EXPANDING QUANTUM FIELD THEORY USING AFFINE QUANTIZATION

- *What is affine quantization?*
- *Eliminating nonrenormalization*
- *An essential Fourier transform*

🙂 🙂 **John R. Klauder** 🙂 🙂

CANONICAL QUANTIZATION – 1

Basic variables and their relations

$$-\infty < p, q < \infty \quad , \quad p, q \;\to P(=P^\dagger), Q(=Q^\dagger) \quad , \quad [Q, P] = i\hbar \mathbb{1}$$

$$|p,q\rangle = e^{-iqP/\hbar}\, e^{ipQ/\hbar}\,|\omega\rangle \quad , \qquad \langle\omega|(Q + iP/\omega)|\omega\rangle = 0$$

🙂 **Dirac : Having $\mathscr{H}$(p, q) = H(p, q), requires Cartesian coordinates in order to obtain physically correct operaters!**

$$H(p,q) = \langle p,q|\mathscr{H}(P,Q)|p,q\rangle$$
$$= \langle\omega|\mathscr{H}(P+p, Q+q)|\omega\rangle = \mathscr{H}(p,q) + \mathscr{O}(\hbar; p,q)$$
$$d\sigma^2 = 2\hbar^2[\,\|d|p,q\rangle\|^2 - |\langle p,q|d|p,q\rangle|^2\,] \,, \quad \mathscr{F} - \mathscr{S}$$
$$= \omega^{-1}\,dp^2 + \omega\,dq^2$$

🙂

CQ → flat surface = constant zero curvature := 0

CANONICAL QUANTIZATION – 2

Classical physics

$$-\infty < p\,, q < \infty$$

$$A = \int [p(t)\,\dot{q}(t) - H(p(t), q(t))]\,dt$$

Favored variables

$$p \rightarrow P\ (= P^{\dagger}) \quad , \quad q \rightarrow Q\ (= Q^{\dagger}) \quad ; \quad [Q, P] = i\hbar\,1\!\!1$$

Schrödinger′s representation

$$Q \rightarrow x \in I\!\!R \quad , \quad P \rightarrow -\,i\hbar(\partial/\partial\,x)$$

Schrödinger′s equation

$$i\hbar\;\partial\,\psi(x,t)/\partial\,t \;=\; H(-i\hbar\,\partial/\partial\,x, x)\;\psi(x,t)$$

🙂

$$\int_{-\infty}^{\infty} |\,\psi(x,t)\,|^2\,dx < \infty$$

CANONICAL QUANTIZATION – 3

Classical physics

$$-\infty < p < \infty \;,\; \underline{0 < q < \infty} \quad , \quad H(p,q) = [p^2 + q^2]/2$$

$$A = \int [p(t)\,\dot{q}(t) - H(p(t),q(t))]\,dt$$

Favored variables

$$p \rightarrow \underline{P\;(\neq P^\dagger})\;,\quad q \rightarrow Q\;(=Q^\dagger > 0)\;,\; [Q!P] = i\hbar 1\!\!1$$

Hamiltonian operator(s)

$$H_0 = [PP^\dagger + Q^2]/2 \neq [P^\dagger P + Q^2]/2 = H_1$$

Hamiltonian spectra

$$E_0 = \hbar[(0,2,4,6,...) + 1/2] \quad , \quad E_1 = \hbar[(1,3,5,7,...) + 1/2]$$

$$E = \hbar[(0,2,3,5,7,8,10...) + 1/2]$$

CANONICAL QUANTIZATION FAILS

AFFINE QUANTIZATION – 1

Basic variables and relations

$$-\infty < p < \infty \;\;,\;\; 0 < q < \infty \;\;,\;\; p, q \;\rightarrow P(\neq P^{\dagger}), Q(= Q^{\dagger} > 0)$$

$$d \equiv pq \rightarrow (P^{\dagger}Q + QP)/2 \equiv D\,(= D^{\dagger}) \;\;,\;\; [Q, D] = i\hbar Q$$

$$|p;q\rangle = e^{ipQ/\hbar}\, e^{-i\ln(q)D/\hbar}\,|b\rangle \;\;,\;\; \langle b|[(Q - 1\!\!1) + iD/b]|b\rangle = 0$$

Favored affine coordinates

$$H'(pq, q) = \langle p;q|\mathcal{H}'(D, Q)|p;q\rangle \qquad q > 0$$

$$= \langle b|\mathcal{H}'(D + pqQ, qQ)|b\rangle = \mathcal{H}'(pq, q) + \mathcal{O}'(\hbar; p, q)$$

$$d\sigma^2 = 2\hbar^2[\,\|d|p;q\rangle\|^2 - |\langle p;q|\,d|p;q\rangle|^2] = b^{-1}q^2dp^2 + b\,q^{-2}dq^2$$

AQ → constant negative curvature $:= -2/b$

AFFINE QUANTIZATION – 2

Classical variables

$$-\infty < p < \infty \ , \ 0 < q < \infty \ , \qquad H' = [(pq)^2/q^2 + q^2]/2$$

$$A = \int \{ p(t)q(t)[\dot{q}(t)/q(t)] - H'(p(t)q(t), q(t)) \} \, dt$$

Favored variables

$$p \rightarrow P \ (\neq P^\dagger) \ [but \ \ P^\dagger Q = PQ] \ , \ \ q \rightarrow Q \ (= Q^\dagger > 0)$$

$$d \equiv pq \rightarrow (PQ + QP)/2 \equiv D \ (= D^\dagger) \ , \quad [Q, D] = i\hbar Q$$

$$\int_a^b [g^*(x) f(x)]' \, dx = \int_a^b [g^*(x)' f(x) + g^*(x) f(x)'] \, dx = g^*(b) f(b) - g^*(a) f(a)$$

$$g^*(b) \ , g^*(a) \quad f(b) = 0, f(a) = 0 \qquad i \int_a^b g^*(x)' f(x) \, dx = - \, i \int_a^b g^*(x) f(x)' \, dx$$

Schrödinger′s representation

$$Q \rightarrow x \in \mathbb{R}^+ \ , \qquad D \rightarrow - \, i\hbar[(\partial/\partial \, x)x + x(\partial/\partial \, x)]/2$$

Schrödinger′s equation

$$i\hbar \, \partial \psi(x, t)/\partial \, t \ = \ H'(-i\hbar[(\partial/\partial \, x)x + x(\partial/\partial \, x)]/2, \, x) \ \psi(x, t)$$

$$Master \ eq \, . \, (Me) : \ D \, x^{-1/2} = 0 \ , \qquad \int_0^\infty |\psi(x, t)|^2 \, dx < \infty$$

AFFINE QUANTIZATION – 3

The half – harmonic oscillator

$$-\infty < d \equiv pq < \infty \ , \quad 0 < q < \infty$$

Classical Hamiltonian $H'(d,q) = (d^2/q^2 + q^2)/2$

Quantum Hamiltonian $\mathscr{H}'(D,Q) = (DQ^{-2}D + Q^2)/2$

Schrödinger′s representation

$$Q \to x \in I\!R^+ \ , \quad D \to -i\hbar[(\partial/\partial x)x + x(\partial/\partial x)]/2 \quad (\cdots)$$

Hamiltonian $\mathscr{H} = [-\hbar^2(\partial^2/\partial x^2) + (3/4)\hbar^2/x^2 + x^2]/2$

Eigenvalues $2\,\hbar[(0,1,2,...) + 1] \ , \quad (\ \hbar[(0,1,2,...) + 1/2]\ )$

L. Gouba, *arXiv* : 2005.08696 🙂

CLASSICAL* FIELD THEORY – 1

classical, and initial quantum issues

$$\pi(x)\ ,\ \ \varphi(x)\ \ ,\ \ \ x \in \mathbb{R}^s \quad (CQ)$$

$$H(\pi,\varphi) = \int \{[\pi(x)^2 + (\vec{\nabla}\varphi(x))^2 + m^2\varphi(x)^2]/2 + g\,\varphi(x)^r\}\ d^sx$$

r = power of interaction term

$n = s + 1$, *number of spacetime dimensions*

🙂 $r < 2n/(n-2)$ *leads to renormalizable behavior* 🙂

$r \geq 2n/(n-2)$ *leads to nonrenormalizable behavior*

$$MC:\ \ \varphi^r_n:\ \ \varphi^4_3 \uparrow,\ \ \varphi^4_4 \downarrow,\ \ \varphi^{12}_3 \downarrow \quad (nf, f, f)$$

domain$(g = 0) \geq$ **domain**$(g > 0)$

CLASSICAL* FIELD THEORY – 2

affine, and initial quantum issues

$$\kappa(x) \equiv \pi(x)\,\varphi(x) \quad , \quad \varphi(x) \neq 0 \qquad (AQ)$$

$$H'(\kappa, \varphi) = \int \{ [\kappa(x)^2 \varphi(x)^{-2} + (\vec{\nabla}\varphi(x))^2 + m^2 \varphi(x)^2]/2 + g\,\varphi(x)^r \}\; d^s x$$

$$r = power\ of\ interaction\ term$$

$$n = s + 1,\ number\ of\ spacetime\ dimensions$$

$$MC: \quad \varphi_n^r: \quad \varphi_3^4 \uparrow, \ \ \varphi_4^4 \uparrow, \ \ \varphi_3^{12} \uparrow \qquad (nf, nf, nf)$$

🙂 **A legitimate pair of classical variables** 🙂

CLASSICAL* FIELD THEORY – 3

affine and canonical quantum issues

$$(AQ) \quad \kappa(x) \equiv \pi(x)\,\varphi(x)\;, \quad \varphi(x) \neq 0$$

$$H'(\kappa,\varphi) = \int \{ [\kappa(x)^2\,\varphi(x)^{-2} + (\overrightarrow{\nabla}\varphi(x))^2 + m^2\,\varphi(x)^2]/2 + g\,\varphi(x)^r \}\; d^sx$$

$$\varphi(x) \neq 0 \rightarrow 0 < \varphi(x)^{-2} < \infty \rightarrow 0 < |\varphi(x)|^r < \infty$$

🙂 **RENORMALIZABLE**

$$\{ \int \varphi(x)^r\; d^nx \}^{2/r} \;\; < or = or > \{ \int [(\overrightarrow{\nabla}\varphi(x))^2 + m^2\varphi(x)^2]/2\; d^nx \}$$

leads to

(CQ) $\quad r < 2n/(n-2)$ *renormalizable*
$\quad r \geq 2n/(n-2)$ *nonrenormalizable*

NONRENORMALIZABLE

QUANTUM FIELD THEORY – 1

Schrödinger representation : $\hat{\varphi}(x) = \varphi(x) \neq 0$

$$\hat{\kappa}(x) = -\,i\hbar[\varphi(x)\ (\partial/\partial\varphi(x)) + (\partial/\partial\varphi(x))\ \varphi(x)]/2$$

Me

such that $\hat{\kappa}(x)\,\varphi(x)^{-1/2} = 0$, $\hat{\kappa}(x)\,\Pi_y \varphi(y)^{-1/2} = 0$

$$|\,\varphi(x)^{-1/2}\rangle \to \varphi(x)^{-1/2} \quad , \qquad \Pi_y\,|\,\varphi(y)^{-1/2}\rangle \to \Pi_y\,\varphi(y)^{-1/2}$$

Selected wave functions : $y \in I\!R^s$

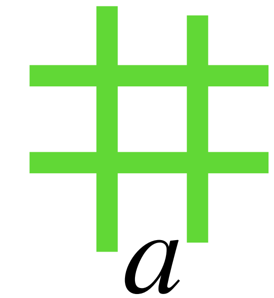
$$\Phi(\varphi) = \Pi_y \textstyle\int v(\varphi(y))\ \varphi(y)^{-1/2}\ dy$$

$$\Psi(\varphi) = \Pi_y \textstyle\int w(\varphi(y))\ \varphi(y)^{-1/2}\ dy$$

$$\int \Phi(\varphi)^*\ \Psi(\varphi)\ \mathscr{D}(\varphi) = \Pi_y \textstyle\int v(\varphi(y))^*\, w(\varphi(y))\ \varphi(y)^{-1}\ d\varphi(y)$$

QUANTUM FIELD THEORY – 2

A regularized representation $\quad k \in a\mathbb{Z}^s$



$$\hat{\varphi}_k = \varphi_k \;\;,\quad \hat{\kappa}_k = -\,i\hbar[\,\varphi_k\,(d/d\varphi_k) + (d/d\varphi_k)\,\varphi_k\,]/2a^s$$

$$\hat{\kappa}_k\,\varphi_k^{-1/2} = 0\;,\quad \hat{\kappa}_k\,\Pi_l\,\varphi_l^{-1/2} = 0 \quad Me$$

Selected regularized wave functions; $(ba^s)^{1/2}$

$$\Phi(\varphi) = \Pi_k\;v(\varphi_k)\,\varphi_k^{-(1-2ba^s)/2}\;,\quad \Psi(\varphi) = \Pi_k\;w(\varphi_k)\;\varphi_k^{-(1-2ba^s)/2}$$

$$\Pi_k\;\int\Phi(\varphi_k)^*\;\Psi(\varphi_k)\;d\varphi_k = \;\Pi_k\int\;v(\varphi_k)^*\,w(\varphi_k)\;\varphi_k^{-(1-2ba^s)}\,d\varphi_k$$

QUANTUM FIELD THEORY – 3

Schrödinger's equation

$$i\hbar(\partial/\partial t)\, \Psi(\varphi, t) = \mathscr{H}(\hat{\kappa}, \varphi)\, \Psi(\varphi, t)$$

A reqularized and normalized |solution|^2

$$1 = \Pi_k\, ba^s \int \{\, |w(\varphi_k)|^2\, |\varphi_k|^{-(1-2ba^s)}\, d\varphi_k \,\}$$

🙂 Introduction of a Fourier transform 🙂

🙂 $$F(f) = \Pi_k\, ba^s \int \{\, e^{i2\pi\, f_k \varphi_k}\, |w(\varphi_k)|^2\, |\varphi_k|^{-(1-2ba^s)}\, d\varphi_k \,\}$$ 🙂

🙂 $$F(0) = \Pi_k\, ba^s \int \{\, |w(\varphi_k)|^2\, |\varphi_k|^{-(1-2ba^s)}\, d\varphi_k \,\}$$ 🙂

QUANTUM FIELD THEORY – 4

$$\Psi(\varphi) = \Pi_k \, w(\varphi_k) \, (ba^s)^{1/2} \, \varphi_k^{-(1-2ba^s)/2}$$

$$F(0) = 1 = \Pi_k \, \{ba^s \int |w(\varphi_k)|^2 \, |\varphi_k|^{-(1-2ba^s)} \, d\varphi_k \, \}$$

$$F(f) = \Pi_k\{1- \; ba^s \int (1 - e^{i\,2\pi f_k \varphi_k}) \, |w(\varphi_k)|^2 \, |\varphi_k|^{-(1-2ba^s)} \, d\varphi_k \, \}$$

🙂 **The continuum limit:** $a \to 0$ 🙂

🙂 $F(f) = \exp\{ \, -b \int d^s x \, \int (1 - e^{i\,2\pi f(x)\,\lambda}) \, |w(\lambda)|^2 \, |\lambda|^{-1} \, d\lambda \, \}$ 🙂

This is a Poisson distribution

QUANTUM FIELD THEORY – 5

This is an example of the CENTRAL LIMIT THEOREM

$$F(f) = \exp\{\, -b \int d^s x \, \int (1 - e^{i2\pi f(x)\,\lambda}) \, |w(\lambda)|^2 \, |\lambda|^{-1} \, d\lambda \,\}$$

🙂 **NOTE** : **If** $g = 0$, $F(f)$ **is NOT a Gaussian**! 🙂

A legitimate pair of quantum variables

$$\hat{\kappa}(x) \;\; \& \;\; \hat{\varphi}(x) \;\; = \varphi(x) \neq 0$$

🙂 $\varphi(x) \neq 0 \rightarrow 0 < \varphi(x)^{-2} < \infty \rightarrow 0 < |\varphi(x)|^r < \infty$ 🙂

$$0 \leq |w(\lambda)|^2 < \infty$$

🙂 **NO NONRENORMALIZABILITY** 🙂

QUANTUM FIELD THEORY – 6

What leads to the final AQ result?

Normalization: $1 = \Pi_k \int \{ \, ba^s |w(\varphi_k)|^2 \, |\varphi_k|^{-(1-2ba^s)} \, d\varphi_k \, \}$

Components:

$$d\,|\varphi_k|^{2ba^s} = 2ba^s \, |\varphi_k|^{-(1-2ba^s)} \, d\varphi_k \quad (\textit{positive derivative})$$

$$0 \le |w(\varphi_k)|^2 < \infty \quad (\textit{well behaved})$$

$$F(f) = \Pi_k \, ba^s \int \{ \, e^{i2\pi f_k \varphi_k} \, |w(\varphi_k)|^2 \, |\varphi_k|^{-(1-2ba^s} \, d\varphi_k \, \} \quad (\textit{initial})$$

🙂 🙂 $$F(f) = \exp\{ \, -b \int d^s x \int (1 - e^{i\,2\pi f(x)\,\lambda}) \, |w(\lambda)|^2 \, |\lambda|^{-1} \, d\lambda \, \} \quad (\textit{final})$$

Bonus: NO NONRENORMALIZABILITY

QUANTUM FIELD THEORY – 7

Suppose we choose $\kappa'(x) = \pi(x)\,(\varphi(x)^2 - 1)$

$$H'(\kappa',\varphi) = \int \{[\kappa'(x)^2\,(\varphi(x)^2 - 1)^{-2} + (\vec{\nabla}\varphi(x))^2 + m^2\varphi(x)^2]/2 + g\,(\varphi(x)^2 - 1)^r\}\; d^s\!x$$

To quantize this we use $\hat{\varphi}(x) \to \varphi(x)$

$$\hat{\kappa}'(x) = [\hat{\pi}(x)\,(\varphi(x)^2 - 1) + (\varphi(x)^2 - 1)\,\hat{\pi}(x)]/2$$

$$\mathscr{H}(\hat{\kappa}',\varphi) = \int \{\,[\hat{\kappa}'(x)\,(\varphi(x)^2 - 1)^{-2}\,\hat{\kappa}'(x) + (\vec{\nabla}\varphi(x))^2 + m^2\,\varphi(x)^2]/2 \,+ g\,(\varphi(x)^2 - 1)^r\,\}\; d^s\!x$$

This now requires that $(\varphi(x)^2 - 1) \neq 0$

$$0 < (\varphi(x)^2 - 1)^{-2} < \infty \;\;\rightarrow\;\; 0 < |\,(\varphi(x)^2 - 1)\,|^r < \infty$$

🙂 **NO NONRENORMALIZABILITY** 🙂

🙂 THANK YOU 🙂

P.S. Another Fourier tranformation is desired to see the results in proper coordinates.

$$G(\varphi) = \int e^{-i2\pi \int \varphi(x) f(x)\, d^s x}\, F(f)\, \mathscr{D}(f)$$

This task is open for others to contribute.

Suggested Publications

1 – *affine quantization* $[Q, D] = i\hbar\, Q$
arXiv : 2006.09156

2 – *half – harmonic oscillators*
arXiv : 2005.08696 $(p^2 + q^2)\ ,\ \ q > 0$

3 – *quantizing a special field* φ^4_4
arXiv : 2012.09991.

🙂 **4 – quantizing more fields** $[\overrightarrow{\varphi}(x)^2 - 1]^p$
arXiv : 2110 . 05952

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SOME FALSE HAMILTONIANS

The classical Hamiltoian

$$H(p,q) = (p^2 + q^2)/2 \quad , \quad q > 0 \quad \rightarrow \quad P^\dagger \neq P$$

False self – adjoint quantum Hamiltonians

$$\mathscr{H}_1(P^\dagger, P, Q) = [(P^\dagger + P)^2/4 + Q^2]/2$$

$$\mathscr{H}_2(P^\dagger, P, Q) = [(2\,P^\dagger - P)(2\,P - P^\dagger) + Q^2]/2$$

$$\mathscr{H}'_n(P^\dagger, P, Q) = [(P^{\dagger\ 4+n}/P^{2+n} + P^{4+n}/P^{\dagger\ 2+n})/2 + Q^2]/2$$

$$with \quad n = 0,1,2,3,...$$

SPIN QUANTIZATION

Spin variable properties

$$[S_2, S_3] = i\hbar S_1 \ , \quad S_1^2 + S_2^2 + S_3^2 = \hbar^2 s(s+1) I_{2s+1} \ , \quad s \in (1/2)\{1,2,3,\cdots\}$$

$$S_3 |s,m\rangle = m\hbar |s,m\rangle \ , \ m \in \{-s,\cdots,s-1,s\} \ , \quad (S_1 + iS_2)|s,s\rangle = 0$$

Spin coherent states

$$|\theta,\varphi\rangle \equiv e^{-i\varphi S_3/\hbar} e^{-i\theta S_2/\hbar} |s,s\rangle$$

$$|p,q\rangle \equiv e^{-i(q/(s\hbar)^{1/2})S_3/\hbar}\, e^{-i\cos^{-1}(p/(s\hbar)^{1/2})S_2/\hbar} |s,s\rangle$$

$$-\pi(s\hbar)^{1/2} < q \leq \pi(s\hbar)^{1/2} \ , \quad -(s\hbar)^{1/2} \leq p \leq (s\hbar)^{1/2}$$

$$d\sigma^2 = 2\hbar[\| d|\theta,\varphi\rangle \|^2 - |\langle\theta,\varphi| d |\theta,\varphi\rangle|^2]$$

$$= (s\hbar)[d\theta^2 + \sin^2(\theta)^2 d\varphi^2]$$ 🙂

$$= (1 - p^2/s\hbar)^{-1} dp^2 + (1 - p^2/s\hbar)\, dq^2$$ 🙂

SQ → constant positive curvature $:= (s\hbar)^{-1}$

ADDED : – CANONICAL & AFFINE PATH INTEGRLS

BOOK: 2010 (Birkhäuser)

"A Modern Approach to Functional Integration"

Chapter 8:

Continuous-Time Regularized Path Integrals

Sec 8.1: Wiener Measure Regularization of Phase Space Path Integrals

Sec 8.3: Continuous-Time Regularization of Affine Variable Path Integrals 🙂

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FAVORED COORDINATES – 2

Dirac: Cartesian coordinates should lead to $\mathscr{H}(p,q) = H(p,q)$

$$|p,q\rangle = e^{-iqP/\hbar}\, e^{ipQ/\hbar}\, |\omega\rangle \quad , \qquad (\omega Q + iP)\,|\omega\rangle = 0$$

$$H(p,q) = \langle p,q\,|\,\mathscr{H}(P,Q)\,|\,p,q\rangle \;,$$

$$= \langle \omega\,|\,\mathscr{H}(P+p, Q+q)\,|\,\omega\rangle = \mathscr{H}(p,q) + \mathscr{O}(\hbar; p,q)$$

$$0$$

$$2\hbar[\,\|\,d|p,q\rangle\,\|^2 - |\langle p,q\,|\,d|p,q\rangle\,|^2\,] = \omega^{-1}dp^2 + \omega\, dq^2$$

$$|p;q\rangle = e^{ipQ/\hbar}\, e^{-i\ln(q)D/\hbar}\, |b\rangle \quad , \quad [(Q - 1\!\!1) + iD/b]\,|b\rangle = 0$$

$$-1/2b$$

$$H'(pq,q) = \langle p;q\,|\,\mathscr{H}'(D,Q)\,|\,p;q\rangle \;, \quad q > 0$$

$$= \langle b\,|\,\mathscr{H}'(D + pqQ, qQ)\,|\,b\rangle = \mathscr{H}'(pq,q) + \mathscr{O}'(\hbar; p,q)$$

$$2\hbar[\,\|\,d|p;q\rangle\,\|^2 - |\langle p;q\,|\,d|p;q\rangle\,|^2\,] = b^{-1}q^2dp^2 + bq^{-2}dq^2$$

🙂 **CQ → flat** , 🙂 **AQ → constant negative curvature**