^1(\mathbb{R})$$

and its  $n$ -dimensional analogue

$$\left( \int_{\mathbb{R}^n} \frac{|F(x+t) + F(x-t) - 2F(x)|^2}{|t|^2} \frac{dt}{|t|^n} \right)^{\frac{1}{2}}.$$

In the real-line case, the Hilbert transform corresponds to the conjugate function operator and is presented by the formula

$$H(f)(x) = p.v. - \frac{1}{\pi} \int_0^\infty \frac{f(x+t) - f(x-t)}{t} dt.$$

It may be rewritten in the form

$$-\frac{1}{\pi} \int_0^\infty \frac{F(x+t) + F(x-t) - 2F(x)}{t^2} dt$$

which is suitable for further applications.

In his seminal paper [123], E. M. Stein generalized the Marcinkiewicz function to higher dimensions and proved similar results by means of the so called real variables method, in the following setting. Let  $\Omega(x)$  be a homogeneous function of degree 0 and satisfy two additional conditions:

- (1)  $\Omega(x)$  is continuous on  $S^n$ —the unit sphere of  $\mathbb{R}^n$  and satisfies a Lipschitz condition of order  $\alpha$  there, i.e.,

$$|\Omega(x') - \Omega(y')| \leq c|x' - y'|^\alpha, \quad x', y' \in S^n.$$

(2)  $\int_{S^n} \Omega(x') dx' = 0.$

For a locally integrable function  $f$  on  $\mathbb{R}^n$  and  $t > 0$ , let  $F_t(f, x) := F_t(x)$  be given by

$$F_t(x) := \int_{\{y: |y| \leq t\}} \frac{\Omega(y)}{|y|^{n-1}} f(x-y) dy, \quad x \in \mathbb{R}^n,$$

and define now  $\mu(f)(x)$  by

$$\mu(f)(x) := \left( \int_0^\infty \frac{|F_t(x)|^2}{t^3} dt \right)^{\frac{1}{2}}.$$

Stein showed that if  $f \in L^p(\mathbb{R}^n)$ ,  $1 \leq p \leq 2$ , then

$$\|\mu(f)\|_p \leq c_p \|f\|_p, \quad 1 < p \leq 2, \text{ and}$$

$$|\{\mu(f) > \lambda\}| \leq \frac{c}{\lambda} \|f\|_1, \quad \text{for } p = 1 \text{ and all } \lambda > 0.$$

The  $\mu(f)$  formula given above greatly impacted a subsequent development of theory of singular integrals, see Stein's book [125]. For the origins of singular integrals, see Calderon–Zygmund paper [9].

Before we close this important section, let us indicate some connections of Marcinkiewicz function with BMO theory. By a *cube* in  $\mathbb{R}^n$  we mean an interval in  $\mathbb{R}^n$  with all edges of equal length. A function  $f$  integrable over a cube  $Q$  is said to be of *bounded mean oscillation* on  $Q$  if there is a constant  $c$  such that

$$\frac{1}{|\tilde{Q}|} \int_Q |f(x) - f_{\tilde{Q}}| dx \leq c$$

for all cubes  $\tilde{Q} \subseteq Q$ , where  $f_{\tilde{Q}} := \frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} f$ . The BMO space over  $Q$  is a collection of such functions.

The least constant  $c$  in the above inequality is a seminorm on this vector space. Similarly, we define the BMO class of locally integrable functions on  $\mathbb{R}^n$ .

Within the past of almost of sixty years, the BMO spaces rose to the prominence as one of the most important classes of functions in harmonic analysis. Introduced by F. John and L. Nirenberg

in 1961 in relation to partial differential equations, in 1971 was characterized by C. Fefferman (see [19], [20] and literature therein) as the dual of the Hardy space  $H^1$  (over the unit disk or  $\mathbb{R}^n$ ).

Assume  $\mathcal{I}_0$  is an open interval (a product of segments) in  $\mathbb{R}^n$ , bounded or not, and  $F$  is locally integrable on  $\mathcal{I}_0$ , we say  $F$  has first partial derivatives (in a distributional sense)  $F_j$ ,  $j = 1, \dots, n$ , in  $\mathcal{I}_0$  if there exist locally integrable functions  $F_j$  on  $\mathcal{I}_0$  such that

$$\int F \frac{\partial}{\partial x_j} \psi = - \int F_j \psi, \quad j = 1, \dots, n,$$

for every  $\psi$  which is infinitely differentiable with compact support in  $\mathcal{I}_0$ .

A. Torchinsky and R. L. Wheeden ([130]) proved the following

**Proposition 5** *Let  $\mathcal{I}_0$  be an open interval, bounded or not, in  $\mathbb{R}^n$ . Let  $F$  be a function defined on  $\mathcal{I}_0$  such that for every closed cube  $Q \subset \mathcal{I}_0$ ,  $F$  is bounded on  $Q$  and*

$$\sup_{x \in Q} \left( \int_{|t| < \delta(x; \mathcal{I})} |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{|t|^{n+2}} \right)^{\frac{1}{2}} < \infty.$$

*Then  $F$  has first partial derivatives in  $\mathcal{I}_0$  which are in BMO on  $Q$ , for every  $Q$  as above. Here,  $\delta(x; \mathcal{I}) := \text{dist}(x, \mathcal{I}^c)$ , where  $\mathcal{I}^c$  is the complement of  $\mathcal{I}$  in  $\mathbb{R}^n$ .*

By a slight modification of the above integral, one may obtain a characterization of such  $F$  which possess first partial derivatives (in a distributional sense) in BMO. For more details and further results, see [130].

Beside Marcinkiewicz interpolation theorem, Marcinkiewicz integral and Marcinkiewicz function are one of the most fundamental inputs of Marcinkiewicz to analysis.

### 3.9 Marcinkiewicz and probability theory

Modern probability theory, based on Kolmogorov's three 'commandments' created in 1933, became a very rapidly developing discipline which attracted many inventive researchers who inspired the whole generation of pre-WWII mathematicians. The timing was perfect, probability theory was founded on the ground of a contemporary function theory, that is, measure theory and integration. Soon, many results that started to appear on this fertile ground were powerful, beautiful, and ready for applications in the neighboring disciplines such as real and complex analysis, functional analysis, quantum physics, statistics, and more.

Marcinkiewicz's interest in probability theory is clearly visible in his writing. The sequel of papers, [66, 68, 72, 73, 76, 77, 85, 91, 97, 99], shows it. Paley and Zygmund earlier publications [114] created a launching pad for Marcinkiewicz's work in probability, including the most recognized [66] and [76] (both with A. Zygmund). The Paley-Zygmund work also motivated J.-P. Kahane to pen his monograph [33].

There are several prominent results in the above papers. We present them here. The first is a theorem on maximal functions:

**Theorem 9** (Marcinkiewicz–Zygmund 1937, [66]) Let  $(X_i(t)), i = 1, \dots, n$ , be a finite sequence of independent random functions on  $[0, 1]$  with zero mean values, i.e.,

$$\int_0^1 X_i(t) dt = 0 \quad (i = 1, \dots, n).$$

Put

$$X(t) := \sum_{i=1}^n X_i(t), \quad X^*(t) := \max_{1 \leq m \leq n} \left| \sum_{i=1}^m X_i(t) \right|.$$

Then, for any  $p > 1$ , one has

$$\int_0^1 [X^*(t)]^p dt \leq 2 \left( \frac{p}{p-1} \right)^p \int_0^1 |X(t)|^p dt.$$

Marcinkiewicz–Zygmund inequality

**Theorem 10** (Marcinkiewicz–Zygmund 1938, [76]) For  $p \geq 1$ , there exist positive constants  $A_p, B_p$  such that for any sequence  $(X_j)$  of independent, 0-mean random variables one has

$$A_p \left\| \left( \sum_{j=1}^{\infty} X_j^2 \right)^{1/2} \right\|_p \leq \left\| \sum_{j=1}^{\infty} X_j \right\|_p \leq B_p \left\| \left( \sum_{j=1}^{\infty} X_j^2 \right)^{1/2} \right\|_p.$$

For  $1 \leq p \leq 2$  the following inequalities hold:  $A_p \geq A > 0$  and  $B_p \leq B < \infty$ , where  $A$  and  $B$  are absolute constants.

For more details and comments on extensions of these theorems and their applications, we recommend the book by Y. S. Chow and H. Teicher, [11].

Finally, we have the *Marcinkiewicz–Zygmund Law of Large Numbers* (LLN)

**Theorem 11** (Marcinkiewicz–Zygmund, 1937) For  $(X_j)$ –identically distributed independent random variables,  $S_n := \sum_{j=1}^n X_j$ , and  $0 < p < 2$ , TFAE:

1.  $\mathbb{E}|X_1|^p < \infty$ ,
2. There exists a constant  $c$  with

$$\frac{(S_n - nc)}{n^{1/p}} \rightarrow 0 \text{ a.e. as } n \rightarrow \infty.$$

(For  $1 \leq p < 2$  put  $c = \mathbb{E}X$ , while  $c$  is arbitrary for  $0 < p < 1$ .)

This result, which provides an extension of the Kolmogorov strong law of large numbers (the  $p = 1$  case only), may be generalized in various directions. For a range of alternative extensions to the Kolmogorov strong law, see Bingham [7].

The LLN was also generalized in the context of geometry of Banach spaces, see A. de Acosta [1], who proved that for any  $p \in [1, 2)$  and any Banach space  $B$  the following are equivalent:

- (1) the Marcinkiewicz–Zygmund Law of Large Numbers holds for  $B$ ,
- (2)  $B$  has *type*  $p$ .

Recall that a Banach space  $B$  is of *type*  $p$  ( $1 \leq p \leq 2$ ) if there exists a constant  $\alpha$  such that for all finite sequences  $(x_j)_{j=1}^n \subset B$

$$\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\| \leq \alpha \left( \sum_{j=1}^n \|x_j\|^p \right)^{1/p},$$

and

a Banach space  $B$  is of *cotype*  $p$  ( $2 \leq p \leq \infty$ ) if there exists a constant  $\alpha$  such that for all finite sequences  $(x_j)_{j=1}^n \subset B$

$$\alpha \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\| \geq \left( \sum_{j=1}^n \|x_j\|^p \right)^{1/p}.$$

Here,  $r_j(t) := \text{sgn} \sin(2^j \pi t)$  for  $t \in [0, 1]$  ( $j \in \mathbb{N}$ ) are *Rademacher functions* on  $[0, 1]$ . It is a model of a sequence of independent, symmetric Bernoulli random variables with identical distributions. Obviously,  $(r_j(t))$  is an orthonormal set of functions in  $L^2[0, 1]$ .

The concepts of *type*  $p$  and *cotype*  $p$  are fundamental in geometry of Banach spaces. Let us point out that the restrictions on  $p$  in the definition *type*  $p$  and *cotype*  $p$  are essential in the sense that for  $p > 2$  (for *type*  $p$ ) and for  $1 \leq p < 2$  (for *cotype*  $p$ ) we will not get meaningful definitions. *Marcinkiewicz–Zygmund Law of the Iterated Logarithm* (LiL) Let  $(X_n)$  be an infinite sequence of independent, bounded and 0-mean random variables. For any  $n \in \mathbb{N}$ , put:  $M_n := \sup |X_n|$ ,  $b_n := \mathbb{E}(X_n^2)$ , and  $B_n := b_1 + b_2 + \dots + b_n$ . Assume that  $B_n \rightarrow \infty$ . By the Kolmogorov's Law of the Iterated Logarithm, if

$$M_n = o\left(\sqrt{B_n / \log \log B_n}\right),$$

then with the probability 1,

$$\limsup_{n \rightarrow \infty} \frac{|\sum_{\nu=1}^n X_\nu|}{\sqrt{2B_n \log \log B_n}} = 1.$$

In [68], Marcinkiewicz and Zygmund proved that the above condition is sharp, that is, one cannot replace 'o' by 'O'. We direct the reader to [8] and [46] (pp. 183–185) for an extensive discussion on questions related to (LiL).

Marcinkiewicz ideas in probability influenced later development of martingale theory; it is well documented in numerous works of D. L. Burkholder and R. A. Gundy.

Finally, the following result, called *Marcinkiewicz theorem on characteristic functions*, is on the borderline of probability theory and complex analysis. The *characteristic function* of a random variable  $\xi : \Omega \rightarrow \mathbb{R}$  is the function  $\phi_\xi : \mathbb{R} \rightarrow \mathbb{C}$  given by

$$\phi_\xi(t) = \mathbb{E}(e^{it\xi}) = \int_{-\infty}^{\infty} e^{its} \mu_\xi(ds), \quad t \in \mathbb{R},$$

where  $\mu_\xi$  is the distribution function of the random variable  $\xi$ .

**Theorem 12** (Marcinkiewicz 1939, [85]) *Let  $P_m(t)$  be a polynomial of degree  $m > 2$ . Then the function  $f(t) = e^{P_m(t)}$  is not a characteristic function.*

In the same paper [85], Marcinkiewicz also established a new property of the Gauss' law:

**Theorem 13** (Marcinkiewicz, 1939) *If  $x_1, x_2, \dots$  are independent random variables with the same distribution function and finite moments of all orders and if for certain numerical sequences  $(a_n)$  and  $(b_n)$  the sums  $\sum a_n x_n$  and  $\sum b_n x_n$  exist and have the same distribution functions, then the sequences  $(|a_n|)$  and  $(|b_n|)$  are merely rearrangements of each other or else the  $(x_n)$  are Gaussian variables.*

Marcinkiewicz's contribution to theory analytic functions, apart from the work that was presented in the preceding sections, includes two papers with Stefan Bergman, [93] and [105], on the limiting behavior of analytic functions  $f(z_1, z_2)$  of two variables in somewhat general domains. The problem is to extend to such domains known results of F. Riesz and Hardy and Littlewood about classes  $H^p$  of analytic functions of a single variable defined in the unit disk.

### 3.10 $L^p$ –Marcinkiewicz–Zygmund inequalities

This important inequality was obtained by the authors in their work [64].

**Theorem 14** (Marcinkiewicz–Zygmund, 1937) *For any  $1 < p < \infty$  there exist two constants  $C_1$  and  $C_2$  such that for any trigonometric polynomial  $T$  of degree  $n$ ,*

$$C_1 \int_0^{2\pi} |T(\theta)|^p d\theta \leq \frac{1}{n} \sum_{k=0}^{2n} \left| T\left(\frac{2\pi k}{2n+1}\right) \right|^p \leq C_2 \int_0^{2\pi} |T(\theta)|^p d\theta.$$

The constants  $C_1$  and  $C_2$  depend only on  $p$ .

It establishes an equivalence between the continuous and discrete  $L^p$ -norms of certain basis functions. This inequality has been extended in several directions. K. Gröchenig [22] generalized Marcinkiewicz–Zygmund (MZ) inequalities to the case of non-uniformly distributed nodes (case of trigonometric polynomials). It was of great interest to come up with MZ inequalities for situations where the underlying set is different from the classical settings like the torus or the real axis. The Euclidean sphere was the manifold which initially attracted major interest in this regard. Mhaskar, Narcowich and Ward did some pioneer work in this direction and devised in [111] *spherical MZ inequalities*.

Let  $SO(3) := \{\mathbf{x} \in GL(3, \mathbb{R}) : \mathbf{x}^T \mathbf{x} = I_3, \det(\mathbf{x}) = 1\}$  be the non-abelian compact group of rotations in the Euclidean space  $\mathbb{R}^3$ . The problem of approximating functions defined on  $SO(3)$  is of great importance in various applications, including crystallography and more specifically, in orientations of a crystal in three dimensional space. In his work [120], D. Schmid developed MZ inequalities on  $SO(3)$  and applied to polynomial approximation from scattered data on  $SO(3)$ .

### 3.11 Marcinkiewicz–Zygmund operator inequalities

In their paper [95], Marcinkiewicz and Zygmund considered bounded (continuous) linear transformations  $T : L^r(\mu_1) \rightarrow L^p(\mu_2)$  and established the following inequalities:

**Theorem 15** (*Marcinkiewicz–Zygmund 1939, [95]*) *For any  $p \in (1, \infty]$ , for any  $f_j \in L^p(\mu_1)$ ,  $1 \leq j \leq n$ , the following inequality holds true:*

$$\left\| \left( \sum_{j=1}^n |Tf_j|^2 \right)^{1/2} \right\|_p \leq \|T\| \left\| \left( \sum_{j=1}^n |f_j|^2 \right)^{1/2} \right\|_p.$$

Following the argument of G. Pisier presented in [115], we have: The case  $p = 2$  is trivial; the case  $p = \infty$  (and hence  $p = 1$  by duality) is obvious, as we have the following linearization of the ‘square function norm:’

$$\left\| \left( \sum |f_j|^2 \right)^{1/2} \right\|_\infty = \sup \left\{ \left\| \sum a_j f_j \right\|_\infty : a_j \in K, \sum |a_j|^2 \leq 1 \right\}.$$

Here,  $\mathbb{K} := \mathbb{R}$  or  $\mathbb{C}$ . The remaining case  $1 < p < \infty$  is an easy consequence of Fubini’s Theorem and the isometric embedding of  $l^2$  into  $L^p$  provided by the independent standard Gaussian variable: Indeed, if,  $(g_j)$  is an independent, identically distributed sequence of Gaussian variables relative to a probability  $\mathbb{P}$ , we have for any scalar sequence  $(\lambda_j) \in l^2$ ,

$$\left( \sum |\lambda_j|^2 \right)^{1/2} = \|g_1\|_p^{-1} \left\| \sum \lambda_j g_j \right\|_p.$$

Raising this to the  $p^{th}$  power (set  $\lambda_j := f_j(t)$ ) and integrating with respect to  $\mu_1(dt)$ , we find for any  $(f_j)$  in  $L^p(\mu_1)$ ,

$$\left\| \left( \sum |f_j|^2 \right)^{1/2} \right\|_p = \|g_1\|_p^{-1} \left( \int \|g_j(\omega) f_j\|_p^p d\mathbb{P}(\omega) \right)^{1/p}.$$

Pisier reformulates the classical Grothendieck’s Theorem in the Marcinkiewicz–Zygmund style:

**Proposition 6** *For any pair of measurable spaces  $(\Omega_1, \mu_1), (\Omega_2, \mu_2)$  and any bounded linear map  $T : L^\infty(\mu_1) \rightarrow L_1(\mu_2)$ , for any  $n \in \mathbb{N}$  and any  $f_j \in L^\infty(\mu_1)$ , for  $j = 1, \dots, n$ , one has*

$$\left\| \left( \sum_{j=1}^n |Tf_j|^2 \right)^{1/2} \right\|_1 \leq K \|T\| \left\| \left( \sum_{j=1}^n |f_j|^2 \right)^{1/2} \right\|_\infty.$$

Moreover, the optimal  $K$  is the Grothendieck constant  $K_G$ . There are several directions in functional analysis in which the original development is still heading; we will not address it here.

### 3.12 Fourier multipliers

As it was indicated earlier, Marcinkiewicz collaborated with Kaczmarz and Schauder while staying in Lwów during the academic year of 1935–36. Both, Kaczmarz and Schauder were working on multiplier operators; Kaczmarz in a more general setup (dealing with orthogonal expansions in the  $L^p$  spaces), while Schauder focused on Fourier multipliers. The work Marcinkiewicz did with Kaczmarz resulted in their paper [75], the only common work of these two authors.

We would like to highlight Marcinkiewicz's achievements on Fourier multipliers. Let  $f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{int}$  be a Fourier expansion of an  $L^p[0, 2\pi]$  function  $f$ , where  $p \geq 1$ . Given a numerical sequence  $(\lambda_n)_{n \in \mathbb{Z}}$ , we consider the series  $\sum \lambda_n c_n e^{int}$  and asking whether it represents another function in  $L^p[0, 2\pi]$ . More formally, one tries to find (necessary and sufficient) conditions on  $(\lambda_n)_{n \in \mathbb{Z}}$ , so that a linear transformation  $T_\lambda(f) \sim \sum \lambda_n c_n e^{int}$  is a bounded linear mapping on  $L^p[0, 2\pi]$ .

**Theorem 16** (Marcinkiewicz 1939, [90]) *The operator  $T_\lambda$  is bounded on  $L^p[0, 2\pi]$  if for some  $M > 0$ , the multiplier  $(\lambda_n)$  satisfies  $|\lambda_n| \leq M$  and*

$$\sum_{2^{k-1} \leq |n| < 2^k} |\lambda_n - \lambda_{n-1}| \leq M \quad (k = 1, 2, \dots).$$

These are only sufficient conditions. The complete characterization exists when  $p = 1$  and  $p = 2$ . The case  $p = 1$  asserts that  $(\lambda_n)_{n \in \mathbb{Z}}$  is a multiplier on  $L^1[0, 2\pi]$  if and only if there exists a finite measure  $\mu$  on  $[0, 2\pi]$  such that  $\lambda_n = \int_0^{2\pi} e^{-int} d\mu(t)$ . The case of  $p = 2$  is easy:  $(\lambda_n)_{n \in \mathbb{Z}}$  is a multiplier on  $L^2[0, 2\pi]$  if and only if  $(\lambda_n) \in l^\infty(\mathbb{Z})$  (see [129], Chap.V.4). The problem of a complete characterization for the remaining exponents  $p$  is still open.

To better understand the nature of multiplier transformations, we have that each  $T_\lambda$  is a translation-invariant operator and vice-versa, any translation-invariant operator can be presented as a multiplier (see Prop. 4.2 and Cor. 4.3 in [129], Chap.V.4). We already met the translation operators while discussing the Marcinkiewicz average.

There are several theorems included in [90]. They are ordered in a somewhat peculiar way, where at first, Theorem 2 was placed. This theorem concerns multipliers of double Fourier series, while Theorem 1, presented above, is for a single variable. Marcinkiewicz provides examples of multipliers in case of double Fourier series:

$$\frac{m^2}{m^2 + n^2}, \quad \frac{n^2}{m^2 + n^2}, \quad \text{and} \quad \frac{mn}{m^2 + n^2}.$$

Let us introduce some more notation. For a double sequence  $(\lambda_{nm})_{n,m \in \mathbb{Z}}$ , consider the multiplier operator  $T_\lambda$  defined as follows: When  $f = \sum c_{nm} e^{i(nx+my)}$ ,  $T_\lambda f := \sum \lambda_{nm} c_{nm} e^{i(nx+my)}$  and dyadic intervals  $I_k := \{i \in \mathbb{Z} : 2^{k-1} \leq |i| < 2^k\}$  and  $J_l := \{j \in \mathbb{Z} : 2^{l-1} \leq |j| < 2^l\}$ . In addition, let  $\Delta_1 \lambda_{nm} = \lambda_{n+1,m} - \lambda_{n,m}$ ,  $\Delta_2 \lambda_{nm} = \lambda_{n,m+1} - \lambda_{n,m}$  be the forward differences in respective indices  $n$  and  $m$ , and  $\Delta_{1,2} = \Delta_1 \cdot \Delta_2$ —the composition of  $\Delta_1$  and  $\Delta_2$ .

A more ‘digestive’ form of Theorem 2 is the following (see [46], p. 201):

**Theorem 17** (Marcinkiewicz 1939, [90]) *Let  $1 < p < \infty$ . Assume that the following four supremum are finite:  $A = \sup_{n,m} |\lambda_{n,m}|$ ,*

$$B_1 = \sup_{k,m} \sum_{n \in I_k} |\Delta_1 \lambda_{n,m}|, \quad B_2 = \sup_{n,l} \sum_{m \in J_l} |\Delta_2 \lambda_{n,m}|, \quad B_{1,2} = \sup_{k,l} \sum_{n \in I_k} \sum_{m \in J_l} |\Delta_{1,2} \lambda_{n,m}|,$$

*then the multiplier operator  $T_\lambda$  is bounded in  $L^p([0, 2\pi]^2)$  and*

$$\|T_\lambda f\|_p \leq A(A + B_1 + B_2 + B_{1,2})\|f\|_p \quad \text{for } f \in L^p.$$

The results of Theorem 3 of [90] are direct extensions of those of Theorem 2 and simply mean that in case of  $n$  variables, one gets next two sets of multipliers in the spaces  $L^p[0, 2\pi]$  for all  $p > 1$ :

$$\frac{m_i^2}{m_1^2 + m_2^2 + \dots + m_n^2} \quad \text{and} \quad \frac{m_i m_j}{m_1^2 + m_2^2 + \dots + m_n^2} \quad (i, j = 1, 2, \dots, n).$$

In the footnotes <sup>(7)</sup> and <sup>(8)</sup>, Marcinkiewicz explicitly stated that Theorems 3 and 4 were solutions to the problems posed to him by J. Schauder. Zygmund writes ([48], p. 3):

*The influence of Schauder was particularly beneficial and would probably have led to important developments had time permitted. For in the field of real variable Marcinkiewicz had exceptionally strong intuition and technique, and the results he obtained in the theory of conjugate functions, had they been extended to functions of several variables might have given (as we see clearly now) a strong push to the theory of partial differential equations. The only visible trace of Schauder's influence is a very interesting paper of Marcinkiewicz on the multipliers of Fourier series, a paper which originated in connection with a problem proposed by Schauder (...).*

After WWII, S. G. Mikhlin [112] and soon after L. Hörmander, were the first who considered the multiplier problem in a non-periodic case (in  $\mathbb{R}^n$ ). Let  $m$  be a bounded measurable function on  $\mathbb{R}^n$ . Hence, one can define a linear transformation  $T_m$  defined on  $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$  by the following relation between Fourier transforms

$$\widehat{(T_m f)}(x) = m(x) \hat{f}(x), \quad f \in L^2 \cap L^p.$$

The function  $m$  is a *multiplier* for  $L^p$  ( $1 \leq p \leq \infty$ ) if for any  $f \in L^2 \cap L^p$ ,  $T_m f \in L^p$  (observe that already  $T_m f \in L^2$ ), and  $T_m$  is bounded, i.e.,  $\|T_m f\|_p \leq A\|f\|_p$  when  $f \in L^2 \cap L^p$  (with  $A$  independent of  $f$ ). The smallest  $A$  with this property is called the *norm* of the multiplier  $m$ . When  $p < \infty$ , the above boundedness condition allows extension  $T_m$  to the entire space  $L^p(\mathbb{R}^n)$ . We denote  $\mathcal{M}_p$  the Banach space of multipliers  $m$  with the above norm. It is also Banach algebra with pointwise multiplication. We have

**Proposition 7** (Mikhlin [112], Hörmander [26]) *Suppose that  $m$  is of class  $C^k$  in the complement of the origin of  $\mathbb{R}^n$ , where  $k$  is an integer  $> \frac{n}{2}$ . Assume also that for every differential monomial  $(\frac{\partial}{\partial x})^\alpha$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ , with  $|\alpha| = \alpha_1 + \dots + \alpha_n$ , we have*

$$\left| \left( \frac{\partial}{\partial x} \right)^\alpha m(x) \right| \leq B|x|^{-|\alpha|}, \quad \text{whenever } |\alpha| \leq k. \quad (1)$$

Then the multiplier operator  $T_m$  maps  $L^p \rightarrow L^p$ ,  $1 < p < \infty$ ; that is,  $\|T_m f\|_p \leq A_p \|f\|_p$ , and  $A_p$  depends only on  $B, p$  and  $n$ .

**Corollary 1** (Stein 1970, [124], p. 96) *Actually, the assumption (1) can be replaced by the weaker assumptions*

$$|m(x)| \leq B,$$

$$\sup_{0 < R < \infty} R^{2|\alpha|+n} \int_{R \leq \|x\| \leq 2R} \left| \left( \frac{\partial}{\partial x} \right)^\alpha m(x) \right|^2 dx \leq B' \quad \text{for all } |\alpha| \leq k.$$

Assume now that the function  $m$  is defined on each open orthant in  $\mathbb{R}^n$  along with all its continuous partial derivatives of order  $\leq n$ , so  $m$  may be left undefined on the set of all points with at least one vanishing coordinate. For any  $k \leq n$  we embed  $\mathbb{R}^k$  in a canonical way into  $\mathbb{R}^n$ . Observe that

$$\mathbb{R}^1 = \bigcup_{k \in \mathbb{Z}} [-2^{k+1}, -2^k] \bigcup \{0\} \bigcup_{k \in \mathbb{Z}} [2^{k+1}, 2^k].$$

We call it the *dyadic decomposition* of  $\mathbb{R}^1$ . We can write  $\mathbb{R}^n$  as a union of almost disjoint rectangles (disjoint interiors) that are products of intervals that appear in the dyadic decomposition of each of the axes. Such a collection we call the dyadic decomposition of  $\mathbb{R}^n$ .

A result presented below may be considered a consequence of Marcinkiewicz multiplier theorem.

**Theorem 18** (Marcinkiewicz 1939, [90], Stein 1970, [124]) *Let  $m$  be a bounded function on  $\mathbb{R}^n$  with the properties listed above. Suppose that*

- (a)  $|m(x)| \leq B$ ,
- (b) for each  $0 < k \leq n$ ,

$$\sup_{x_{k+1}, \dots, x_n} \int_{\rho} \left| \frac{\partial^k m}{\partial x_1 \partial x_2 \cdots \partial x_k} \right| dx_1 \cdots dx_k \leq B$$

as  $\rho$  ranges over dyadic rectangles of  $\mathbb{R}^k$ . (In case of  $k = n$ , the ‘sup’ sign is omitted).

(c) The condition analogous to (b) is valid for every one of the  $n!$  permutations of the variables  $x_1, x_2, \dots, x_n$ .

Then  $m \in \mathcal{M}_p$  for  $1 < p < \infty$ , and more precisely, if  $f \in L^2 \cap L^p$ ,  $\|T_m f\|_p \leq A_p \|f\|_p$ , where  $A_p$  depends only on  $B, p$ , and  $n$ .

Up today, the characterization of multiplier transformations has not been completed yet.

### 3.13 On Riemannian sums and Marcinkiewicz–Salem conjecture

Raphaël Salem (1898-1963) co-authored one publication, [103], with J. Marcinkiewicz. They met in Paris while Marcinkiewicz was on leave during 1938–39 academic year. This work is about the conduct of Riemann sums when function  $f$  is Riemann integrable or merely Lebesgue integrable. The authors considered functions on the unit interval  $[0, 1)$  identified with the torus  $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$ . Let

$\lambda$  be the standard Lebesgue measure on  $\mathbb{T}$ . For a  $\lambda$ -measurable function  $f$  any  $n \in \mathbb{N}$ , we consider the  $n^{th}$  Riemann sum associated with the uniformly distributed sequence of nodes on  $\mathbb{T}$ :

$$T_n f(x) := \frac{1}{n} \sum_{i=0}^{n-1} f\left(x + \frac{i}{n}\right), \quad x \in \mathbb{T}.$$

Jessen proved that

**Proposition 8** (*Jessen, 1934*) *If  $f \in L^1(\mathbb{T})$  and  $(n_k)$  is an increasing sequence of natural numbers such that  $n_{k+1}$  is divisible by  $n_k$  (for all  $k \in \mathbb{N}$ ), then*

$$\lim_{k \rightarrow \infty} T_{n_k} f(x) = \int_0^1 f(t) d\lambda(t) \quad \text{for almost all } x.$$

Marcinkiewicz and Salem showed

**Theorem 19** (*Marcinkiewicz–Salem [103], 1940*) *For any positive increasing function  $\omega$  satisfying the condition*

$$\lim_{x \rightarrow \infty} \frac{\omega(x)}{\log x} = 0,$$

*one can find a function  $f$  such that*

$$\int_{\mathbb{T}} |f| \omega(|f|) d\lambda < \infty \quad \text{and} \quad \int_{\mathbb{T}} \sup_{k \geq 0} |T_{2^k} f| d\lambda = +\infty.$$

They also proved the existence of a function  $f \in L^1(\mathbb{T})$  such that  $\limsup_{n \rightarrow \infty} |T_n f(x)| = +\infty$  for any  $x$ . One cannot have convergence almost everywhere even for bounded functions.

Other results demonstrated in this paper we present here as parts of one theorem:

**Theorem 20** (*Marcinkiewicz–Salem, 1940*)

(1) *If*

$$\int_{\mathbb{T}} (f(x+t) - f(x))^2 d\lambda(x) = O(t^\varepsilon), \quad \varepsilon > 0,$$

*then the sequence  $(T_n f)_{n \geq 1}$  is convergent almost everywhere to  $\int_{\mathbb{T}} f d\lambda$ .*

(2) *If*

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \frac{(f(x+t) - f(x))^2}{t |\log(t/2)|} d\lambda(t) d\lambda(x) < \infty,$$

*then the sequences of averages  $(A_n f)_{n \geq 1}$  is convergent to  $\int_{\mathbb{T}} f d\lambda$ , where  $A_n f = \frac{1}{n} \sum_{k=1}^n T_k f$ .*

(3) *If*

$$\int_{\mathbb{T}} |f(x+t) - f(x)| d\lambda(x) = O\left(\frac{1}{|\log t|^p}\right), \quad p > 1,$$

*then the sequence of averages  $(A_n f)_{n \geq 1}$  is convergent almost everywhere to  $\int_{\mathbb{T}} f d\lambda$ .*

Intimately related to the theorems presented above is

**Marcinkiewicz–Salem Conjecture (1940):** If  $f \in L^1(\mathbb{T})$ , then the sequence of averages  $A_n f$  is convergent almost everywhere.

For related ideas and further detailed discussion on Marcinkiewicz–Salem Conjecture, we refer to [46], pp. 204–206.

### 3.14 General orthogonal systems of functions and the Haar functions

In parallel to his initial primary interest in the trigonometric system, J. Marcinkiewicz was also working with general orthogonal sets of functions. His results in this direction are mainly in the papers [60], [79], [89], as well as in [65] and [80] (with A. Zygmund).

Recall that a system of (real) functions  $(\phi_n)_{n=1}^\infty$  is *orthonormal* in  $L^2(a, b)$  if

$$\int_a^b \phi_m \phi_n dx = \delta_{m,n} \quad \text{for } m, n \in \mathbb{N}.$$

The symbol  $\delta_{m,n}$  is the Kronecker's delta. Given any function  $f \in L^2(a, b)$  one computes, in a natural way, its Fourier coefficients

$$c_n = \int_a^b f \phi_n dx \quad \text{for } n \in \mathbb{N}.$$

Bessel's inequality guarantees  $\sum_{n=1}^\infty |c_n|^2 < \infty$ . When  $(\phi_n)_{n=1}^\infty$  is linearly dense in  $L^2(a, b)$ , one has  $f = \sum_{n=1}^\infty c_n \phi_n$  ( $\|\cdot\|_2$ -norm convergence). When the system  $(\phi_n)$  is uniformly bounded, one can compute  $c_n$ 's even for  $f \in L^1(a, b)$ .

Two theorems given below are of quite general nature and were published in separate articles.

**Theorem 21** (Marcinkiewicz 1936, [60]) *For any orthonormal system  $(\phi_n)_{n=1}^\infty$  of functions on the interval  $(a, b)$ , there exists an increasing sequence  $(n_i)_{i=1}^\infty$  of natural numbers such that for any series*

$$\sum_{n=1}^\infty a_n \phi_n(t) \quad \text{with} \quad \sum_{n=1}^\infty |a_n|^2 < \infty,$$

*the subsequence of partial sums  $(S_{n_i}(t))_{i=1}^\infty$  is almost everywhere convergent for  $t \in (a, b)$ ; here  $S_{n_i}(t) := \sum_{n=1}^{n_i} a_n \phi_n(t)$ . Moreover, the following inequality holds*

$$\left\| \sup_{i \geq 1} |S_{n_i}(t)| \right\|_2 \leq C \left( \sum_{n=1}^\infty |a_n|^2 \right)^{1/2},$$

*for some universal constant  $C$ , that is, independent of the system  $(\phi_n)_{n=1}^\infty$ .*

Despite the fact that the constant  $C$  is independent of the orthogonal system  $(\phi_n)_{n=1}^\infty$  used in the above theorem, the subsequence  $(n_i)$  of the sequence of natural numbers in general depends on

$(\phi_n)$ . If we demand more than  $(a_n) \in l^2$ , e.g.,  $\sum_{n=1}^{\infty} |a_n|^2 \log n < \infty$ , then the sequence  $S_{2^n}(t) := \sum_{k=1}^{2^n} a_k \phi_k(t)$  converges on  $(a, b)$  almost everywhere to  $f(t)$ , where  $f \in L^2$  and  $f = \sum a_i \phi_i$ .

Assuming merely  $(a_n) \in l^2$ , we cannot guarantee universality of  $(n_i)_{i=1}^{\infty}$ , e.g., for trigonometric system it is enough to put  $n_i := 2^i$  and for the Haar system it is enough to put  $n_i = i$ . For further discussion on everywhere convergence of orthogonal systems, see Sikorski [121] and Kashin and Saakyan [34].

Another result was published jointly with A. Zygmund in [65]. The authors provided the following generalization of the Hausdorff–Young theorem (for trigonometric system and  $\nu = \infty$ ) and the F. Riesz theorem (1923, for uniformly bounded orthonormal system and  $\nu = \infty$ ).

**Theorem 22** (*Marcinkiewicz–Zygmund, 1937*) *Let  $\{\phi_n\}_{n=1}^{\infty}$  be an orthonormal system in  $L^2(a, b)$  such that  $\|\phi_n\|_{\nu} \leq M_n < +\infty$  for some  $\nu > 2$  and arbitrary  $n \in \mathbb{N}$ . Assume also that  $p$  and  $q$  satisfy the condition  $\frac{\nu'}{p} + \frac{2-\nu'}{q} = 1$ . Then*

(1) *If  $\nu' \leq p \leq 2$  and  $f \in L^p(a, b)$ , then  $(\sum_{n=1}^{\infty} |c_n|^q M_n^{2-q})^{1/q} \leq \|f\|_p$ .*

(2) *If  $2 \leq p \leq \nu'$  and the sequence  $(a_n)$  satisfies the conditions*

$$\left( \sum_{n=1}^{\infty} |a_n|^q M_n^{2-q} \right)^{1/q} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} < \infty,$$

*then there exists an  $f \in L^p(a, b)$  such that  $f = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k \phi_k$  in  $L^p(a, b)$  and*

$$\|f\|_p \leq \left( \sum_{n=1}^{\infty} |a_n|^q M_n^{2-q} \right)^{1/q}.$$

At first, the inequality in part (1) may look strange, as it seems impossible to control the magnitude of  $(M_n)_{n=1}^{\infty}$ . However, when  $(a, b)$  is a bounded interval,  $\inf_{n \geq 1} M_n > 0$ . Since  $2 - q = \frac{\nu'(p-2)}{p-\nu'} \leq 0$ , the left-hand side of the inequality is of a reasonable size (one may simply assume that  $\|\phi_n\|_{\nu} = M_n$ ). The authors provide a thorough discussion on various aspects of the inequalities they proved.

#### *The Haar system revisited*

The Haar system  $(h_n)_{n=1}^{\infty}$  is one of the classical orthogonal systems defined on the interval  $[0, 1]$ . To define it, put  $h_1(t) \equiv 1$  and for  $n = 2^k + i$  ( $i = 1, 2, \dots, 2^k$ ,  $k = 0, 1, \dots$ ),

$$h_n(t) = \begin{cases} 2^{\frac{k}{2}} & \text{for } t \in \left( \frac{i-1}{2^k}, \frac{2i-1}{2^{k+1}} \right), \\ -2^{\frac{k}{2}} & \text{for } t \in \left( \frac{2i-1}{2^{k+1}}, \frac{i}{2^k} \right), \\ 0 & \text{for } t \text{ elsewhere in } [0, 1]. \end{cases}$$

Modifying a rather profound result of Paley [113], Marcinkiewicz proved

**Theorem 23** (*Marcinkiewicz 1937, [69]*) *The Haar system is an unconditional (Schauder) basis in the spaces  $L^p[0, 1]$  for  $1 < p < \infty$ .*

This means that for any  $f \in L^p[0, 1]$ , there exists a unique sequence of coefficients  $(a_n)_{n=1}^\infty$  such that  $f = \sum_{n=1}^\infty a_n h_n$  (convergence in the  $L_p$ -norm) and for any permutation  $\sigma$  of indices  $n \in \mathbb{N}$ , the series  $\sum_{n=1}^\infty a_{\sigma(n)} h_{\sigma(n)}$  is also convergent (to the same element  $f$ ).

The scale of spaces  $L^p[0, 1]$  for  $1 < p < \infty$  is complete in the sense that  $L^1[0, 1]$  does not have any unconditional basis (Pełczyński, 1961) and  $L^\infty[0, 1]$  is not separable. In [119], Schauder (who was the first to introduce a concept of basis in Banach space) proved that the Haar system is a basis in  $L^p[0, 1]$  for  $1 \leq p < \infty$ .

For other details and discussion related to this section, especially on generalization of Paley's theorem (1931), see [46], pp. 215–218.

### 3.15 A result on universal function

Marcinkiewicz generalized the following result by N. Lusin: Every measurable function is almost everywhere the derivative of a continuous function.

**Theorem 24** (Marcinkiewicz 1935, [51]) *For any nonzero null sequence  $(h_n)_{n=1}^\infty \subset \mathbb{R}$ , there exists a continuous function  $F : [0, 1] \rightarrow \mathbb{R}$  having the property:*

*For any measurable function  $\phi : [0, 1] \rightarrow \mathbb{R}$ , there is a subsequence  $(h_{n_k})_{k=1}^\infty$  of  $(h_n)_{n=1}^\infty$  such that*

$$\lim_{k \rightarrow \infty} \frac{F(x + h_{n_k}) - F(x)}{h_{n_k}} = \phi(x)$$

*almost everywhere on  $[0, 1]$ .*

The above result A. Zygmund called "minor", see [48], p. 8 of the Introduction; bearing in mind the length of Marcinkiewicz's note, it is indeed. A generalized multidimensional version was established by X.-X. Gan and K. Stromberg in [21] in 1994.

### 3.16 Miscellaneous results on Fourier series

We display here a selection of Marcinkiewicz's theorems on convergence of Fourier series.

**Theorem 25** (Marcinkiewicz 1933, [49]) *If  $(s_n(x))$  is the sequence of partial sums of the Fourier series of a function  $f \in L^2[0, 1]$ , then  $(s_{\lambda_n}(x))$  converges almost everywhere to  $f$ , given that  $\lambda_{n+1}/\lambda_n > q > 1$ .*

**Theorem 26** (Marcinkiewicz 1935, [54]) *If  $f \in L^1$  is periodic and for each point  $x$  of a set  $E$  satisfies*

$$\frac{1}{h} \int_0^h |f(x+t) - f(x)| dt = O(1/\log 1/|h|)$$

*as  $h \rightarrow 0$  for all  $x \in E$ , then the Fourier series of  $f$  converges a.e. on  $E$ .*

**Theorem 27** (Marcinkiewicz 1936, [57]) *There exists an integrable function whose Fourier series diverges almost everywhere, though its sequence of partial sums is bounded (non-uniformly) at almost all points.*

**Theorem 28** (Marcinkiewicz 1938, [82]) *Suppose that  $f \in L^1(\mathbb{T})$  and that*

$$\int_0^\pi \omega_1(f, t) \frac{dt}{t} < \infty.$$

*Then  $\lim_n s_n(x) = f(x)$ , a.e. in  $\mathbb{T}$ .*

Here, as before,  $\mathbb{T}$  is a unit circle, which we identify with the segment  $[0, 2\pi]$  and

$$\omega_1(f, x) := \frac{1}{2\pi} \int_{\mathbb{T}} |f(x+t) - f(t)| dt$$

is the  $L^1$ -modulus of continuity of the function  $f$ .

The first of the theorems presented above is at heart of a one-page note where Marcinkiewicz presented an alternative proof of an earlier theorem by A. Kolmogorov [36]. At that time, only a few analysts really expected a.e. convergence of the full sequence  $(s_n(x))$ , but in both notes it was shown that its subsequence with indices growing faster than a certain geometric sequence (lacunary sequence) is already convergent almost everywhere. We have known for more than half of the century that indeed, the whole sequence  $(s_n(x))$  converges a.e., but it was not proved until 1966, when L. Carleson resolved in the positive the so-called *Luzin conjecture* (1915) that a Fourier series of an arbitrary square-integrable function is a.e. convergent. Thus, Marcinkiewicz's result [49], which was his first publication, is now of mere historical importance.

The last theorem displays a Dini-type condition that guarantees an a.e. convergence of a sequence of partial sums  $(s_n(x))$  of a function  $f$ . In fact, this theorem easily follows from the classical Dini's test.

### 3.17 Marcinkiewicz holomorphic symbols

Y. Meyer and R. Coifman in their book [110] (section 12.4), discuss the so-called Marcinkiewicz holomorphic symbols. This concept somewhat echoes Marcinkiewicz multipliers theorem for Fourier series that we met earlier. The setup is in a complex plane. Let  $\Gamma$  be the graph of the Lipschitz function  $a : \mathbb{R} \rightarrow \mathbb{R}$ . We let  $M := \|a'\|_\infty$  and let  $S$  be a sector  $y \geq M'|x|$ , where  $M' > M$ . Let  $K(z, w)$  be a function of two complex variables  $z$  and  $w$ , which is holomorphic in the open set  $W \subset \mathbb{C}^2$  defined by  $w - z \notin S$  and which satisfies

$$|K(z, w)| \leq |z - w|^{-1} \quad \text{for } w - z \notin S.$$

The following holds true.

**Proposition 9** *With the above hypotheses, for each  $f \in L^2(\Gamma, ds)$ , the function  $F : \Omega_1 \rightarrow \mathbb{C}$  defined by  $F(z) = \int_\Gamma K(z, w) f(w) dw$  belongs to  $H^2(\Omega_1)$ . Here,  $\Omega_1$  is the open set above  $\Gamma$ .*

The authors are going to use the above proposition to deduce the existence of a commutative algebra of operators  $T : L^2(\Gamma) \rightarrow L^2(\Gamma)$  for which they have a symbolic calculus. The symbols, denoted by  $m(\xi)$ , are *holomorphic Marcinkiewicz symbols*. That is, they are defined on  $\mathbb{R} \setminus \{0\}$  and satisfy

$$\left| \left( \frac{\partial}{\partial \xi} \right)^k m(\xi) \right| \leq C \alpha^k k! |\xi|^{-k}, \quad k \in \mathbb{N},$$

where  $C \geq 0$  and  $\alpha > 1$  are constants.

### 3.18 Marcinkiewicz and the Scottish Book

Marcinkiewicz spent his 1935–36 academic year at the Jan Kazimierz University in Lwów as a recipient of the fellowship from the Fund for National Culture and became an assistant at the chair of Stefan Banach. He was visiting the Scottish Café and his name became related to four problems presented in the Scottish Book ([107], [108]). He solved problems 83 of H. Auerbach, 106 of S. Banach, and 131 of A. Zygmund from the Scottish Book. Moreover, he posed his own problem 124. Let us shed some light on problems 106, 124, and 131.

*Problem 106.* Let  $X$  be a Banach space and  $\sum_{k=1}^{\infty} x_k$  be a series in  $X$  and  $S := S(\sum_{k=1}^{\infty} x_k)$  be the set of the sums of this series, i.e., the set

$$\left\{ x \in X : \text{there exists a permutation } \sigma : \mathbb{N} \rightarrow \mathbb{N} \text{ such that } x = \sum_{k=1}^{\infty} x_{\sigma(k)} \right\}.$$

Problem 106 in The Scottish Book, posed by S. Banach in 1935, asks whether the set  $S$  of sums under all possible permutations of a series in an infinite-dimensional Banach space must be a linear manifold. If  $X = \mathbb{R}$ , then  $S$  is either empty (divergent series) or single point (absolutely convergent series) or whole  $\mathbb{R}$  (for any conditionally convergent series by the Riemann theorem). If  $X = \mathbb{C}$ , the set  $S$  is either an empty set or a single point, a straight line in  $\mathbb{C}$ , or the whole  $\mathbb{C}$ .

When  $X$  is finite-dimensional, the well-known Lévy–Steinitz theorem on rearrangements of series yields that  $S(\sum_{k=1}^{\infty} x_k)$  is a linear manifold (affine subspace) in  $X$ , that is,  $S = x_0 + M$ , where  $x_0 \in X$  and  $M$  is a linear subspace of  $X$ . P. Lévy offered the first proof in 1905, but in 1913 E. Steinitz pointed out that it was incomplete, especially when  $X$  became higher dimensional. Steinitz filled the gap of Lévy’s proof and also found an entirely different approach. Banach thus asked whether Lévy–Steinitz theorem is valid in infinite-dimensional normed spaces. He was anticipating a positive answer to his question. Next to Problem 106, Marcinkiewicz wrote down an elegant counter-example, but did not sign his solution. So, when the Scottish Book was published ([107]), the Marcinkiewicz’s authorship of that example was determined by his handwriting (a comment V. M. Kadets made in a discussion on the ResearchGate in Fall of 2019).

Here is Marcinkiewicz’s solution who constructed an example of a conditionally convergent series in infinite-dimensional Hilbert space  $L^2[0, 1]$  with a nonconvex set of sums, since series of integer-valued functions cannot converge in the strong  $L^2$  topology to  $1/2$ .

Consider the following sequence (where  $\chi_A$  is an indicator function of a set  $A$ )

$$x_{2^n+k} := \chi_{[k/2^n, (k+1)/2^n]}, \text{ where } 0 \leq n < \infty, 0 \leq k < 2^n.$$

Then  $x_1 = \chi_{[0,1]} = 1, x_2 = \chi_{[0,1/2]}, x_3 = \chi_{[1/2,1]}, x_4 = \chi_{[0,1/4]}, x_5 = \chi_{[1/4,1/2]}, x_6 = \chi_{[1/2,3/4]}, x_7 = \chi_{[3/4,1]}, x_8 = \chi_{[0,1/8]}$ , etc. Consider the series  $\sum_{n=1}^{\infty} y_n$ , where  $y_{2n-1} = x_n, y_{2n} = -x_n$  ( $n \geq 1$ ). Since  $\|x_{2^n+k}\|_2^2 = 2^{-n} \rightarrow 0$  as  $n \rightarrow \infty$ , it follows that  $\sum_{n=1}^{\infty} y_n = (x_1 - x_1) + (x_2 - x_2) + \dots = 0$ . Also, since  $x_2 + x_3 - x_1 = x_4 + x_5 - x_2 = x_6 + x_7 - x_3 = \dots = 0$  it follows that  $x_1 + (x_2 + x_3 - x_1) + (x_4 + x_5 - x_2) + \dots = 1$ . However, no rearrangement will make this series convergent to the function identically equal to  $1/2$  on  $[0, 1]$ , because each of the partial sums of the series is an integer-valued function. Thus, the set of sums  $S$  is not a convex set since  $0, 1 \in S$  but  $1/2 \notin S$ . By the same token, no constant function  $l, 0 < l < 1$  is in  $S$ .

Note that Marcinkiewicz's construction will show nonconvexity of the set of sums  $S$  also in  $L^p[0, 1]$  for  $0 < p < \infty$ , in  $C[0, 1]$  which was mentioned by Marcinkiewicz, since by the Banach theorem [3], the space  $L^2[0, 1]$  can be embedded isometrically in the Banach space  $C[0, 1]$  and in  $L^\infty[0, 1]$  (since it has  $C[0, 1]$  as a subspace). Examples of series in  $L^p[0, 1]$  with nonconvex set of sums  $S$  were given independently in 1971 by Nikishin for  $p = \infty$  and, in 1980, by Kornilov for  $1 \leq p \leq 2$ , see [39].

Kadets in [31], using the Marcinkiewicz–Kornilov example together with the Dvoretzky theorem, has shown that in any infinite-dimensional Banach space there exists a series with a nonconvex set of sums  $S$ . Nowadays, even more is known: for a given finite subset  $A$  of an infinite-dimensional Banach space  $X$  there is a series in  $X$  whose sum range equals  $A$ , proved by J. O. Wojtaszczyk [131] (so far, no simpler proof has been offered).

The detailed comments and discussion on the history of Lévy–Steinitz theorem and Problem 106 can be found in [46], pp. 172–175, in [107] (commentary by R. D. Mauldin and W. A. Beyer) and in [108] (commentary by J. Diestel).

*Problem 124.* The problem asks, what can one say about the uniqueness for the integral equation

$$\int_0^1 y(t)f(x-t)dt = 0, \quad 0 \leq x \leq 1? \quad (2)$$

‘I know that if the sequence of integrals  $f_k(x) = \int_0^x f_{k-1}(t)dt, f_0 = f, k = 1, 2, 3, \dots$ , is complete in  $L^2$  then the only solution of Eq. (2) is  $y \equiv 0$ . This is the case also if  $f$  is of bounded variation and  $f(0) \neq 0$ . Finally, if Eq. (2) possesses even one nonzero solution,  $y$ , then every (iterated) integral of  $y$  also satisfies this equation.

I conjecture that if  $f(0) \neq 0$  and  $f$  is continuous, then Eq. (2) has only the solution  $y \equiv 0$ .

There was no commentary in the first edition of the Scottish Book [107], while in the second edition [108] Z. Buczolich and M. Laczkovich indicated that the solution of the above problem follows from the (now) famous Titchmarsh theorem, published in 1926: If  $\Phi$  and  $\Psi$  are integrable functions such that

$$\int_0^x \Phi(x)\Psi(x-t)dt = 0$$

almost everywhere in the interval  $0 < x < K$ , then  $\Phi(t) = 0$  almost everywhere in  $(0, \lambda)$  and  $\Psi(t) = 0$  almost everywhere in  $(0, \mu)$ , where  $\lambda + \mu > K$ .

*Problem 131.* (posed by Zygmund): Given is a function  $f(x)$ , continuous (for simplicity), and such that

$$\overline{\lim}_{h \rightarrow 0} \left| \int_h^1 \frac{f(x+t) - f(x)}{t} dt \right| < \infty, \text{ for } x \in E, |E| > 0.$$

Is it true that the integral

$$\int_0^1 \frac{f(x+t) - f(x)}{t} dt$$

may not exist almost everywhere in  $E$ ? Similarly, for other Dini integrals?

It was remarked in [108] that this question was raised by A. Zygmund in a lecture given in Lwów in early thirties. The solution in the positive was given by J. Marcinkiewicz in his paper [78] and published in *Bull. Math. Seminar at the University of Wilno*, 1938. There were only several volumes this journal published. The journal ceased to exist at the outbreak of the WWII. The copies of this bulletin are practically inaccessible, but the paper was republished in [48].

## 4 The epilogue

If Józef Marcinkiewicz could be described as a mathematician by only one expression, it would be: *The Master of Averaging*. At USB, he was too young faculty member to have already his own promoted doctoral students or, maybe, he did not show yet a need to guide some young souls towards a doctoral degree. Nevertheless, in Poland, there are at least two living mathematicians, Professors Zbigniew Ciesielski and Stanisław Kwapień, who launched their research carriers based on the themes very close to Marcinkiewicz's interest. They certainly are the post mortem scientific descendants of Marcinkiewicz. Both started in probability theory, however their interests naturally evolved and they later contributed to new contemporary disciplines. One should not forget the schools they created in Gdańsk and in Warsaw. In addition, there are active groups in Poznań, Wrocław, and Łódź working in probability, function theory and functional analysis.

Since 1957, the Toruń Branch of the Polish Mathematical Society organizes the so-called Józef Marcinkiewicz Competition for the best college student paper. Awards (1<sup>st</sup>, 2<sup>nd</sup> and the 3<sup>rd</sup> place) are granted in several categories. Many contemporary Polish mathematicians became laureates of this award.

Scientific life of Józef Marcinkiewicz was extremely intense. Bearing in mind such a short period of his research activity, roughly less than 6 years, and the gravity of his intellectual production, it is clear that Marcinkiewicz was rushing as if his end was imminent. Some try to explain it by eruption of his talent. One thing is for sure: life of a genius is short. Marcinkiewicz was no exception.

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