Climbing the Diagonal Clifford Hierarchy

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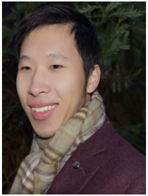
FFT Online Seminar

arXiv: 2109.13481 and 2110.11923



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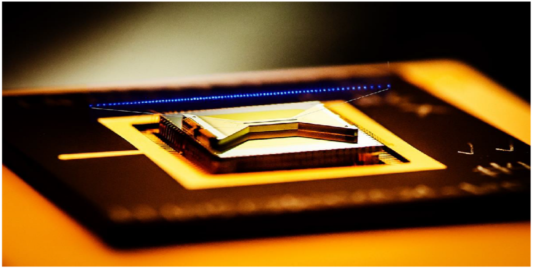
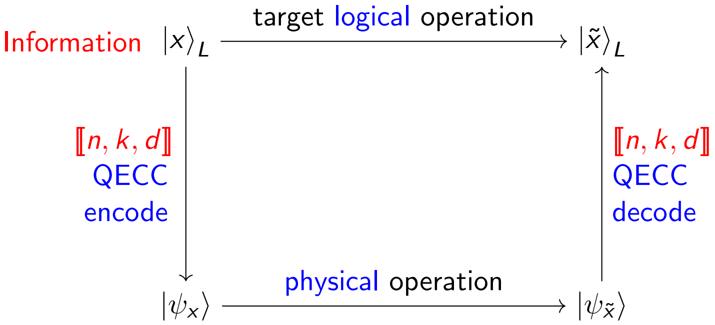


Ion trap fabricated by Sandia National Laboratories
and used by experimental teams at Duke and IonQ

Photo Credit: Kai Hudek (IonQ) and Sandia National Laboratories

Quantum computers are moving out of physics labs and becoming generally programmable. In this talk, we start from quantum protocols like magic state distillation and Shor factoring algorithm that make essential use of diagonal logical gates. The difficulty of reliably implementing these gates in some quantum error correcting code (QECC) is measured by their level in the Clifford hierarchy, a mathematical framework that was defined by Gottesman and Chuang when introducing the teleportation model of quantum computation. We describe a method of working backwards from a target logical diagonal gate at some level in the Clifford hierarchy to a quantum error correcting code (CSS code) in which the target logical can be implemented reliably.

Role of Quantum Error Correcting Code (QECC):



Generator Coefficients: Mathematical Framework
for reasoning about Diagonal Logical Channels

Pauli Group $\mathcal{P}_2$: Generated by $\imath I_2$, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Pauli Group $\mathcal{P}_N = \mathcal{P}_2^{\otimes m}$: Generated by $\imath I_N$, and matrices $D(\boldsymbol{a}, \boldsymbol{b})$ parametrized by binary vectors $\boldsymbol{a} = [a_1, \ldots, a_m]$, $\boldsymbol{b} = [b_1, \ldots, b_m]$

$$D(\boldsymbol{a}, \boldsymbol{b}) := X^{a_1} Z^{b_1} \otimes \cdots \otimes X^{a_m} Z^{b_m}$$

$$D(a, b) \lvert v \rangle = (-1)^{vb^T} \lvert v \oplus a \rangle \text{ for all } v \in \mathbb{F}_2^m$$

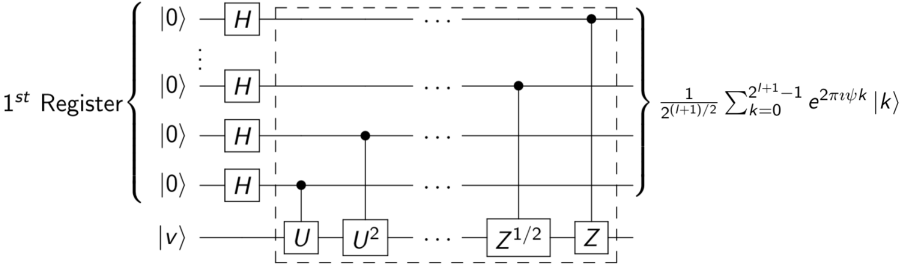
Hermitian Paulis: $E(a, b) := \imath^{ab^T \pmod 4} D(a, b)$, $E(a, b)^2 = I_N$

Pauli Group $\mathcal{P}_N := \{\imath^\kappa D(a, b) \colon a, b \in \mathbb{F}_2^m, \kappa \in \mathbb{Z}_4\}$ $(\imath = \sqrt{-1})$

$E(a, b), a, b \in \mathbb{F}_2^m$: $\underbrace{X \otimes Z \otimes Y}_{m=3 \text{ qubits}} = E(\underbrace{1\,0\,1}_{a}, \underbrace{0\,1\,1}_{b})$

$a =$	1	0	1
$b =$	0	1	1
$E(a, b) =$	X_1	Z_2	Y_3

1. Magic State Distillation
2. Period finding in the Shor factoring algorithm
   Unitary operator $U = Z^{1/2^l}$, eigenvector $\boldsymbol{v}$, eigenvalue $e^{2\pi\imath\psi}$



Measured by level in the Clifford Hierarchy

l-th level: Unitaries

$$\mathcal{C}^{(l)} := \{U \in \mathbb{U}_N : U\mathcal{P}_N U^\dagger \subset \mathcal{C}^{(l-1)}\}$$

$\vdots$

3^{rd} level

|

2^{nd} level: The Clifford Group

$$Cliff := \{U \in \mathbb{U}_N : U\mathcal{P}_N U^\dagger \subset \mathcal{P}_N\}$$

|

1^{st} level: The Pauli Group

$$\mathcal{P}_N := \mathcal{P}_2^{\otimes m},\ \mathcal{P}_2 := \langle \imath^\kappa I, X, Z\rangle$$

Diagonal Gates

$$Z^{\frac{1}{2^{l-1}}},\ \mathrm{C}Z^{\frac{1}{2^{l-2}}},\ \ldots,\ \mathrm{C}^{l-1}Z$$

$\vdots$

$$T = \sqrt{P} = Z^{\frac{1}{4}},\ \mathrm{C}P,\ \mathrm{CC}Z$$

|

$$P = Z^{\frac{1}{2}},\ \mathrm{C}Z$$

|

$$Z$$

Representing Diagonal Gates

Quantum Gate: $2^n \times 2^n$ complex diagonal unitary matrix

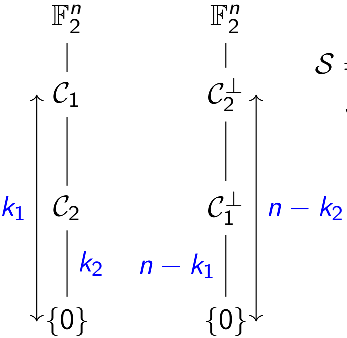
Walsh-Hadamard matrix: $H_{2^n} = H_2 \otimes H_2 \otimes \cdots \otimes H_2 = H_2^{\otimes n}$

$$\text{where } H_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Pauli Basis $\xleftrightarrow{H_{2^n}}$ Dirac Basis

$$U_Z = \sum_{\mathbf{v} \in \mathbb{F}_2^n} f(\mathbf{v}) E(0, \mathbf{v}) \qquad U_Z = \sum_{\mathbf{u} \in \mathbb{F}_2^n} d_{\mathbf{u}} \left|\mathbf{u}\right\rangle \left\langle \mathbf{u}\right|$$

$$[f(\mathbf{v})]_{\mathbf{v} \in \mathbb{F}_2^n} = [d_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{F}_2^n} H_{2^n}$$



Stabilizer Group $\mathcal{S} \subseteq \mathcal{P}_N$

$$\mathcal{S} = \{\epsilon_{(\boldsymbol{a},0)} E\left(\boldsymbol{a}, 0\right) \epsilon_{(0,\boldsymbol{b})} E\left(0, \boldsymbol{b}\right) : \boldsymbol{a} \in \mathcal{C}_2, \boldsymbol{b} \in \mathcal{C}_1^\perp\},$$

where $\epsilon_{(\boldsymbol{a},0)} = (-1)^{\boldsymbol{a}\boldsymbol{r}^T}$ and $\epsilon_{(0,\boldsymbol{b})} = (-1)^{\boldsymbol{b}\boldsymbol{y}^T}$.

Matrix Representation

$$G_{\mathcal{S}} = \left[\begin{array}{c|c} & G_1^\perp \\ \hline G_2 & \end{array}\right],$$

where G_2 generates $\mathcal{C}_2 = \mathcal{S}_X$
and $G_1^\perp$ generates $\mathcal{C}_1^\perp = \mathcal{S}_Z$.

Code projector $\Pi_{\mathcal{S}}$ for $\mathcal{S} = \langle \epsilon_{(\boldsymbol{c_i},0)} E\left(\boldsymbol{c_i}, 0\right), \epsilon_{(0,\boldsymbol{d_j})} E\left(0, \boldsymbol{d_j}\right)\rangle$

$$\Pi_{\mathcal{S}} = \prod_{i=1}^{k_2} \frac{(I_N + \epsilon_{(\boldsymbol{c_i},0)} E(\boldsymbol{c_i}, 0))}{2} \prod_{j=1}^{n-k_1} \frac{(I_N + \epsilon_{(0,\boldsymbol{d_j})} E(0, \boldsymbol{d_j}))}{2} =: \Pi_{\mathcal{S}_X} \Pi_{\mathcal{S}_Z}$$

$[\![n, k = k_1 - k_2, d]\!]$ CSS$(X, \mathcal{C}_2, \boldsymbol{r}; Z, \mathcal{C}_1^{\perp}, \boldsymbol{y})$: Stabilizer $\mathcal{S}$

Codespace $\mathcal{V}(\mathcal{S}) := \{|\psi\rangle \in \mathbb{C}^N : g|\psi\rangle = |\psi\rangle \text{ for all } g \in \mathcal{S}\}$

Undetectable Errors: $g \in \mathcal{P}_N, g \notin \mathcal{S}, gh = hg$ for all $h \in \mathcal{S}$

Distance d: Undetectable error acts on at least d qubits

X-distance d_X: $\min_{\boldsymbol{x} \in \mathcal{C}_1 \setminus \mathcal{C}_2} w_H(\boldsymbol{x})$

Z-distance d_Z: $\min_{\boldsymbol{z} \in \mathcal{C}_2^{\perp} \setminus \mathcal{C}_1^{\perp}} w_H(\boldsymbol{z})$

General Encoding Map g_e: $|\psi\rangle_L \in \mathbb{F}_2^k \to |\overline{\psi}\rangle \in \mathcal{V}(\mathcal{S})$

$$|\overline{\psi}\rangle := \frac{1}{\sqrt{|\mathcal{C}_2|}} \sum_{\boldsymbol{a} \in \mathcal{C}_2} (-1)^{\boldsymbol{a}\boldsymbol{r}^T} |\psi G_{\mathcal{C}_1/\mathcal{C}_2} \oplus \boldsymbol{a} \oplus \boldsymbol{y}\rangle$$

CSS Codes Oblivious to Coherent Noise

Coherent Noise: Z-rotation $R_Z(\theta)$ through angle θ on each qubit

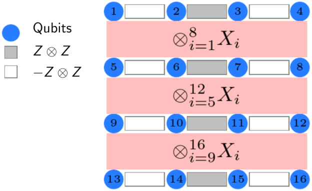
Obliviousness: Logical operator induced by $R_Z(\theta)$ is the identity
$\mathcal{V}(\mathcal{S})$ is a Decoherence Free Subspace (DFS)

Necessary and Sufficient conditions for $\mathcal{V}(\mathcal{S}) \subseteq$ DFS:

- Product Structure: weight-2 Z-stabilizers partition qubits into clumps
- Negative Signs: character vector $\boldsymbol{y}$ supported on half of each clump

$[\![16, 1, 4]\!]$ Shor Code with negative signs:

- Four Rows: Four Clumps $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$
- Character Vector: $\boldsymbol{y}|_{\Gamma_k} = [0\,1\,1\,0]$



For details see: https://arxiv.org/pdf/2011.00197

Code States that Support Logical Transversal T

Bravyi and Haah (2012): Triorthogonal CSS codes

A binary matrix $[G_{i,j}]_{\{i\in\{1,\cdots,m\},j\in\{1,\cdots,n\}\}}$ is triorthogonal if for all $1 \le f < g < h \le m$,

$$\sum_{i=1}^{n} G_{f,i}G_{g,i} = 0 \pmod 2 \text{ and } \sum_{i=1}^{n} G_{f,i}G_{g,i}G_{h,i} = 0 \pmod 2.$$

Triorthogonal CSS$(X, \mathcal{C}_2 = \langle G_0 \rangle, \boldsymbol{r} = 0; Z, \mathcal{C}_1^{\perp} = \langle G^{\perp} \rangle, \boldsymbol{y} = 0)$ code:

G is a triorthogonal matrix and G_0 is the submatrix of all even-weight rows

$$G = \left[\begin{array}{c} \boldsymbol{w}_1 \\ \vdots \\ \boldsymbol{w}_k \\ \hline \text{even-weight submatrix } G_0 : X\text{-Stabilizers} \end{array}\right]$$

Triorthogonality is Sufficient: Physical transversal T preserves the codespace and induces logical transversal T (up to logical Clifford gates)

Necessary and Sufficient Conditions

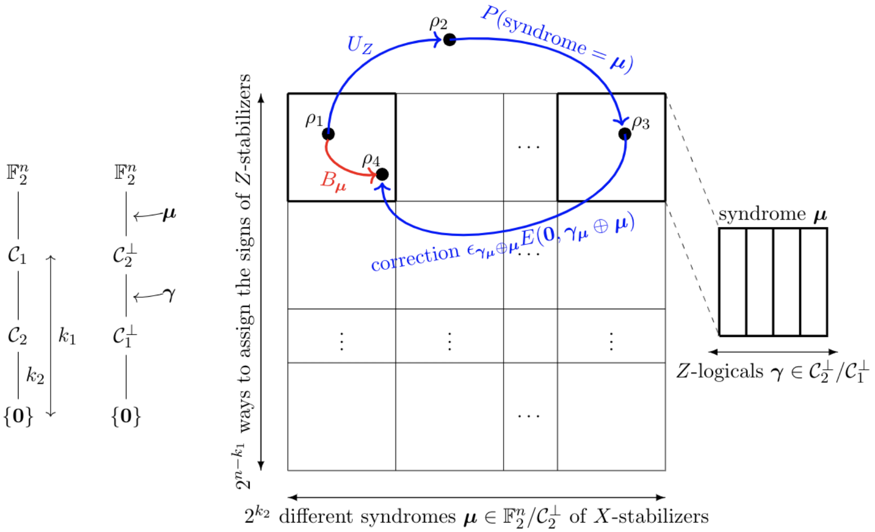
Stabilizer Perspective: Preserving a code is equivalent to preserving the projector that defines the code.

$$U\left|\overline{\psi}\right\rangle = UE(\boldsymbol{a},\boldsymbol{b})\left|\overline{\psi}\right\rangle = (UE(\boldsymbol{a},\boldsymbol{b})U^{\dagger})U\left|\overline{\psi}\right\rangle \Rightarrow UE(\boldsymbol{a},\boldsymbol{b})U^{\dagger} \in \mathcal{S}(U\left|\overline{\psi}\right\rangle)$$

Rengaswamy, Calderbank, Newman, and Pfister (2020) show that triorthogonality is necessary if physical transversal T is to induce logical transversal T (up to logical Clifford gates)

Hu, Liang, and Calderbank (2021) derive necessary and sufficient conditions for a physical diagonal gate to preserve a stabilizer code and induce a target logical diagonal gate

Creating an Average Logical Channel



2^{n-k_1} **rows**: the $[\![n, k_1 - k_2, d]\!]$ CSS codes corresponding to all possible signings of the Z-stabilizer group

2^{k_2} **columns**: all possible X-syndromes $\boldsymbol{\mu}$

Logical operator $\mathrm{B}_{\boldsymbol{\mu}}$ is induced by

1. preparing a code state ρ_1
2. applying a diagonal physical gate U_Z to obtain ρ_2
3. using X-stabilizers to measure ρ_2, obtaining the syndrome $\boldsymbol{\mu}$ with probability $p_{\boldsymbol{\mu}}$, and the post-measurement state ρ_3
4. applying a Pauli correction to ρ_3, obtaining ρ_4

$$\rho_4 = \sum_{\boldsymbol{\mu} \in \mathbb{F}_2^n / \mathcal{C}_2^{\perp}} B_{\boldsymbol{\mu}} \rho_1 B_{\boldsymbol{\mu}}^{\dagger} \text{ and } B_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\gamma} \in \mathcal{C}_2^{\perp} / \mathcal{C}_1^{\perp}} A_{\boldsymbol{\mu},\boldsymbol{\gamma}} \underbrace{\epsilon_{(0,\boldsymbol{\gamma} \oplus \boldsymbol{\gamma}_{\boldsymbol{\mu}})} E(0, \boldsymbol{\gamma} \oplus \boldsymbol{\gamma}_{\boldsymbol{\mu}})}_{\text{Logical Pauli } Z \text{ Operator}}$$

$A_{\boldsymbol{\mu},\boldsymbol{\gamma}}$ are the generator coefficients determined by U_Z

Diagonal Physical Gate: $U_Z = \sum_{\boldsymbol{v}\in\mathbb{F}_2^n} f(\boldsymbol{v})E(0,\boldsymbol{v}) = \sum_{\boldsymbol{u}\in\mathbb{F}_2^n} d_{\boldsymbol{u}} \left|\boldsymbol{u}\right\rangle\left\langle\boldsymbol{u}\right|$

For a CSS$(X, \mathcal{C}_2; Z, \mathcal{C}_1^{\perp})$ code, let

- $\boldsymbol{\mu} \in \mathbb{F}_2^n/\mathcal{C}_2^{\perp}$ be any X-syndrome
- $\boldsymbol{\gamma} \in \mathcal{C}_2^{\perp}/\mathcal{C}_1^{\perp}$ be any Z-logical

The generator coefficient $A_{\boldsymbol{\mu},\boldsymbol{\gamma}}$ corresponding to U_Z is

$$A_{\boldsymbol{\mu},\boldsymbol{\gamma}} = \sum_{\boldsymbol{z}\in\mathcal{C}_1^{\perp}+\boldsymbol{\mu}+\boldsymbol{\gamma}} \epsilon_{(0,\boldsymbol{z})} f(\boldsymbol{z}) = \frac{1}{|\mathcal{C}_1|}\sum_{\boldsymbol{u}\in\mathcal{C}_1}(-1)^{(\boldsymbol{\mu}\oplus\boldsymbol{\gamma})\boldsymbol{u}^T} d_{\boldsymbol{u}\oplus\boldsymbol{y}}.$$

Here $\epsilon_{(0,\boldsymbol{z})} = (-1)^{\boldsymbol{z}\boldsymbol{y}^T}$ is the sign of Z-stabilizer $E(0,\boldsymbol{z})$ for $\boldsymbol{y} \in \mathbb{F}_2^n/\mathcal{C}_1$

When the Physical gate is a Transversal Z-Rotation

$$R_Z(\theta) = \left[\exp\left(-\imath\tfrac{\theta}{2}Z\right)\right]^{\otimes n} = \left(\cos\tfrac{\theta}{2}I - \imath\sin\tfrac{\theta}{2}Z\right)^{\otimes n} = \textstyle\sum_{\boldsymbol{v}\in\mathbb{F}_2^n} f(\boldsymbol{v})E(0,\boldsymbol{v})$$

$$f(\boldsymbol{v}) = \left(\cos\frac{\theta}{2}\right)^{n-w_H(\boldsymbol{v})}\left(-\imath\sin\frac{\theta}{2}\right)^{w_H(\boldsymbol{v})}$$

Generator Coefficients:

$$A_{\boldsymbol{\mu},\boldsymbol{\gamma}}(\theta) = \sum_{\boldsymbol{z}\in\mathcal{C}_1^\perp+\boldsymbol{\mu}+\boldsymbol{\gamma}} \epsilon_{(0,\boldsymbol{z})}\left(\cos\frac{\theta}{2}\right)^{n-w_H(\boldsymbol{z})}\left(-\imath\sin\frac{\theta}{2}\right)^{w_H(\boldsymbol{z})}$$
$$= \frac{1}{|\mathcal{C}_1|}\sum_{\boldsymbol{z}\in\mathcal{C}_1}(-1)^{(\boldsymbol{\mu}\oplus\boldsymbol{\gamma})\boldsymbol{z}^T}\left(e^{-\imath\frac{\theta}{2}}\right)^{n-2w_H(\boldsymbol{z}\oplus\boldsymbol{y})}.$$

for all syndromes $\boldsymbol{\mu}\in\mathbb{F}_2^n/\mathcal{C}_2^\perp$ and Z-logicals $\boldsymbol{\gamma}\in\mathcal{C}_2^\perp/\mathcal{C}_1^\perp$

$w_H(\boldsymbol{z}\oplus\boldsymbol{y}) = \frac{n}{2}$: Induced logical gate is the identity

Stabilizer $\mathcal{S} = \langle -Z_1 Z_2, Z_2 Z_3 \rangle$: $\mathcal{C}_2 = \langle 000 \rangle$, $\mathcal{C}_1^{\perp} = \langle 110, 011 \rangle$

Character Vector: $\boldsymbol{y} = [\,1\,0\,0\,]$

Generator Coefficients $A_{0,\gamma}(\theta)$: $\gamma \in \mathcal{C}_2^{\perp}/\mathcal{C}_1^{\perp} = \{000, 111\}$

$$A_{0,\gamma}(\theta) = \sum_{\boldsymbol{z} \in \mathcal{C}_1^{\perp} + \gamma} \epsilon_{(0,\boldsymbol{z})} \left(\cos\frac{\theta}{2}\right)^{n - w_H(\boldsymbol{z})} \left(-\imath \sin\frac{\theta}{2}\right)^{w_H(\boldsymbol{z})}$$

$$= \begin{cases} \cos\frac{\theta}{2}, & \text{if } \gamma = [\,0\,0\,0\,] \\ -\imath \sin\frac{\theta}{2}, & \text{if } \gamma = [\,1\,1\,1\,] \end{cases}$$

$\sum_{\gamma} |A_{0,\gamma}(\theta)|^2 = 1$: $R_Z(\theta)$ preserves the codespace

Induced Logical Operator: $\exp(-\imath \frac{\theta}{2} Z_L)$

$$R_Z^L(\theta) = A_{0,000}(\theta) I_L + A_{0,111}(\theta) Z_L$$

U_Z: physical diagonal gate

Theorem: U_Z preserves $\text{CSS}(X, \mathcal{C}_2; Z, \mathcal{C}_1^\perp)$ codespace if and only if

$$\sum_{\gamma \in \mathcal{C}_2^\perp / \mathcal{C}_1^\perp} |A_{0,\gamma}|^2 = 1$$

Intuition: Invariance of the codespace is equivalent to requiring the logical operator induced by the trivial syndrome $B_{\mu=0}$ is unitary

Logical operator induced by U_Z is

$$U_Z^L = \sum_{\alpha \in \mathbb{F}_2^k} A_{0,h(\alpha)} E(0, \alpha)$$

where $h : \mathbb{F}_2^k \to \mathcal{C}_2^\perp / \mathcal{C}_1^\perp$ defined by $g(\alpha) = \alpha G_{\mathcal{C}_2^\perp / \mathcal{C}_1^\perp}$ and $G_{\mathcal{C}_2^\perp / \mathcal{C}_1^\perp}$ is the generator matrix of $\mathcal{C}_2^\perp / \mathcal{C}_1^\perp$

Example: $[\![7,1,3]\!]$ Steane Code with the trivial signs

wt	0	3	4	7
#	1	7	7	1

Table: Weight Dist of $\mathcal{C}_1 = \mathcal{C}_2^{\perp}$=RM(1,3)

wt	0	4
#	1	7

Table: Weight Dist of $\mathcal{C}_1^{\perp} = \mathcal{C}_2$

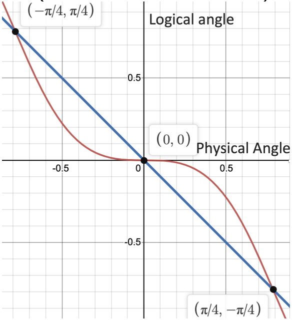
Generator Coefficients for the trivial syndrome $\boldsymbol{\mu} = 0$:

$$A_{0,0}(\theta) = \frac{1}{8}\left(\cos\frac{7\theta}{2} + 7\cos\frac{\theta}{2}\right),\ A_{0,1}(\theta) = \frac{\imath}{8}\left(-\sin\frac{7\theta}{2} + 7\sin\frac{\theta}{2}\right)$$

Logical angle θ_L in terms of physical angle θ:

$$\theta_L = 2\tan^{-1}\left(\imath\frac{A_{0,1}(\theta)}{A_{0,0}(\theta)}\right)$$

$$\theta = \frac{\pi}{4} \Rightarrow \theta_L = -\frac{\pi}{4}$$



Necessary and Sufficient Divisibility Conditions

Quadratic Form Diagonal (QFD) Gate: $\tau_R^{(l)} = \sum_{\mathbf{v} \in \mathbb{F}_2^n} \xi_l^{\mathbf{v} R \mathbf{v}^T \bmod 2^l} |\mathbf{v}\rangle \langle \mathbf{v}|$

Here $l \geq 1$ is the level in the Clifford hierarchy, $\xi_l = e^{\imath \frac{\pi}{2^{l-1}}}$, and R is an $n \times n$ symmetric matrix with entries in $\mathbb{Z}_{2^l} = \{0, 1, \ldots, 2^l - 1\}$

Rengaswamy, Calderbank, and Pfister (2019): QFD Gates include all 1-local and 2-local diagonal gates in the Clifford hierarchy.

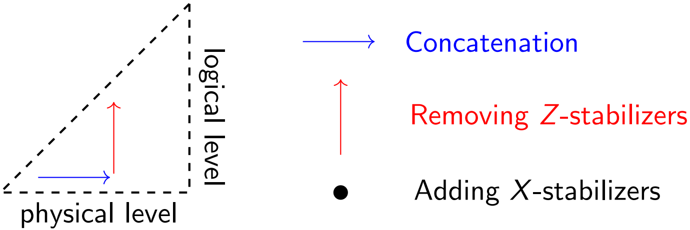
Theorem: If $U_Z = \tau_R^{(l)}$, then $\sum_{\gamma \in C_2^\perp / C_1^\perp} |A_{0,\gamma}|^2 = 1$ if and only if

$$2^l \mid (\mathbf{v}_1 R \mathbf{v}_1^T - \mathbf{v}_2 R \mathbf{v}_2^T)$$

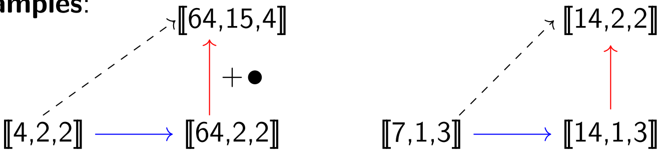
for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{C}_1 + \mathbf{y}$ such that $\mathbf{v}_1 \oplus \mathbf{v}_2 \in \mathcal{C}_2$

Divisibility conditions corresponding to successive levels differ only by a factor of 2

Climbing the Diagonal Clifford Hierarchy



Examples:



Quadratic Form Diagonal Gate: $\tau_R^{(l)} = \sum_{\mathbf{v} \in \mathbb{F}_2^n} \xi_l^{\mathbf{v} R \mathbf{v}^T \bmod 2^l} |\mathbf{v}\rangle \langle \mathbf{v}| \in \mathcal{C}^{(l)}$

$[\![n, k, d]\!]$ $\quad\quad$ $[\![2n, k, d' \geq d]\!]$ $\quad\quad$ $\mathbf{y}' = [1,1] \otimes \mathbf{y}$

$$\mathbb{F}_2^n \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \{0\} \qquad \mathbb{F}_2^n \overset{\leftarrow \boldsymbol{\mu}}{\supseteq} \mathcal{C}_2^{\perp} \overset{\leftarrow \boldsymbol{\gamma}}{\supseteq} \mathcal{C}_1^{\perp} \supseteq \{0\}$$

$\rightarrow$

$$\mathbb{F}_2^{2n} \supseteq \mathcal{C}_1' = [1,1] \otimes \mathcal{C}_1 \supseteq \mathcal{C}_2' = [1,1] \otimes \mathcal{C}_2 \supseteq \{0\}$$

$$\mathbb{F}_2^{2n} \overset{\leftarrow \boldsymbol{\mu}' = (\boldsymbol{\mu}, 0)}{\supseteq} (\mathcal{C}_2')^{\perp} \overset{\leftarrow \boldsymbol{\gamma}' = (\boldsymbol{\gamma}, 0)}{\supseteq} (\mathcal{C}_1')^{\perp} \supseteq \{0\}$$

$$A_{\boldsymbol{\mu},\boldsymbol{\gamma}}\left(\tau_R^{(l)}\right) = A'_{\boldsymbol{\mu}',\boldsymbol{\gamma}'}\left(\tau_{I_2 \otimes R}^{(l+1)}\right)$$

$U_Z \in \mathcal{C}^{(l)}$ and $U_Z' = \sqrt{U_Z}^{\otimes 2} \in \mathcal{C}^{(l+1)}$: $A_{\boldsymbol{\mu},\boldsymbol{\gamma}}(U_Z) = A'_{\boldsymbol{\mu}',\boldsymbol{\gamma}'}(U_Z')$

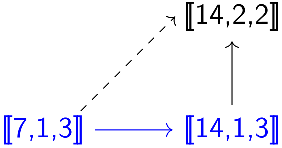
Climbing the Physical Hierarchy: $[\![7,1,3]\!] \rightarrow [\![14,1,3]\!]$

When $R = I_n$, $\tau_R^{(2)} = P^{\otimes n} = R_Z\left(\frac{\pi}{2}\right)$ and $\tau_{I_2 \otimes R}^{(3)} = T^{\otimes 2n} = R_Z\left(\frac{\pi}{4}\right)$

- $A_{\boldsymbol{\mu},\gamma}\left(\frac{\pi}{2}\right)$: $R_Z\left(\frac{\pi}{2}\right)$ acts on $[\![7,1,3]\!]$ code
- $A'_{\boldsymbol{\mu}',\gamma'}\left(\frac{\pi}{4}\right)$: $R_Z\left(\frac{\pi}{4}\right)$ acts on $[\![14,1,3]\!]$ code

$$A_{\boldsymbol{\mu}=0,\gamma=0}\left(\frac{\pi}{2}\right) = A'_{\boldsymbol{\mu}'=(\boldsymbol{\mu},0),\gamma'=(\gamma,0)}\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$$
$$A_{\boldsymbol{\mu}=0,\gamma=1}\left(\frac{\pi}{2}\right) = A'_{\boldsymbol{\mu}'=(\boldsymbol{\mu},0),\gamma'=(\gamma,0)}\left(\frac{\pi}{4}\right) = \imath\sin\left(\frac{\pi}{4}\right)$$

	$[\![7,1,3]\!]$	$[\![14,1,3]\!]$
Physical Gates	$P^{\otimes 7}$	$T^{\otimes 14}$
Logical Gates	$P^{\dagger}$	$P^{\dagger}$



$U_Z = \sum_{\boldsymbol{u} \in \mathbb{F}_2^n} d_{\boldsymbol{u}} |\boldsymbol{u}\rangle \langle \boldsymbol{u}|$: Physical diagonal gate that preserves an $[\![n, k, d]\!]$ CSS$(X, \mathcal{C}_2; Z, \mathcal{C}_1^{\perp}, \boldsymbol{y})$ code.

Theorem: The $[\![2n, k, d' \geq d]\!]$ CSS$(X, \mathcal{C}_2'; Z, (\mathcal{C}_1')^{\perp}, \boldsymbol{y}')$ code is preserved by any physical diagonal gate

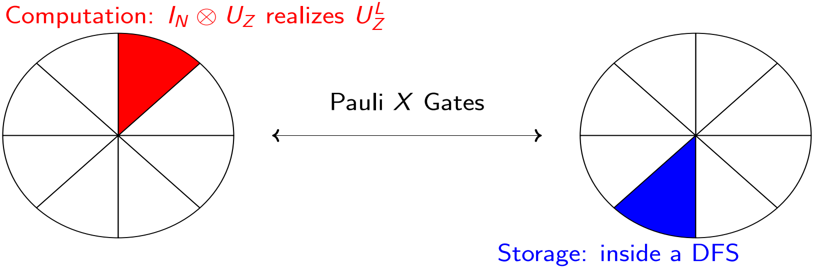
$$U_Z' = \sum_{\boldsymbol{u}' \in \mathbb{F}_2^{2n}} d'_{\boldsymbol{u}'} |\boldsymbol{u}'\rangle \langle \boldsymbol{u}'|$$

for which $d'_{[\boldsymbol{u},\boldsymbol{u}]} = d_{\boldsymbol{u}}$ for all $\boldsymbol{u} \in \mathbb{F}_2^n$

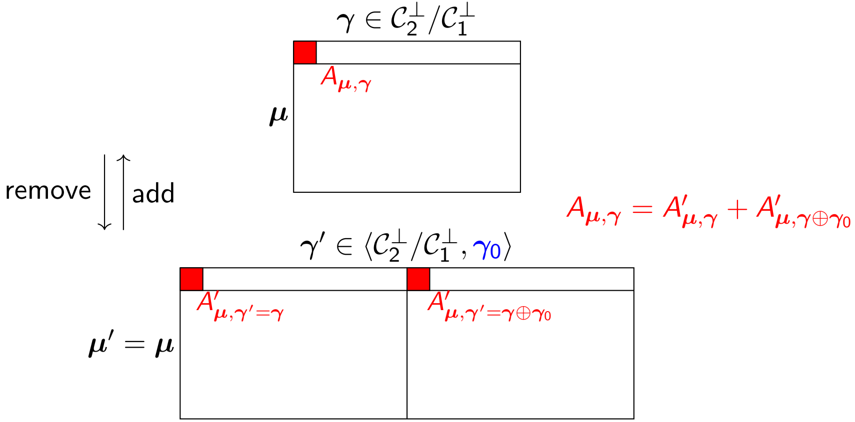
$(U_Z')^L = U_Z^L$: The induced logical operator are the same, and many degrees of freedom are available to design U_Z'

A CSS code supporting U_Z to realize U_Z^L: 2^{n-k_1} Z-resolutions of identity

After Concatenation: the CSS code with 2^{2n-k_1} Z-resolutions of identity

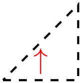


Remove a Z-stabilizer generator $\boldsymbol{z}$ = Add $\boldsymbol{z}$ as a new Z-logical γ_0



Climbing the Logical Hierarchy: $[\![14,1,3]\!] \rightarrow [\![14,2,2]\!]$

$$[\![14,1,3]\!] \xrightarrow[\text{Add } \gamma_0 = (1,1,\ldots,1)]{U_Z = T^{\otimes 14}} [\![14,2,2]\!]$$



$P^\dagger$:

- $A_{0,0} = \cos\frac{\pi}{4}$ → $A'_{0,0} = \left(\cos\frac{\pi}{8}\right)^2$, $\quad A'_{0,\gamma_0} = \left(\imath \sin\frac{\pi}{8}\right)^2$
- $A_{0,\gamma\neq 0} = \imath \sin\frac{\pi}{4}$ → $A'_{0,\gamma} = \imath \sin\frac{\pi}{8}\cos\frac{\pi}{8}$, $\quad A'_{0,\gamma\oplus\gamma_0} = \imath \sin\frac{\pi}{8}\cos\frac{\pi}{8}$

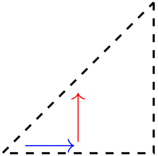
$(T^\dagger)^{\otimes 2}$

Magic of Trigonometry: Double-Angle Identities

When to Remove a Z-Stabilizer

U_Z: a fixed diagonal physical gate

Assume U_Z preserves the original CSS code: $\sum_{\gamma \in \mathcal{C}_2^\perp / \mathcal{C}_1^\perp} |A_{0,\gamma}|^2 = 1$



Admissible Split: U_Z preserves the CSS$(X, \mathcal{C}_2; Z, (\mathcal{C}_1')^\perp, \boldsymbol{y})$ code obtained by removing the Z-stabilizer $\boldsymbol{\gamma}_0$

Theorem: The necessary and sufficient condition for admissibility is

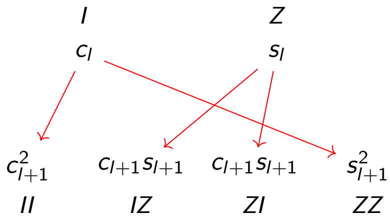
$$\sum_{\boldsymbol{\gamma}' \in \langle \mathcal{C}_2^\perp / \mathcal{C}_1^\perp, \boldsymbol{\gamma}_0 \rangle} |A'_{0,\boldsymbol{\gamma}'}|^2 = \sum_{\boldsymbol{\gamma} \in \mathcal{C}_2^\perp / \mathcal{C}_1^\perp} |A'_{0,\boldsymbol{\gamma}}|^2 + |A'_{0,\boldsymbol{\gamma} \oplus \boldsymbol{\gamma}_0}|^2 = 1$$

Example: $|(\cos\theta)^2|^2 + |(\imath \sin\theta)^2|^2 + |\imath \sin\theta \cos\theta|^2 + |\imath \sin\theta \cos\theta|^2 = 1$

Climbing from $Z^{1/2^{l-1}}$ to $Z^{1/2^l} \otimes Z^{1/2^l}$

$$x_l = e^{\imath\pi/2^l}$$
$$c_l = \cos\frac{\pi}{2^l}, \; s_l = -\imath \sin\frac{\pi}{2^l}$$

$$\begin{aligned} Z^{1/2^{l-1}} &= \frac{1+x_{l-1}}{2} I + \frac{1-x_{l-1}}{2} Z \\ &= x_l(c_l I + s_l Z) \\ &\equiv c_l I + s_l Z \end{aligned}$$



One Step for uniform rotations:

$$Z^{1/2^{l-1}} \rightarrow Z^{1/2^l} \otimes Z^{1/2^l}$$

$$Z^{1/2^{l-1}} \;\Big\downarrow\; \left(Z^{1/2^l}\right)^{\otimes 2}$$

Double-Angle Formulas

Multi-step for non-uniform rotations: $Z \to Z \otimes P^\dagger \to Z \otimes P^\dagger \otimes T^\dagger \to \cdots$

$$Z^{1/2^{l-1}}$$

$$\downarrow$$

$$Z^{1/2^{l-1}} \otimes \left(Z^{1/2^l}\right)^\dagger$$

$$\vdots$$

$$Z^{1/2^{l-1}} \otimes \left(Z^{1/2^l}\right)^\dagger \otimes \cdots \otimes \left(Z^{1/2^{l-1+j}}\right)^\dagger$$

$$x_l = e^{\imath\pi/2^l}$$

$$c_l = \cos\frac{\pi}{2^l},\, s_l = -\imath \sin\frac{\pi}{2^l}$$

Euler's Formula

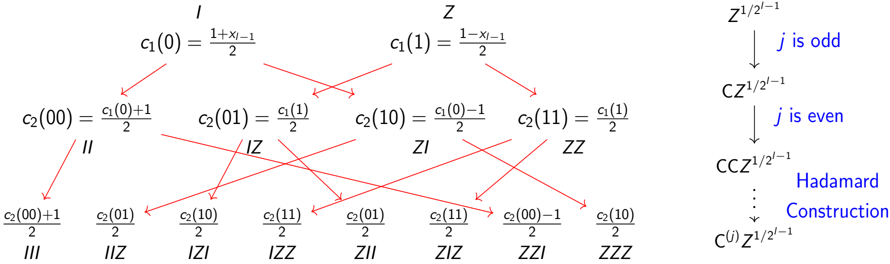
$$x_l = x_{l+1}x_{l+1} = x_{l+1}(c_{l+1} - s_{l+1})$$

$$I \qquad x_l c_l \;\to\; x_{l+1} c_l c_{l+1} \;(II), \quad -x_{l+1} c_l s_{l+1} \;(IZ)$$

$$Z \qquad x_l s_l \;\to\; x_{l+1} s_l c_{l+1} \;(ZI), \quad -x_{l+1} s_l s_{l+1} \;(ZZ)$$

$$C^{(j-1)}Z^{1/2^{l-1}} = \text{diag}[\boldsymbol{d_j}] \text{ with } \boldsymbol{d_j} = [1_{2^{j-1}}, 1_{2^{j-1}-1}, x_{l-1}]^T \ \& \ x_l = e^{\imath\pi/2^{l-1}}$$

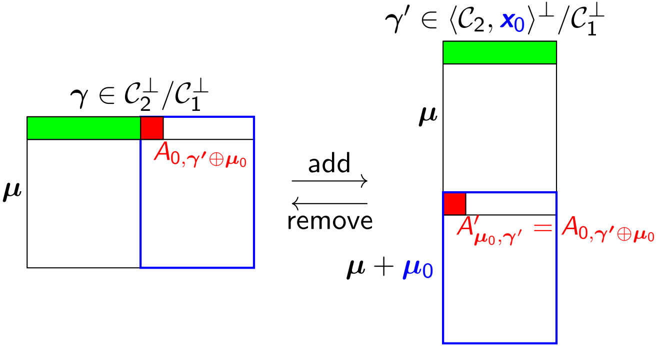
$$\boldsymbol{d_j} \to \boldsymbol{c_j} = H_{2^j}\boldsymbol{d_j} \to \boldsymbol{c_{j+1}} = \frac{1}{2}\begin{bmatrix} H_{2^j} & H_{2^j} \\ H_{2^j} & -H_{2^j} \end{bmatrix}\begin{bmatrix} 1_{2^j} \\ \boldsymbol{d_j} \end{bmatrix}$$



Recursion for the coefficients $c_j(\boldsymbol{v})$ comes from the recursive construction for the Walsh-Hadamard matrix

Adding an X-stabilizer Permutes Generator Coefficients

Adding an X-stabilizer $\boldsymbol{x}_0$ transforms the Z-logical $\boldsymbol{\mu}_0$ to a X-syndrome



Introducing $\boldsymbol{x}_0$: doubles the number of X-syndromes and halves the number of Z-logicals

The blue rectangle shifts as the generator coefficients evolve

U_Z: a fixed diagonal physical gate

Assume U_Z preserves the original CSS code: $\sum_{\gamma \in \mathcal{C}_2^\perp / \mathcal{C}_1^\perp} |A_{0,\gamma}|^2 = 1$

Admissible Addition: U_Z preserves the CSS$(X, \langle \mathcal{C}_2, \boldsymbol{x}_0 \rangle; Z, \mathcal{C}_1^\perp, \boldsymbol{y})$ code obtained by adding the X-stabilizer $\boldsymbol{x}_0$

Theorem: Addition of $\boldsymbol{x}_0$ is admissible if and only if

$$A_{0,\gamma} = 0 \text{ for all } \boldsymbol{\gamma} \in D + \boldsymbol{\mu}_0, \text{ where } \mathcal{C}_2^\perp / \mathcal{C}_1^\perp = \langle D, \boldsymbol{\mu}_0 \rangle$$

Admissibility requires that half the generator coefficients $A_{0,\gamma}$ vanish

Quantum Reed-Muller (QRM) Code Family

Physical levels:	m/r $\longrightarrow$	$(m+r)/r = m/r + 1$	$m/r + 1$
Logical levels:	m/r	m/r $\longrightarrow$	$m/r + 1$

$\mathbb{F}_2^{2^m}$	$\mathbb{F}_2^{2^{m+r}}$	$\mathbb{F}_2^{2^{m+r}}$
$\mathcal{C}_1 = \mathrm{RM}(r, m)$	$\mathcal{C}_1' = 1_{2^r} \otimes \mathrm{RM}(r, m)$	$\mathcal{C}_1'' = \mathrm{RM}(r, m+r)$
$\mathcal{C}_2 = \mathrm{RM}(r-1, m)$	$\mathcal{C}_2' = 1_{2^r} \otimes \mathrm{RM}(r-1, m)$	$\mathcal{C}_2'' = \mathrm{RM}(r-1, m+r)$
$\{0\}$	$\{0\}$	$\{0\}$
$[\![2^m, \binom{m}{r}, 2^{\min\{r,m-r\}}]\!]$	$[\![2^{m+r}, \binom{m}{r}, 2^{\min\{r,m-r\}}]\!]$	$[\![2^{m+r}, \binom{m+r}{r}, 2^{\min\{r,m\}}]\!]$

Concatenating r times (first code $\rightarrow$ second code)

removing Z-stabilizers
adding X-stabilizers (second code $\rightarrow$ third code)

- Prior work focuses on CSS code states and derives sufficient conditions for a CSS code to be fixed by a transversal Z-rotation. The generator coefficient framework provides necessary and sufficient conditions. It has the advantage of tracking the logical operator induced by a physical diagonal gate.
- Generator coefficients support synthesis of a target logical diagonal operator by combining three basic operations.
    1. Concatenation: increases physical level
    2. Removal of Z-stabilizers: increases logical level and code rate
    3. Addition of X-stabilizers: increases the distance
- When coherent noise dominates, Pauli X matrices can be used to switch between computation and storage of intermediate results in a decoherence-free subspace