Exponential Bases for Partitions of Intervals.

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Outline

Orthogonal Bases of Exponentials

Definition

Given a countable set $\Lambda \subseteq \mathbb{R}^d$, define $\mathcal{E}(\Lambda)$ to be the exponential system

$$\mathcal{E}(\Lambda) = \{e_\lambda(t) \colon \lambda \in \Lambda\} = \{e^{2\pi i \langle \lambda, t \rangle} \colon \lambda \in \Lambda\}.$$

Theorem

$\mathcal{E}(\mathbb{Z}^d)$ is an orthonormal basis for $L^2[0,1]^d$. $f \in L^2[0,1]^d$ can be written

$$f(t) = \sum_{n \in \mathbb{Z}^d} \langle f, e_n \rangle \, e_n(t).$$

Orthogonal Bases of Exponentials

We will be working in $d = 1$ with the following easy variant.

Theorem

Given $\alpha \in \mathbb{R}$ and $a > 0$. Then with $\Lambda = \frac{\mathbb{Z}+\alpha}{a}$, $\mathcal{E}(\Lambda)$ is an orthogonal basis for $L^2(I)$ where I is any interval with $|I| = a$.

What is most important here is the *length* of the interval, and less so the interval itself.

Existence of Orthogonal Bases of Exponentials

- Fundamental Question: Given a domain $\Omega \subseteq \mathbb{R}^d$, does there exist a countable set Λ such that $\mathcal{E}(\Lambda)$ is an orthogonal basis for $L^2(\Omega)$?
- Fuglede (1974) conjectured the following: *A domain $\Omega \subseteq \mathbb{R}^d$ admits an orthogonal basis of the form $\mathcal{E}(\Lambda)$ if and only if Ω tiles $\mathbb{R}^d$ by Λ, that is,*
  - $(\Omega+\lambda)\cap(\Omega+\lambda')=\emptyset$, *a.e.* if λ, λ' are distinct elements of Λ,
  - $\mathbb{R}^d = \bigcup_{\lambda\in\Lambda}(\Omega+\lambda)$.

Existence of Orthogonal Bases of Exponentials

- Fuglede proved that the conjecture held for Ω a fundamental domain for a lattice Λ.
- The conjecture is false for $d \geq 5$ (Tao, 2004), for $d = 4$ (Matolcsi, 2005), and for $d = 3$ (Matolcsi, Kouluntzakis, Balint, Mora, 2005). However, the conjecture remains unsolved in full generality for $d = 1, 2$.
- If Ω is a convex body in $\mathbb{R}^d$, then the conjecture holds in all dimensions. (Lev, Matolcsi, 2019).

Riesz Bases

By passing from an orthogonal to a non-orthogonal basis, we open up new possibilities.

Definition

A Riesz basis of a Hilbert space $\mathcal{H}$ is the image of an orthonormal basis under a bounded, invertible operator on $\mathcal{H}$.

Theorem

Given $\Omega \subseteq \mathbb{R}^d$, $\mathcal{E}(\Lambda)$ is a Riesz basis of $L^2(\Omega)$ if and only if

(1) $\overline{\operatorname{span}}\, \mathcal{E}(\Lambda) = L^2(\Omega)$ *and*

(2) *for some* $0 < A, B < \infty$ *and every* $\{c_\lambda\} \in \ell^2(\Lambda)$,

$$A \sum_{\lambda} |c_\lambda|^2 \leq \int_\Omega \left| \sum_{\lambda \in \Lambda} c_\lambda e^{2\pi i \langle \lambda, x \rangle} \right|^2 dx \leq B \sum_{\lambda} |c_\lambda|^2$$

Riesz Bases

Theorem

For a Riesz basis $\mathcal{E}(\Lambda)$ *of* $L^2(\Omega)$ *exists* $\{g_\lambda\}_{\lambda\in\Lambda}$ *so that for all* $f \in L^2(\Omega)$ *we have*

$$f(x) \overset{L^2}{=} \sum_{\lambda\in\Lambda} \langle f, g_\lambda\rangle\, e^{2\pi i\lambda x}$$

Theorem (Kadec 1/4-theorem)

For $\varphi : \frac{\mathbb{Z}+\alpha}{a} \to \mathbb{R}$, $\mathcal{E}(Range(\varphi))$ *is a Riesz basis for* $L^2(I)$ *for any interval* I *with* $|I| = a$ *if*

$$\sup_{k\in\mathbb{Z}} \left|\varphi\left(\frac{k+\alpha}{a}\right) - \frac{k+\alpha}{a}\right| < \frac{1}{4a}.$$

Fundamental Questions on Riesz Bases

- Given a domain $\Omega \subseteq \mathbb{R}^d$, does there exist a countable set Λ such that $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2(\Omega)$?
- There is no Ω for which such a Riesz basis is known not to exist.
- In relatively few cases is it known how to construct such a basis.

Kozma and Nitzan (2015)

Theorem

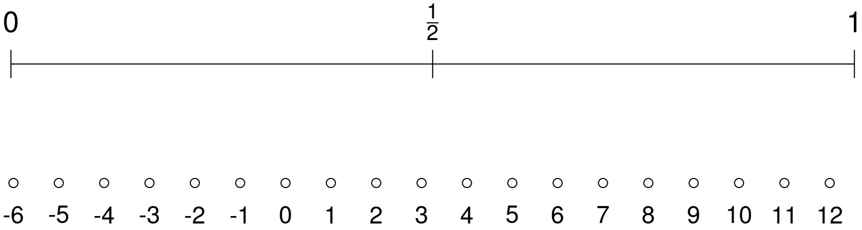
Let $\{I_1, I_2, \ldots, I_n\}$ be a collection of disjoint subintervals of $[0,1]$. Then there exists $\Lambda \subseteq \mathbb{Z}$ such that $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2(I_1 \cup I_2 \cup \cdots \cup I_n)$.

In the paper the authors recount an imaginary conversation with a graduate student who asks: *Why not just find sets Λ_k such that $\mathcal{E}(\Lambda_k)$ is a Riesz basis for $L^2(I_k)$, and let*

$$\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_n?$$

Are unions of Riesz bases Riesz bases of unions?

Sometimes they are.



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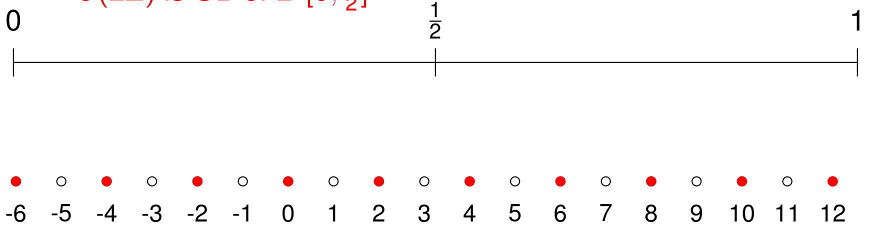
$\mathcal{E}(2\mathbb{Z})$ is OB of $L^2[0, \frac{1}{2}]$



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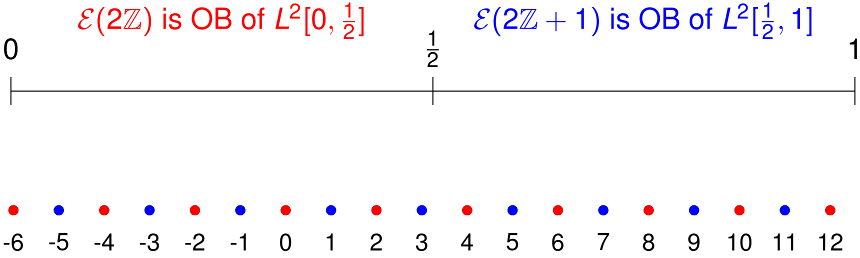
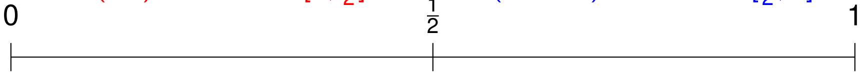
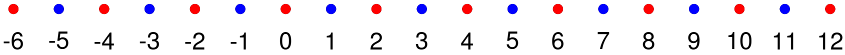


Sometimes they are.

$\mathcal{E}(2\mathbb{Z})$ is OB of $L^2[0, \frac{1}{2}]$ $\mathcal{E}(2\mathbb{Z}+1)$ is OB of $L^2[\frac{1}{2}, 1]$



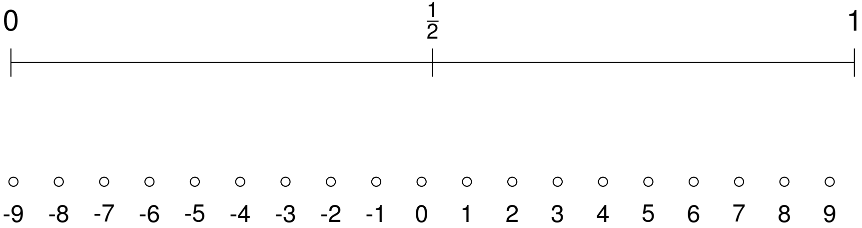
$$\mathcal{E}(\mathbb{Z}) = \mathcal{E}(2\mathbb{Z}+1) \cup \mathcal{E}(2\mathbb{Z}) \text{ is ONB of } L^2[0, 1]$$



Sometimes they're not.

$$\Lambda_1 = \{0\} \cup \{2n - \tfrac{1}{4}\}_{n>0} \cup \{2n + \tfrac{1}{4}\}_{n<0}$$

$$\Lambda_2 = \{2n + 1 - \tfrac{1}{4}\}_{n>0} \cup \{2n - 1 + \tfrac{1}{4}\}_{n<0}$$



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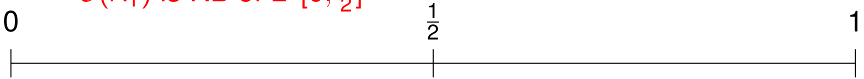
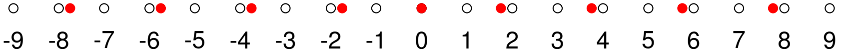
$\mathcal{E}(\Lambda_1)$ is RB of $L^2[0, \frac{1}{2}]$





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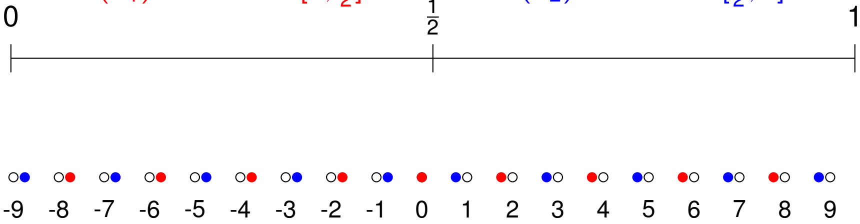

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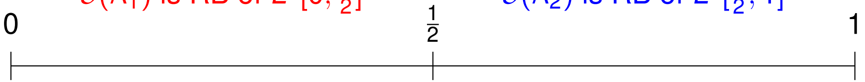
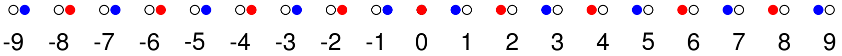


$\mathcal{E}(\Lambda_1 \cup \Lambda_2)$ is not RB of $L^2[0, 1]$



Basis extraction (S. Avdonin)

Theorem (Basis extraction)

Suppose that for $\Lambda \subseteq \mathbb{R}$, $\mathcal{E}(\Lambda)$ *is a Riesz basis for* $L^2[0,1]$. *Then for every* $0 < \alpha < 1$ *there exists* $\Lambda' \subseteq \Lambda$ *such that* $\mathcal{E}(\Lambda')$ *is a Riesz basis for* $L^2[0,\alpha]$.

Question: Is it necessarily true that $\mathcal{E}(\Lambda \setminus \Lambda')$ is a Riesz basis for $L^2[\alpha, 1]$?

Basis complementation (S. Avdonin)

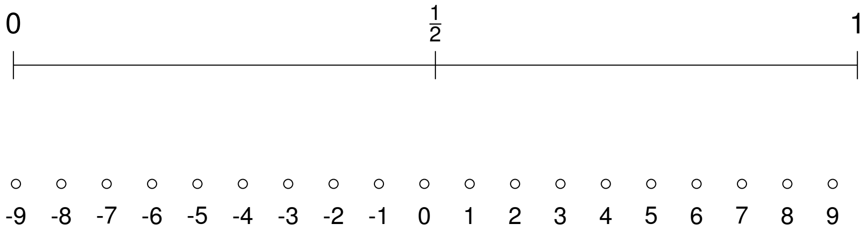
Theorem (Basis complementation)

Let $0 < \alpha < 1$ and suppose that for $\Lambda \subseteq \mathbb{R}$, $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2[0, \alpha]$. Then there exists $\Lambda' \supseteq \Lambda$ such that $\mathcal{E}(\Lambda')$ is a Riesz basis for $L^2[0, 1]$.

Question: Is it necessarily true that $\mathcal{E}(\Lambda' \setminus \Lambda)$ is a Riesz basis for $L^2[\alpha, 1]$?

$$\Lambda = \{2n\}_{n\leq 0} \cup \{2n - 1 + \tfrac{1}{8}\}_{n>0}$$

Λ is a perturbation of $\{2n - \frac{7}{16}\}_{n\in\mathbb{Z}}$



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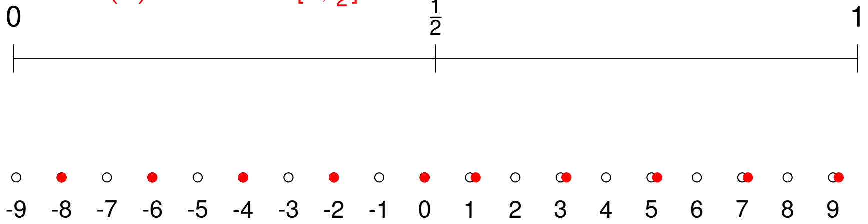

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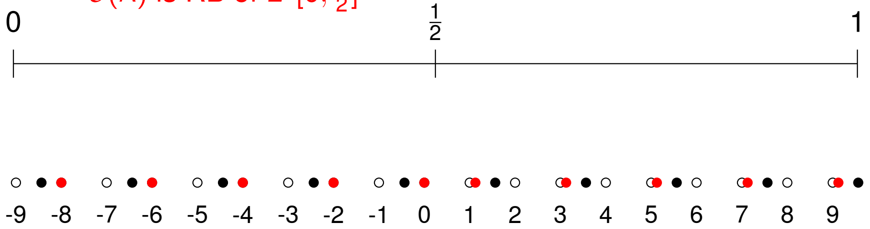

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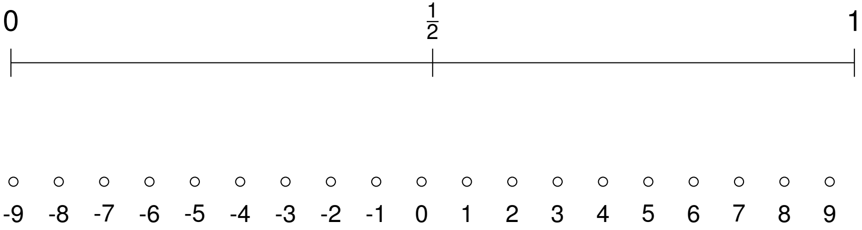

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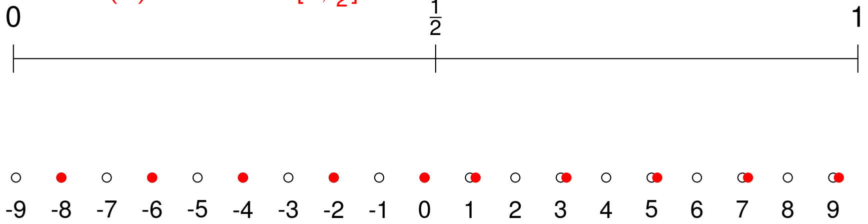

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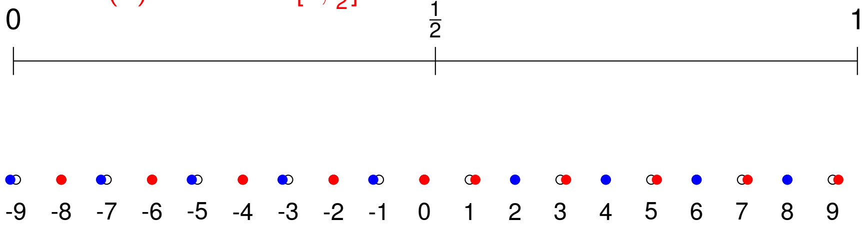

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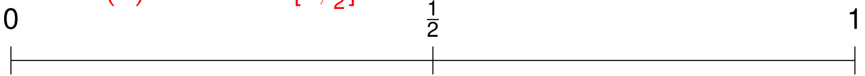
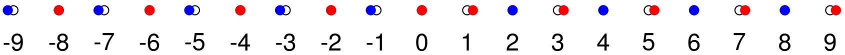


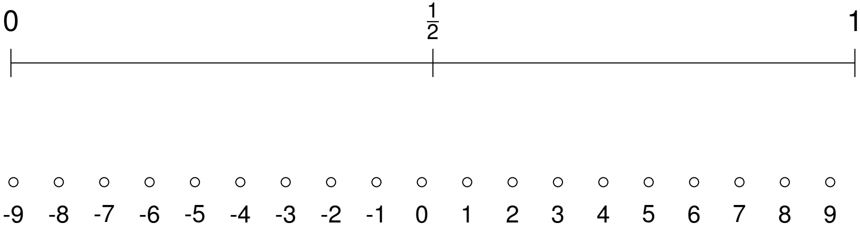
$\mathcal{E}(\Lambda^\circ)$ is RB of $L^2[0, 1]$



Answer: No. (Dae Gwan Lee)

$$\Lambda^\circ \setminus \Lambda = \{2n\}_{n>0} \cup \{2n+1-\tfrac{1}{8}\}_{n<0}$$

$(\Lambda^\circ \setminus \Lambda) \cup \{0\}$ is a perturbation of $\{2n + \frac{7}{16}\}_{n\in\mathbb{Z}}$



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$\mathcal{E}(\Lambda)$ is RB of $L^2[0, \frac{1}{2}]$

0 $\quad \frac{1}{2} \quad$ 1

$\mathcal{E}(\Lambda^\circ)$ is RB of $L^2[0, 1]$

-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9

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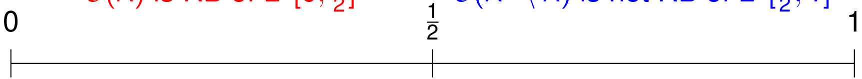
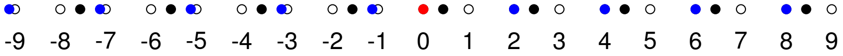
$\mathcal{E}(\Lambda)$ is RB of $L^2[0,\frac{1}{2}]$

$\mathcal{E}(\Lambda^\circ \setminus \Lambda)$ is not RB of $L^2[\frac{1}{2},1]$



$\mathcal{E}(\Lambda^\circ)$ is RB of $L^2[0,1]$



The following result of Lyubarski and Seip (2001) shows that extraction is always possible.

Theorem

Let $\mathcal{E}(\Lambda)$ be a Riesz basis of exponentials for $L^2[0,1]$. For each $0 < a < 1$, there is a splitting

$$\Lambda = \Lambda' \cup \Lambda'', \ \Lambda' \cap \Lambda'' = \emptyset$$

such that $\mathcal{E}(\Lambda')$ and $\mathcal{E}(\Lambda'')$ are Riesz bases for $L^2[0,a]$ and $L^2[a,1]$ respectively.

If $\mathcal{E}(\Lambda)$ is an *orthogonal* basis, then extraction and complementation always go together.

Theorem (Meyer, Matei (2009), Bownik, Casazza, Marcus, Speegle (2016))

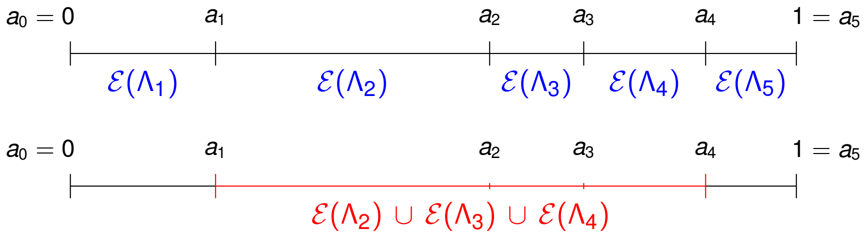
Let $S \subseteq [0,1]$ and suppose that for some $\Lambda \subseteq \mathbb{Z}$, $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2(S)$. Then $\mathcal{E}(\mathbb{Z} \setminus \Lambda)$ is a Riesz basis for $L^2([0,1] \setminus S)$.

Interestingly, Lee's example seems to show that the assumption of orthogonality cannot be weakened.

Main Results

Theorem (Pfander, Revay, DW 2018)

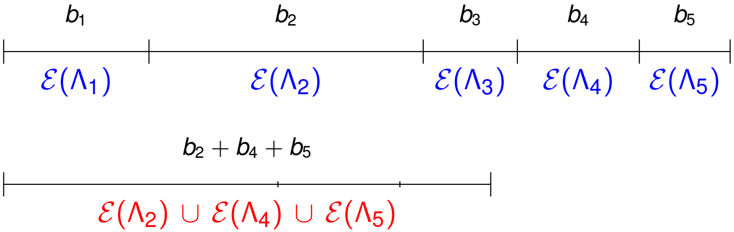
Given a partition $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ *of* $[0,1]$, *there exists a partition of* $\mathbb{Z}$ *into* $\Lambda_1, \ldots, \Lambda_n$ *such that for each* k, $\mathcal{E}(\Lambda_k)$ *is a Riesz basis of* $L^2[a_{k-1}, a_k]$. *In addition* $\bigcup_{r=k}^{\ell} \mathcal{E}(\Lambda_r)$ *is a Riesz basis of* $L^2[a_{k-1}, a_\ell]$.



Main Results

Theorem (Pfander, Revay, DW 2018)

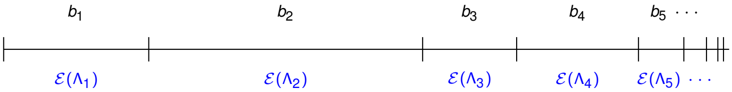
Let $b_1, \ldots, b_n > 0$ with $\sum_{j=1}^{n} b_j = 1$. Then there exist pairwise disjoint sets $\Lambda_1, \ldots, \Lambda_n \subseteq \mathbb{Z}$ such that $\bigcup_{j=1}^{n} \Lambda_j = \mathbb{Z}$ and for any $J \subseteq \{1, \ldots, n\}$, $\bigcup_{j \in J} \mathcal{E}(\Lambda_j)$ is a Riesz basis for any interval of length $\sum_{j \in J} b_j$.



Main Results

Theorem (Pfander, Revay, DW 2018)

Let $b_1, b_2, \ldots > 0$ with $\sum_{j=1}^{\infty} b_j = 1$ and $K \in \mathbb{N}$. Then there exist pairwise disjoint sets $\Lambda_1, \Lambda_2, \ldots \subseteq \mathbb{Z}$ such that for any $J \subseteq \mathbb{N}$ with $|J| \leq K$ or $|\mathbb{N} \setminus J| \leq K$, $\bigcup_{j \in J} \mathcal{E}(\Lambda_j)$ is a Riesz basis for any interval of length $\sum_{j \in J} b_j$.



Avdonin "Average 1/4 Theorem"

Theorem (Avdonin 1974)

For $\varphi : \frac{\mathbb{Z}+\alpha}{a} \to \mathbb{R}$ *injective with separated range,* $\mathcal{E}\left(\mathrm{Range}\left(\varphi\right)\right)$ *is a Riesz basis for* $L^2[0,a]$ *if there exists* $R > 0$ *such that*

$$\sup_{m\in\mathbb{Z}} \left| \frac{1}{R} \sum_{\frac{k+\alpha}{a} \in [mR,(m+1)R)} \varphi\left(\tfrac{k+\alpha}{a}\right) - \frac{k+\alpha}{a} \right| < \frac{1}{4a}.$$

- Says essentially that if a separated set Λ is "on average" close to a set whose exponentials form a Riesz basis for $L^2(I)$ (I an interval), then $\mathcal{E}(\Lambda)$ is also a Riesz basis for $L^2(I)$.
- The above is not the most general statement of the theorem, but is more than good enough for our purposes.

Avdonin maps

Definition (Avdonin map)

Let ϵ, $a > 0$ and $\alpha \in \mathbb{R}$. An injective map $\varphi \colon \dfrac{\mathbb{Z}+\alpha}{a} \to \mathbb{R}$ with separated range is an ϵ-*Avdonin map* for $\frac{\mathbb{Z}+\alpha}{a}$ if for all $R > 0$ sufficiently large,

$$\sup_{m \in \mathbb{Z}} \left| \frac{1}{R} \sum_{\frac{k+\alpha}{a} \in [mR,(m+1)R)} \varphi\left(\frac{k+\alpha}{a}\right) - \left(\frac{k+\alpha}{a}\right) \right| < \epsilon. \qquad (1)$$

Theorem

If φ is an ϵ-Avdonin map for $\frac{\mathbb{Z}+\alpha}{a}$ with $\epsilon \leq 1/4$, then $\mathcal{E}\left(Range(\varphi)\right)$ is a Riesz basis of exponentials for $L^2(I)$ for any interval I with $|I| = a$.

One interval (Weyl-Khinchin Theorem)

Our first goal is to prove the following theorem.

Theorem (Avdonin 1991, Seip 1995)

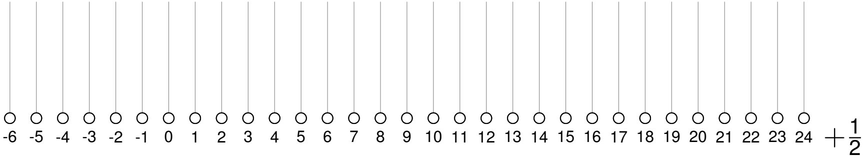
Given $0 < a \leq 1$*, there exists* $\Lambda \subseteq \mathbb{Z}$ *such that* $\mathcal{E}(\Lambda)$ *is a Riesz basis for* $L^2[0, a]$*.*

- a irrational is the interesting case.
- We know that if $\Gamma = \dfrac{\mathbb{Z} + \frac{1}{2}}{a}$, then $\mathcal{E}(\Gamma)$ is a Riesz basis for $L^2[0, a]$.
- Round each element of Γ to the nearest element of $\mathbb{Z} + \frac{1}{2}$. For any $x \in \mathbb{R}$, this is just $\lfloor x \rfloor + \frac{1}{2}$.

Look at $a = \sqrt{2}/2$.

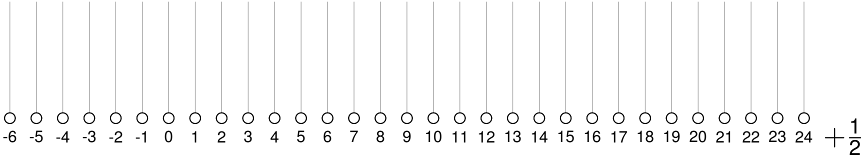
$a = \frac{1}{\sqrt{2}}$

$$\frac{\mathbb{Z}+\frac{1}{2}}{a} = \sqrt{2}\mathbb{Z} + \frac{1}{\sqrt{2}}$$



Look at $a = \sqrt{2}/2$.

$$\frac{\mathbb{Z}+\frac{1}{2}}{a} = \sqrt{2}\mathbb{Z} + \frac{1}{\sqrt{2}}$$



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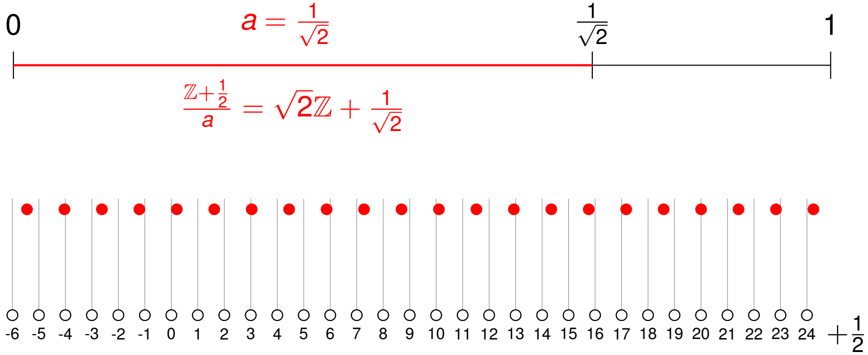

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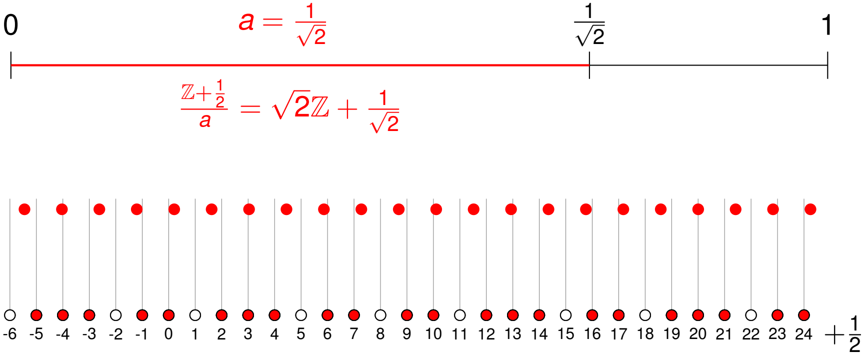


What is the “average perturbation”?

Theorem (Weyl Equidistribution Theorem)

Given α irrational,

$$\lim_{R\to\infty} \frac{1}{R} \sum_{k=1}^{R} k\,\alpha \mod 1 \quad = \frac{1}{2}$$

This is not quite what we want.

Theorem (Weyl-Khinchin)

Let a irrational and $\epsilon > 0$. Then for all R sufficiently large,

$$\sup_{m\in\mathbb{Z}} \left| \frac{1}{R} \sum_{k=mR}^{(m+1)R-1} \frac{k+\frac{1}{2}}{a} \mod 1 \; - \frac{1}{2} \right| < \epsilon.$$

Note that

$$\left| \frac{1}{R} \sum_{k=mR}^{(m+1)R-1} \frac{k+\frac{1}{2}}{a} \mod 1 \; - \frac{1}{2} \right|$$

$$= \left| \frac{1}{R} \sum_{k=mR}^{(m+1)R-1} \frac{k+\frac{1}{2}}{a} - \left(\left\lfloor \frac{k+\frac{1}{2}}{a} \right\rfloor_{\mathbb{Z}} + \frac{1}{2} \right) \right| < \epsilon.$$

Consequently,

$$\varphi\colon \frac{\mathbb{Z}+\frac{1}{2}}{a} \to \mathbb{R}$$

given by

$$\varphi\Big(\frac{k+\frac{1}{2}}{a}\Big) = \Big\lfloor \frac{k+\frac{1}{2}}{a} \Big\rfloor_{\mathbb{Z}} + \frac{1}{2}$$

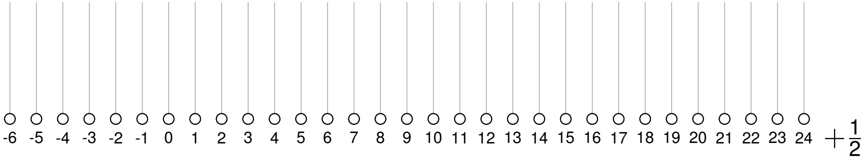
is an ϵ-Avdonin map, *for every* $\epsilon > 0$.

Taking $\epsilon \le \frac{1}{4a}$ gives the result.

Two Intervals

In our previous example, look what happens when we also consider the interval $[a, 1]$.

$$\frac{\mathbb{Z}+\frac{1}{2}}{a} = \sqrt{2}\mathbb{Z} + \frac{1}{\sqrt{2}}$$



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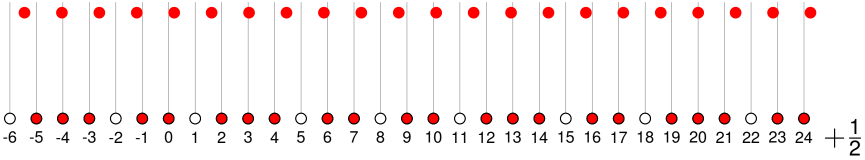

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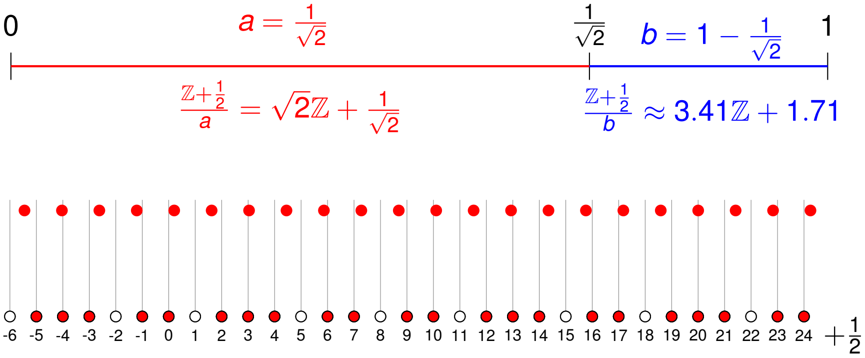

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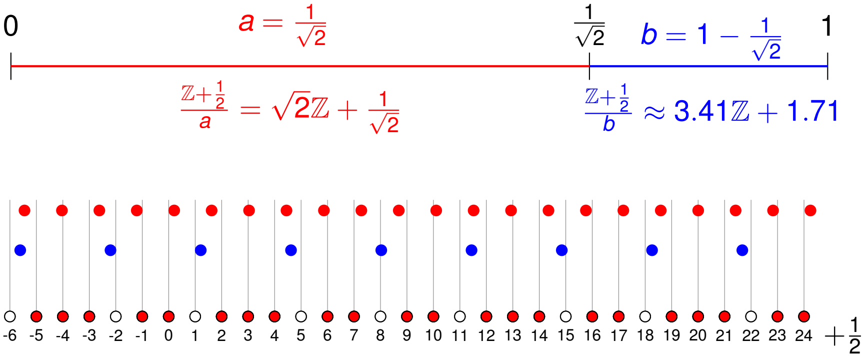

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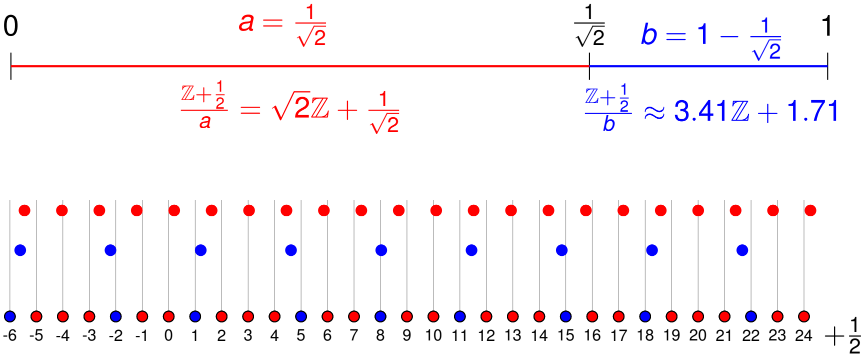


Beatty sequences

Theorem

For a, b irrational with $a + b = 1$, the sets $\mathcal{A} = \left\{ \left\lfloor \frac{k}{a} \right\rfloor \right\}_{k \in \mathbb{N}}$ and $\mathcal{B} = \left\{ \left\lfloor \frac{k}{b} \right\rfloor \right\}_{k \in \mathbb{N}}$ partition $\mathbb{N}$.

Proof.

Clearly $\mathcal{A}$ and $\mathcal{B}$ can never coincide.
Given $N \in \mathbb{N}$, $|\mathcal{A} \cap [0, N)| = \lfloor aN \rfloor$ and $|\mathcal{B} \cap [0, N)| = \lfloor bN \rfloor$

$$aN - 1 < \lfloor aN \rfloor < aN, \quad \text{and} \quad bN - 1 < \lfloor bN \rfloor < bN$$

and summing

$$N - 2 = aN - 1 + bN - 1 < \lfloor aN \rfloor + \lfloor bN \rfloor < aN + bN = N.$$

Hence $|(\mathcal{A} \cup \mathcal{B}) \cap [0, N)| = N - 1$. □

- Beatty sequences arose as a solution to a problem posed in the Mathematical Monthly in 1926, but were known and mentioned in the nineteenth century by Lord Rayleigh in relation to the study of sound waves.
- In 1969, Fraenkel considered sequences of the form $\{\lfloor n\alpha+\gamma\rfloor : n\in\mathbb{Z}\}$ which he referred to as *inhomogeneous Beatty sequences*. It is his results that we need here.

Theorem (Beatty-Fraenkel)

Let a, b irrational with $a+b=1$. Then the sets $\left\{\left\lfloor \frac{k+\frac{1}{2}}{a}\right\rfloor\right\}_{k\in\mathbb{Z}}$ and $\left\{\left\lfloor \frac{\ell+\frac{1}{2}}{b}\right\rfloor\right\}_{\ell\in\mathbb{Z}}$ partition $\mathbb{Z}$.

Combining Beatty-Fraenkel and Weyl-Khinchin:

Theorem (Pfander, Revay, DW, 2018)

Given a, $b > 0$, there exist injective maps

$$\varphi\colon \frac{\mathbb{Z}+\frac{1}{2}}{a} \longrightarrow \frac{\mathbb{Z}+\frac{1}{2}}{a+b}, \qquad \psi\colon \frac{\mathbb{Z}+\frac{1}{2}}{b} \longrightarrow \frac{\mathbb{Z}+\frac{1}{2}}{a+b}$$

such that

(1) *Range(φ) and Range(ψ) partition $\frac{\mathbb{Z}+\frac{1}{2}}{a+b}$,*

(2) *for every $\epsilon > 0$, φ and ψ are ϵ-Avdonin maps for $\frac{\mathbb{Z}+\frac{1}{2}}{a}$ and $\frac{\mathbb{Z}+\frac{1}{2}}{b}$ (resp.).*

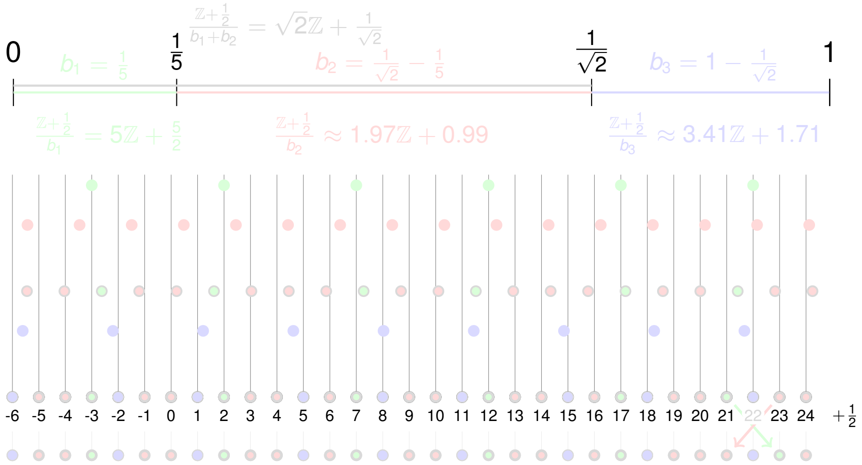
Proof

- By Beatty-Fraenkel, (1) is satisfied with

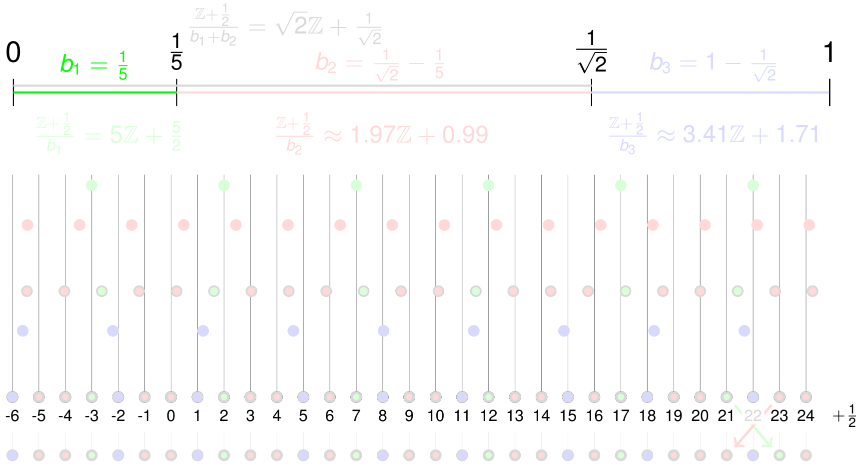
$$\varphi\left(\frac{k+\frac{1}{2}}{a}\right) = \left\lfloor \frac{k+\frac{1}{2}}{a} \right\rfloor + \frac{1}{2}, \quad \psi\left(\frac{k+\frac{1}{2}}{b}\right) = \left\lfloor \frac{k+\frac{1}{2}}{b} \right\rfloor + \frac{1}{2}$$

- Since both $\mathrm{Range}\,(\varphi)$ and $\mathrm{Range}\,(\psi)$ come from rounding, Weyl-Khinchin implies that the average perturbation from the lattices $\frac{\mathbb{Z}+\frac{1}{2}}{a}$ and $\frac{\mathbb{Z}+\frac{1}{2}}{b}$ can be made as small as desired. This is (2).
- Taking $a + b = 1$ gives the partition result for two intervals.

Three intervals



Three intervals



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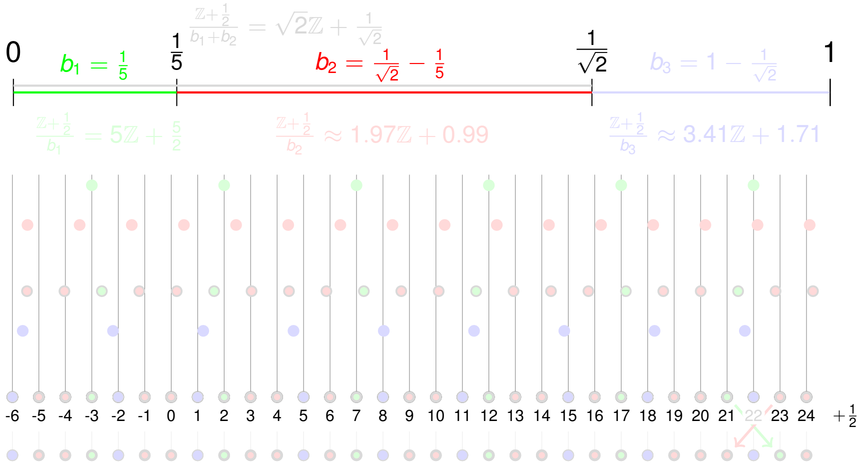

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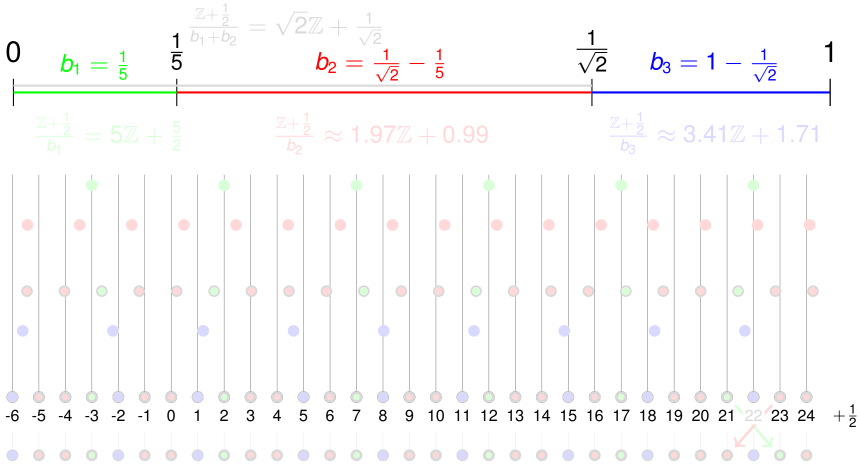

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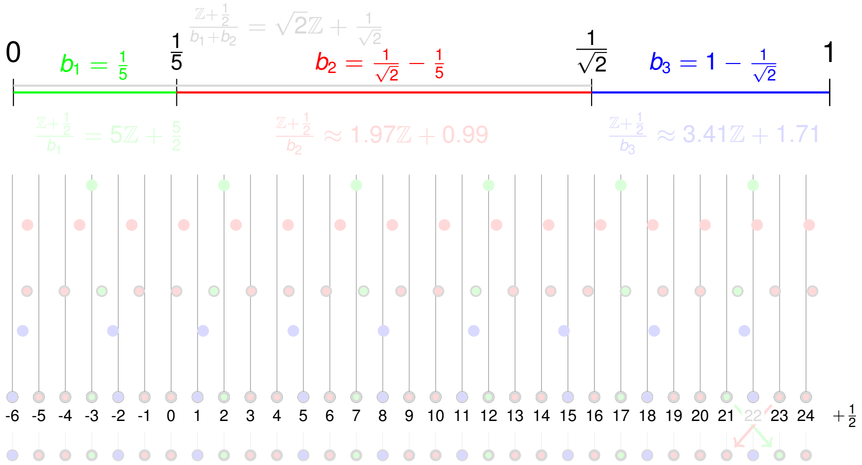

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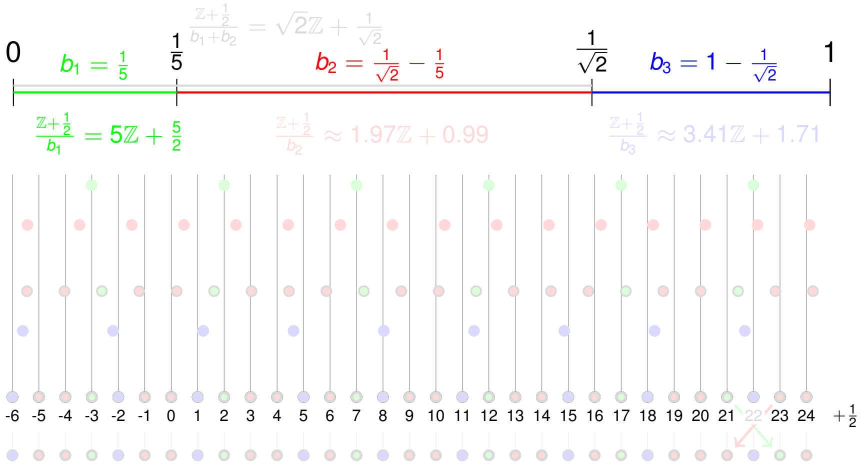

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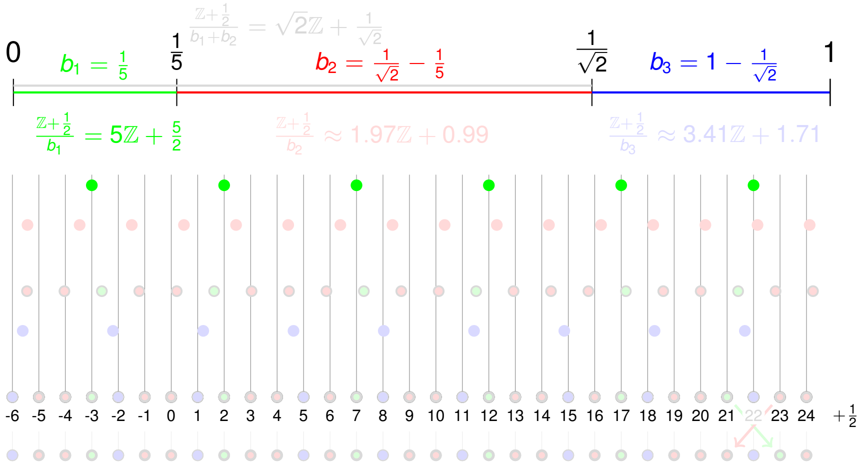

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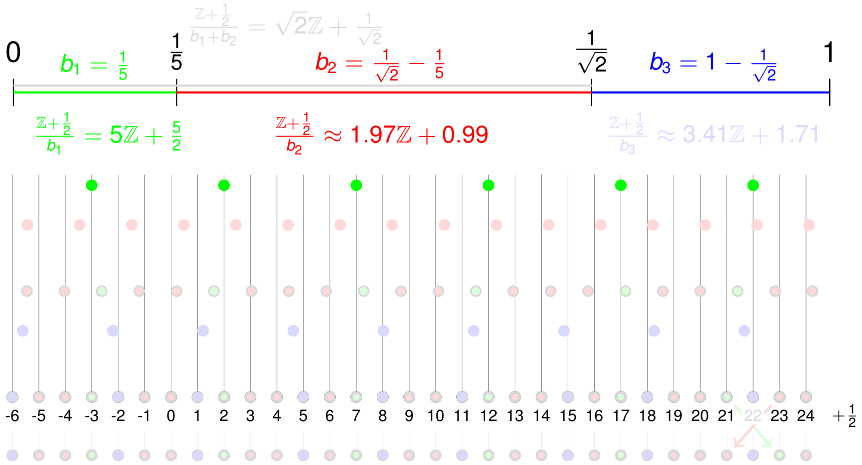

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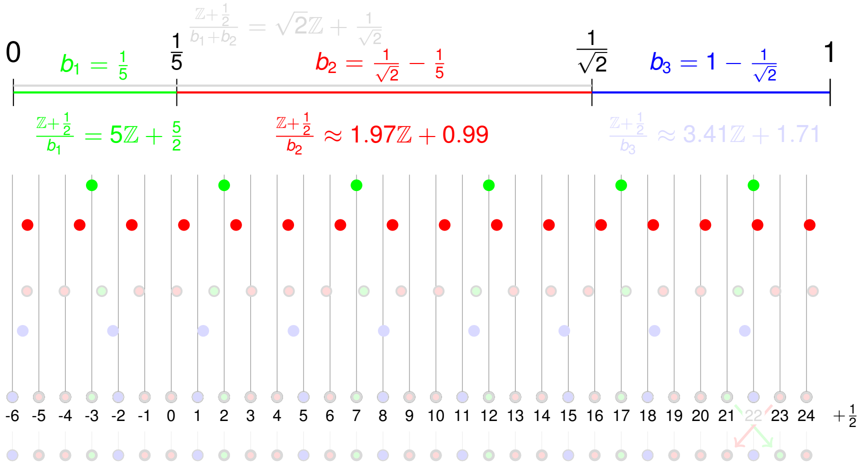

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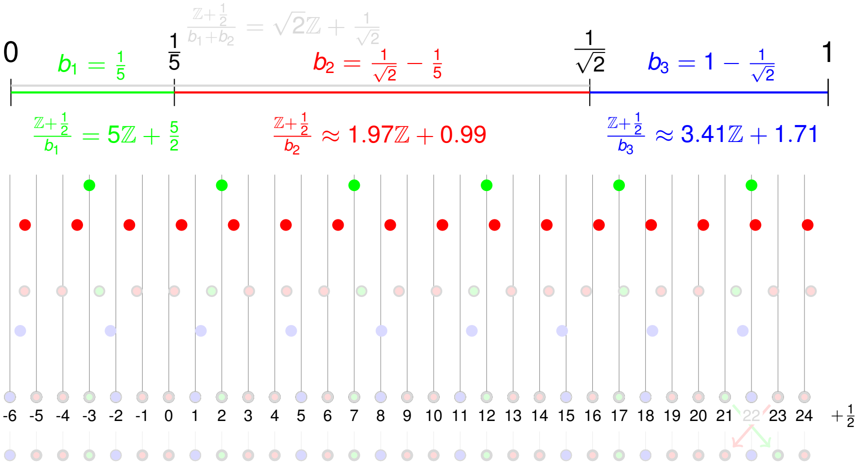

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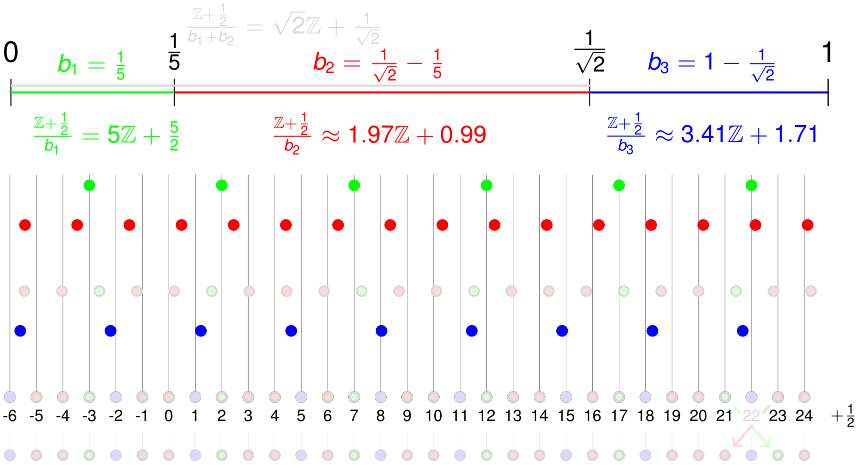

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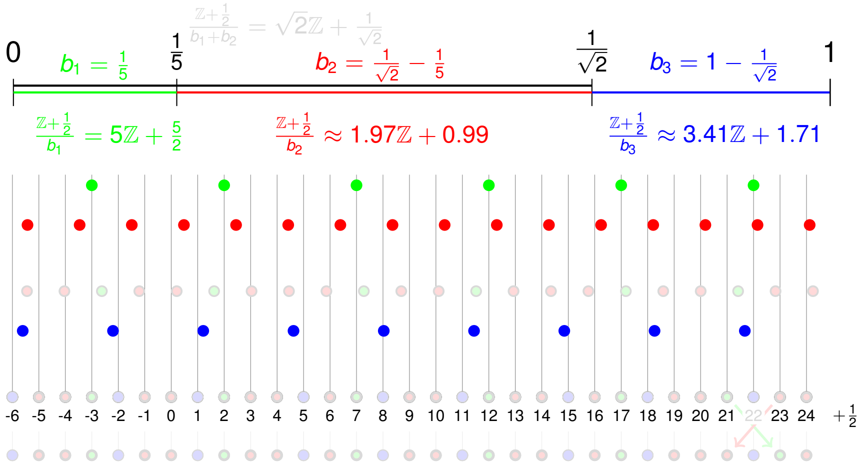

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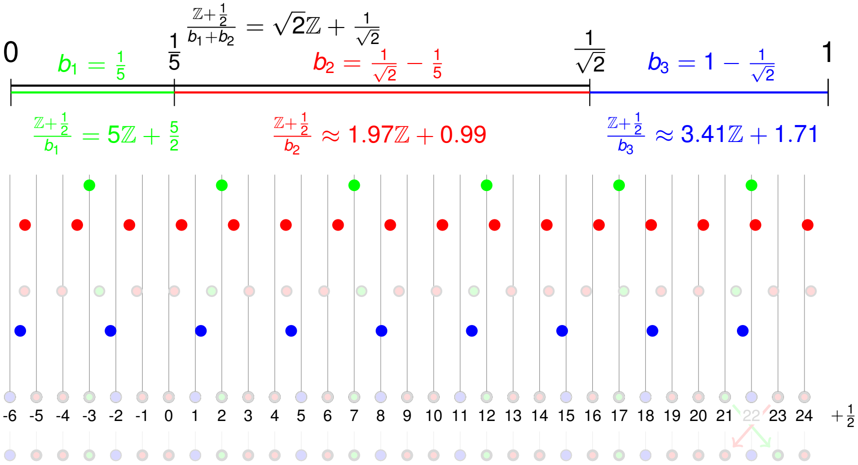

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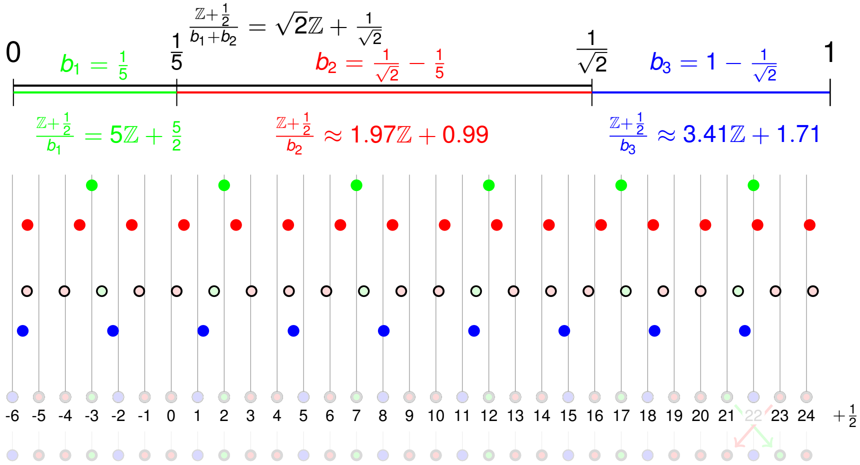

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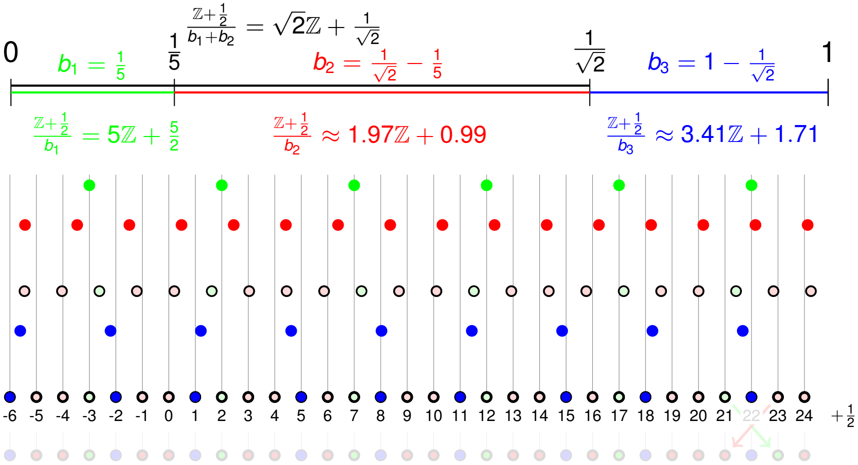

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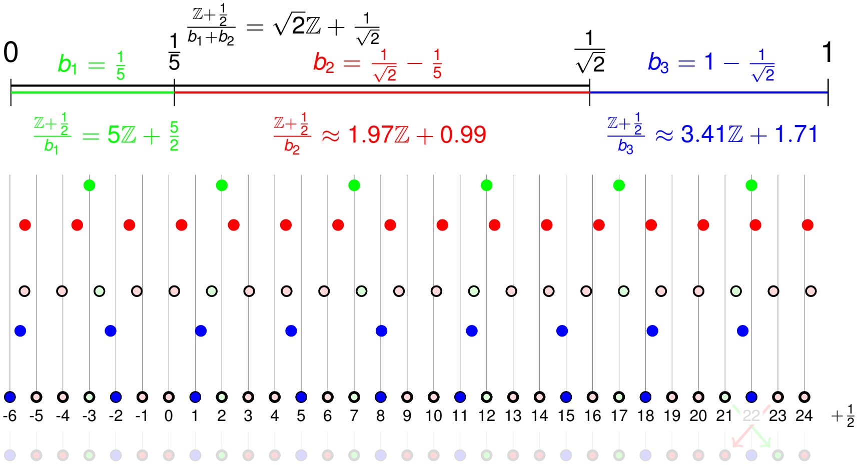

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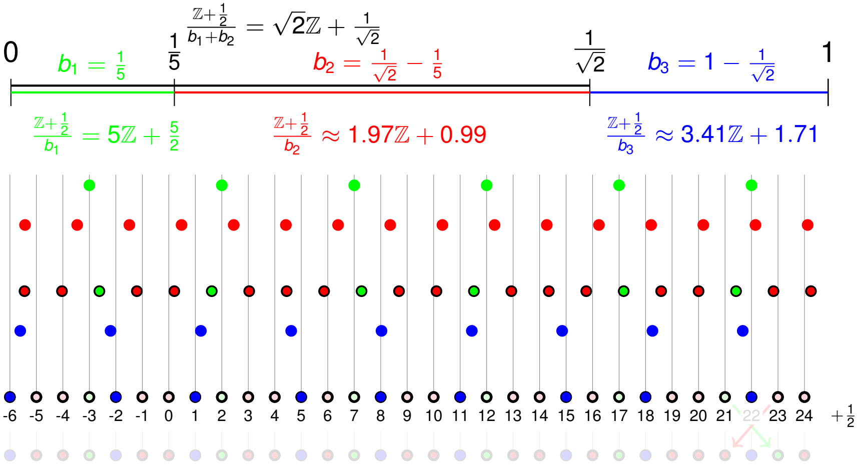

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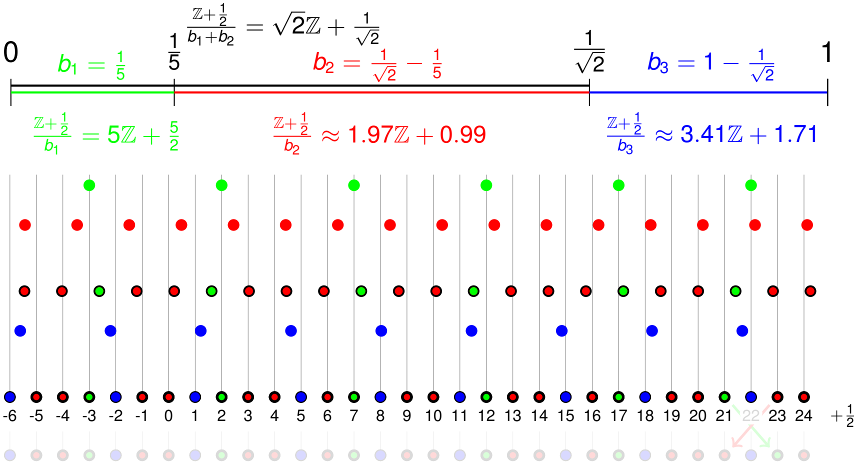

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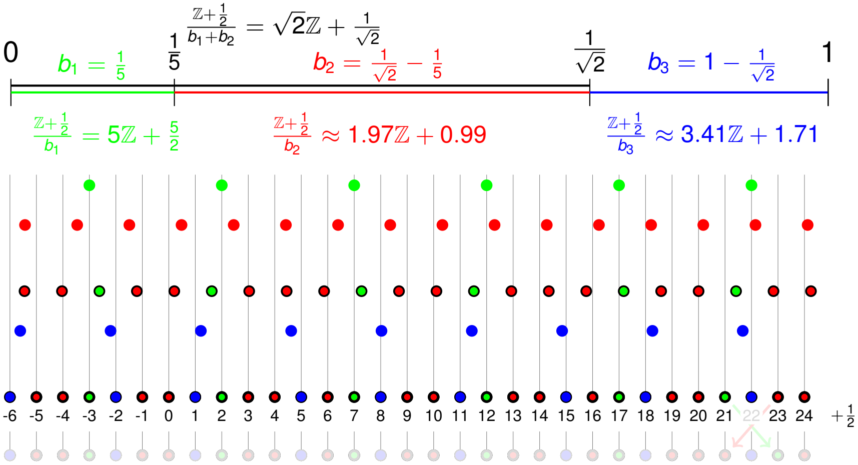

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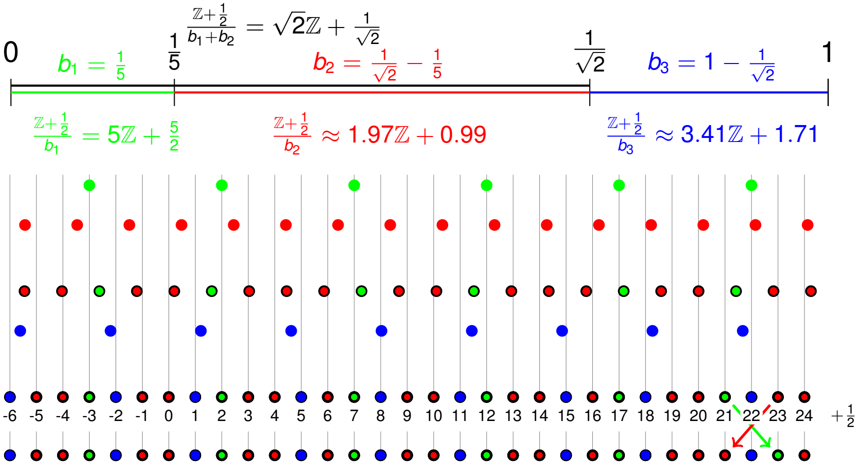

- By working inductively, we can obtain a partition of $\mathbb{Z}$ into three sets.
- However, there is no guarantee that the mappings so defined satisfy Avdonin's Theorem
- To get around this, we deploy a *calculus of Avdonin maps*.

Three or more intervals

Lemma

Suppose that there exist injective maps

$$\widehat{\varphi}\colon \frac{\mathbb{Z}+\frac{1}{2}}{b_1} \to \frac{\mathbb{Z}+\frac{1}{2}}{b_1+b_2}, \quad \widehat{\psi}\colon \frac{\mathbb{Z}+\frac{1}{2}}{b_2} \to \frac{\mathbb{Z}+\frac{1}{2}}{b_1+b_2}, \quad \sigma\colon \frac{\mathbb{Z}+\frac{1}{2}}{b_1+b_2} \to \mathbb{Z}+\frac{1}{2}$$

such that

- $\mathrm{Range}\,(\widehat{\varphi}) \,\dot{\cup}\, \mathrm{Range}\,(\widehat{\psi}) = \frac{\mathbb{Z}+\frac{1}{2}}{b_1+b_2}$
- *φ and ψ are δ-Avdonin maps for $\frac{\mathbb{Z}+\frac{1}{2}}{b_1}$ and $\frac{\mathbb{Z}+\frac{1}{2}}{b_2}$ (resp.), and*
- *σ is an ϵ-Avdonin map for $\frac{\mathbb{Z}+\frac{1}{2}}{b_1+b_2}$.*

Then $\widehat{\varphi}, \widehat{\psi}$ can be locally modified to φ, ψ so that in addition $\sigma \circ \varphi$ and $\sigma \circ \psi$ are $(\epsilon + 3\delta)$-Avdonin maps.

Partitioning into three intervals

- Suppose that we are given $b_1, b_2, b_3 > 0$ so that $b_1 + b_2 + b_3 = 1$.
- We can define ϵ-Avdonin maps

$$\varphi_1 : \frac{\mathbb{Z} + \frac{1}{2}}{b_1} \to \mathbb{Z} + \frac{1}{2}, \quad \psi_1 : \frac{\mathbb{Z} + \frac{1}{2}}{b_2 + b_3} \to \mathbb{Z} + \frac{1}{2}$$

  thereby partitioning $\mathbb{Z} + \frac{1}{2}$ into $\Lambda_1 = \mathrm{Range}\,(\varphi_1)$ and $\Gamma_1 = \mathrm{Range}\,(\psi_1)$.
- With ϵ small enough, we immediately have that

$$\mathcal{E}(\Lambda_1) \text{ is a Riesz basis for } L^2(I) \text{ with } |I| = b_1$$

  and

$$\mathcal{E}(\Gamma_1) \text{ is a Riesz basis for } L^2(I) \text{ with } |I| = 1 - b_1 = b_2 + b_3$$

- Next define δ-Avdonin maps

$$\varphi_2 \colon \frac{\mathbb{Z}+\frac{1}{2}}{b_2} \to \frac{\mathbb{Z}+\frac{1}{2}}{b_2+b_3}, \quad \psi_2 \colon \frac{\mathbb{Z}+\frac{1}{2}}{b_3} \to \frac{\mathbb{Z}+\frac{1}{2}}{b_2+b_3}$$

thereby partitioning $\frac{\mathbb{Z}+\frac{1}{2}}{b_2+b_3}$ into $\mathrm{Range}\,(\varphi_2)$ and $\mathrm{Range}\,(\psi_2)$.
- Applying the Lemma, we can adjust φ_2 and ψ_2 in such a way that

$$\psi_1 \circ \varphi_2 \colon \frac{\mathbb{Z}+\frac{1}{2}}{b_2} \to \mathbb{Z}+\frac{1}{2}, \quad \psi_1 \circ \psi_2 \colon \frac{\mathbb{Z}+\frac{1}{2}}{b_3} \to \mathbb{Z}+\frac{1}{2}$$

are $\epsilon + 3\delta$-Avdonin maps, and Γ_1 is partitioned into $\Lambda_2 = \mathrm{Range}\,(\psi_1 \circ \varphi_2)$ and $\Lambda_3 = \mathrm{Range}\,(\psi_1 \circ \psi_2)$.
- With δ small enough, we immediately have $\mathcal{E}(\Lambda_2)$ RB for $L^2(I)$, $|I| = b_2$ and $\mathcal{E}(\Lambda_3)$ RB for $L^2(I)$, $|I| = b_3$
- $\Lambda_1, \Lambda_2, \Lambda_3$ is our desired partition.

Theorem (Pfander, Revay, DW, 2018)

Let $b_1, b_2, \ldots > 0$ *with* $\sum_{k=1}^{\infty} b_k = 1$, *and* $c_j = \sum_{k=j+1}^{\infty} b_k$ *for* $j \in \mathbb{N}$ *so that* $c_j + b_j = c_{j-1}$. *Let* $\delta > 0$ *be given. Then there exist injective maps*

$$\Phi_j : \frac{\mathbb{Z} + \frac{1}{2}}{b_j} \to \mathbb{Z} + \frac{1}{2}, \quad \Psi_j : \frac{\mathbb{Z} + \frac{1}{2}}{c_j} \to \mathbb{Z} + \frac{1}{2}$$

such that

(a) $\left\{ \mathrm{Range}\,(\Phi_k), \mathrm{Range}\,(\Psi_j) \right\}_{k=1}^{j}$ *is a partition of* $\mathbb{Z} + \frac{1}{2}$,

(b) $\left\{ \mathrm{Range}\,(\Phi_{j+1}), \mathrm{Range}\,(\Psi_{j+1}) \right\}$ *is a partition of* $\mathrm{Range}\,(\Psi_j)$, *and*

(c) Φ_j *and* Ψ_j *are* $(1 - 2^{-j})\delta$*-Avdonin maps for* $\frac{\mathbb{Z}+\frac{1}{2}}{b_j}$ *and* $\frac{\mathbb{Z}+\frac{1}{2}}{c_j}$ *(resp.)*

- For each j, define maps

$$\widehat{\varphi}_j : \frac{\mathbb{Z}+\frac{1}{2}}{b_j} \to \frac{\mathbb{Z}+\frac{1}{2}}{b_j+c_j} = \frac{\mathbb{Z}+\frac{1}{2}}{c_{j-1}},$$

$$\widehat{\psi}_j : \frac{\mathbb{Z}+\frac{1}{2}}{c_j} \to \frac{\mathbb{Z}+\frac{1}{2}}{b_j+c_j} = \frac{\mathbb{Z}+\frac{1}{2}}{c_{j-1}}$$

- These can be simple rounding maps that we can take to be ϵ_j-Avdonin maps with $\epsilon_1 = \frac{\delta}{2}$ and $\epsilon_j = \frac{\delta}{3\cdot 2^j}$ if $j \geq 2$.

- If $j = 1$ then $\Phi_1 = \widehat{\varphi}_1$ and $\Psi_1 = \widehat{\psi}_1$.
- For $j = 2$, adjust the maps $\widehat{\varphi}_2$ and $\widehat{\psi}_2$ to φ_2 and ψ_2 so that

$$\Phi_2 = \Psi_1 \circ \varphi_2 \colon \frac{\mathbb{Z} + \frac{1}{2}}{b_2} \to \mathbb{Z} + \frac{1}{2},$$

$$\Psi_2 = \Psi_1 \circ \psi_2 \colon \frac{\mathbb{Z} + \frac{1}{2}}{c_2} \to \mathbb{Z} + \frac{1}{2}$$

  are $\epsilon_1 + 3\epsilon_2 = (1 - \frac{1}{4})\delta$-Avdonin maps.
- Proceed inductively.

Combining Intervals from Partition

Lemma

Let $a, b > 0$ and suppose that

$$\tau\colon \frac{\mathbb{Z}+\frac{1}{2}}{a} \to \mathbb{Z}+\frac{1}{2}, \qquad \textit{and} \qquad \eta\colon \frac{\mathbb{Z}+\frac{1}{2}}{b} \to \mathbb{Z}+\frac{1}{2}$$

are ϵ-Avdonin maps. Then there exists a 4ϵ-Avdonin map

$$\rho\colon \frac{\mathbb{Z}+\frac{1}{2}}{a+b} \to \mathbb{Z}+\frac{1}{2}$$

such that

$$\rho\Big(\frac{\mathbb{Z}+\frac{1}{2}}{a+b}\Big) = \tau\Big(\frac{\mathbb{Z}+\frac{1}{2}}{a}\Big) \cup \eta\Big(\frac{\mathbb{Z}+\frac{1}{2}}{b}\Big).$$

Theorem (Pfander, Revay, DW 2018)

Let $b_1, b_2, \ldots > 0$ with $\sum_{k=1}^{\infty} b_k = 1$ and $K \in \mathbb{N}$. Then there exist pairwise disjoint sets $\Lambda_1, \Lambda_2, \ldots \subseteq \mathbb{Z}$ such that for any $J \subseteq \mathbb{N}$ with $|J| \leq K$, $\bigcup_{j\in J} \mathcal{E}(\Lambda_j)$ is a Riesz basis for $L^2(I)$, I an interval with $|I| = \sum_{j\in J} b_j$.

- Let $\delta = 4^{-K}$.
- Previous theorem allows us to obtain

$$\Phi_j\colon \frac{\mathbb{Z}+\frac{1}{2}}{b_j} \to \mathbb{Z}+\frac{1}{2}$$

  a 4^{-K}-Avdonin map.
- Letting

$$\{\Lambda_j = \mathrm{Range}\,(\Phi_j)\}_{j=1}^{\infty},$$

  gives pairwise disjoint subsets of $\mathbb{Z}$ such that $\mathcal{E}(\Lambda_j)$ is a Riesz basis for $L^2(I)$, I an interval with $|I| = b_j$.

- Given $J \subseteq \mathbb{N}$ with $|J| \leq K$, use the Lemma to combine bases pairwise to obtain a $4^{K-1}4^{K} = \frac{1}{4}$-Avdonin map from

$$P_J\colon \frac{\mathbb{Z}+\frac{1}{2}}{\sum_{j\in J} b_j} \to \mathbb{Z}+\frac{1}{2}$$

- Hence

$$\bigcup_{j\in J} \mathcal{E}(\Lambda_j)$$

  is a Riesz basis for $L^2(I)$, I an interval with $|I| = \sum_{j\in J} b_j$.
- The rapid growth of the Avdonin constants makes the a priori choice of $K \in \mathbb{N}$ necessary.