# Transition operators on trees, boundaries and spectra

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# Abstract

Since infinite homogeneous trees are discrete analogues of the hyperbolic disc, these trees are a natural environment for studying free group actions and also spectra of transition operators. We outline their introduction in harmonic analysis, discrete potential theory and random walks, and review old and new results on strictly related subjects: the spectrum of the Laplace operator on a homogeneous tree, uniformly bounded representations of free groups, boundary behaviour of harmonic functions, nearest neighbour and finite step transition operators, and the Poisson and Martin boundaries.

A great amount of deep progress on representations of free groups and on spectra of transition operators on groups and graphs has followed these preliminary steps in the course of the years: substantial results have been obtained by K. Aomoto, T. Steger, S. Lalley, W. Woess, V. Kaimanovich, L. Saloff-Coste, Th. Coulhon, N.Th. Varopoulos, T. Nagnibeda, Cohen, Colonna & Singman and many others, not to mention the deep theory of Gromov and hyperbolic graphs, but all these results are beyond the scope of this presentation.

# Random walks for groups acting on the disc

In an enlightening paper of 1971, H. Furstenberg $I-[1]$ studied random walks on $SL(2,\mathbb{R})$ and on some of its discrete subgroups, typically $SL(2,\mathbb{Z})$, in order to describe their Poisson boundary.

- As a discrete subgroup of $SL(2,\mathbb{R})$, $SL(2,\mathbb{Z})$ acts on the disc: it maps the origin to a discrete subset.
- Consider a transient random walk on $SL(2,\mathbb{Z})$: its Poisson boundary is the circle.
- $SL(2,\mathbb{Z}) \approx \mathbb{Z}_2 * \mathbb{Z}_3$ (or better, $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$, but we can pass to the quotient $SL(2,\mathbb{R})/\{\pm\mathbb{I}\}$). So $SL(2,\mathbb{Z})$ contains a free group with 2 generators as a subgroup of finite index.

- In particular, there's a fundamental domain $D$ for $SL(2,\mathbb{Z})$, a tile that $SL(2,\mathbb{Z})$ moves around to cover the whole disc.
  $D \sim SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ is not compact, but has finite hyperbolic volume.
- In this tiling, the orbit of the origin under $SL(2,\mathbb{Z})$ is a sort of hyperbolic lattice: it is a semi-homogeneous tree $T$ of homogeneities 2 and 3.
- Furstenberg showed that the embedding is such that every harmonic function on $D$ restricts to a function on this tree that is harmonic with respect to a suitable transition operator $P$ on it.
- This transition operator is not known explicitly and is not finitely supported.
- The Poisson boundary of the tree with respect to $P$ is the circle; the circle is also the Poisson boundary of $SL(2,\mathbb{R})$ acting on $D$, but a countable number of its points are reached in a two-fold ways with random walks moving along $T$ along disjoint *ends*.

- Lyons and Sullivan $I-[3]$ and Kaimanovich $I-[4]$ gave conditions under which the space of bounded harmonic functions of a Riemannian manifold $M$ is isomorphic to the space of harmonic functions of a Markov chain on a discrete net $T$ arising from a discretization of $(M, T)$.
- The Martin boundary for finite range random walks on Fuchsian groups was studied by C. Series $VII-[2]$, and later generalised by A. Ancona $VII-[3]$; on free groups, it was studied by Y. Derriennic many years before ($VII-[1]$; see later) and it turned out to be again the set of ends.
- The free group $\mathbb{F}$ can be realized as a lattice in $SL(2, \mathbb{R})$: specializing attention to this lattice, Ballmann and Ledrappier [5] produced a random walk on $\mathbb{F}$ whose boundary is the circle. The transition operator is not known explicitly.

- Cohen and Colonna $I-[6]$ have found embeddings of a homogeneous tree $T$ of even valency in the disc $D$ such that the associated free group acts upon $T$ as a set of automorphisms of the hyperbolic disc: if the distance between two neighbours is large enough, then the combinatorial boundary of $T$ (its set of ends) is homeomorphic to the circle, but a countable number of boundary points are reached in a two-fold way, and there is a transition operator on $T$ (not explicitly known but not finitely supported) such that its harmonic functions are the restrictions of harmonic functions on $D$.
- Kaimanovich and Woess ($IX-[6]$; see later) proved that, under reasonable assumptions, the Martin boundary of transition operators with unbounded jumps is still the set of ends: this greatly generalises many of the previous results.
- Denker and Sato $VIII-[2, 3]$ have constructed (non-irreducible) random walks on a graph whose Martin boundary is the Sierpinsky fractal. See also Imai $IX-[4]$ and its references for generalizations to other fractals.

# Trees as simplicial complexes

Instead of considering trees as subset of the disc, Figà-Talamanca and Picardello *III*−[3, 4] considered them intrinsically, as simplicial complexes: that is, instead of considering free groups as lattices in $SL(2, \mathbb{R})$, they considered free groups combinatorically, as the groups of all non-reduced words in an alphabet with finitely many letters and their inverses. Of course these trees are homogeneous: all vertices have the same valency $\mathfrak{h} = q + 1$. In this way, without any embedding, they developed (by means of the Poisson boundary and Poisson kernels on these trees) a theory of unitary representations of free groups that is dramatically similar to that for $SL(2, \mathbb{R})$ acting on the disc, as we shall now outline.
An important step in this direction had previously been taken by P. Cartier *II*−[3, 4, 5].

# Notation for trees

## Definition

We now consider an infinite tree $T$. We shall also write $T$ for its set of vertices.

Natural distance $d(x, y)$: length of the direct path from $x$ to $y$.
Write $x \sim y$ if $x$, $y$ are neighbors: distance 1.
We fix a reference vertex $o \in T$ and call it the origin. The choice of $o$ induces a partial ordering in $T$: $x \leqslant y$ if $x$ belongs to the geodesic from $o$ to $y$. Length: $|x| = d(o, x)$.

For any vertex $x$ and $k \leqslant |x|$, $x_k$ is the vertex of length $k$ in the geodesic $[o, x]$.

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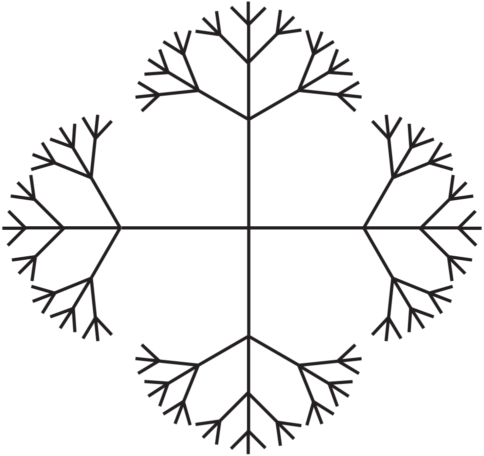

# The boundary of a tree and its topology

## Definition (Boundary)

Let $\Omega$ be the set of infinite geodesics starting at $o$ (*ends*). $\omega_n$ is the vertex of length $n$ in the geodesic $\omega$. For $x \in T$ the interval $I(x) \subset \Omega$, generated by $x$, is the set $I(x) = \left\{\omega \in \Omega : x = \omega_{|x|}\right\}$. The sets $I(\omega_n)$, $n \in \mathbb{N}$, form an open basis at $\omega \in \Omega$. Equipped with this topology $\Omega$ is compact and totally disconnected.

For every vertex $x$, the set of vertices $v > x$ form the sector $S(x)$. Call closed sector the set $\overline{S(x)} := S(x) \cup I(x) \subset T \cup \Omega$. The closed sectors induce on $T \cup \Omega$ a compact topology.

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# The simple transition operator and harmonic measure on homogeneous trees

On the vertices of $T$ we consider the adjacency operator $A$ and its normalization, the isotropic nearest neighbor transition operator $P = \frac{1}{\mathfrak{h}} A$.

Transition probabilities $p(u, v) = 1/\mathfrak{h} = 1/(q+1)$ if $u \sim v$

Exponential growth $\Rightarrow$ probability of moving forwards larger than probability of moving backwards $\Rightarrow$ transience.

The sets $\{I(x) : |x| = n\}$ generate a $\sigma-$algebra on $\Omega$. On this $\sigma-$algebra:

**Definition (Harmonic measure on homogeneous trees)**

$$\begin{aligned}\nu(I(x)) \equiv \nu_o(I(x)) &:= \frac{1}{\text{number of vertices of length } |x|} \\ &= \frac{1}{\mathfrak{h}(\mathfrak{h}-1)^{|x|-1}} = \frac{1}{(q+1)q^{|x|-1}} = \Pr[X_\infty \in I(x)]\,.\end{aligned}$$

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# Laplacian and harmonic functions

**Definition (Laplace operator)**

The Laplace operator is $\triangle = P - \mathbb{I}$.

**Definition (Harmonic functions)**

$f : T \to \mathbb{R}$ is harmonic at $x \in T$ if $\triangle f = 0$ that is, if
$Pf(x) \equiv \sum_{y \sim x} p(x, y)\, f(y) =$ Average of $f$ on neighbors $= f(x)$.

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# In which sense are homogeneous trees analogous to discs?

Compactify $T$ with this boundary $\Omega$, equipped with the nested $\sigma-$algebras induced on $\Omega$ by intervals spanned by vertices of fixed length. Then we have that

$T \cup \Omega$ is a discrete analog of the closed hyperbolic disc.

The natural boundary measure (harmonic measure) and the notion of distance that these intervals generate on $\Omega$ correspond to Lebesgue measure and arc distance on the unit circle. The group of automorphisms $\mathcal{G} = \mathrm{Aut}(T)$ has a subgroup isomorphic with a free group $\mathbb{F}$: namely, if $\mathcal{K}$ is the stabilizer of a reference vertex $o$, then $\mathbb{F} \equiv /\mathcal{G}\mathcal{K}$.

Then eigenspaces of the Laplace operator (that is invariant under automorphisms) are representation spaces of $\mathcal{G}$. In this way, unitary and uniformly bounded representations of $\mathcal{F}$ can be parameterized by the eigenvalues of the Laplace operator, and give rise to a representation theory with extraordinary similarity with rank one semisimple Lie groups $III-$[4]. In order to have a non-trivial automorphism group, the tree must be homogeneous or semi-homogeneous (with two alternating valences). We shall not devote attention to the semi-homogeneous trees here.

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# Poisson and Martin boundaries of a transition operator

Let us now consider a general space $X$ (in most of the following it will be a graph or tree).

## Terminology

*Given a stochastic transition operator on a space $X$, The Martin compactification $\widetilde{X}$ is the minimal compactifiction to which all Poisson kernels extend continuously. All positive harmonic functions are Poisson integrals of positive Borel measures on the Martin boundary $M = \widetilde{X} \setminus X$. The Poisson boundary is the subset of $M$ given by the support of the representing measure of the constants. So all bounded harmonic functions can be represented as Poisson integrals of bounded functions on the Poisson boundary, and all harmonic functions with $L^1$ boundary limits are reconstructed as Poisson integrals of finite boundary measures.*

As we shall see, for a tree, and for many graphs but not all, these two boundaries coincide with the space of **ends**: i.e., the classes of equivalence of rays that remain connected after removal of arbitrarily large sets of vertices.

# Green kernel for general trees

On any tree, homogeneous or not, consider a stochastic transition operator $P$, that is, $\sum_{v \in T} p(u,v) = 1$ for every vertex $u$. For the moment, and for most of this presentation, $P$ will be nearest neighbour, that is $p(u,v) = 0$ if $u \not\sim v$.

## Definition (Green kernel)

The Green kernel $G(u,v)$ is the expected number of visits to $v$ of the random walk starting at $u$:

$$G(u,v) = \sum_{n=0}^{\infty} P^n(u,v)\,. \tag{1}$$

It satisfies $PG = G - \mathbb{I}$, that is, $\sum_y p(x,y)G(y,z) = G(x,z) - \delta_x(z)$.

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# Harmonic measure

Given two positive functions $f$ and $g$, we write $f \approx g$ if $f < Cg$ and $g < Cf$ for some constant $C$.
The harmonic measure $\nu$ (the representing measure of the constants) verifies:

**Proposition**

*For $x \in T$*

$$\nu(I(U(x)) \approx G(o,x)\,.$$

# Poisson kernel

## Definition (Poisson kernel)

For every $x$, $v \in T$ the Martin kernel $K(x, v)$ s defined as

$$K(x, v) \equiv \frac{G(x, v)}{G(o, v)} \approx \frac{d\nu_x}{d\nu_o}(I(v)).$$

For every $x \in T$, $\omega \in \Omega$ the Poisson kernel $K(x, \omega)$ is defined as

$$K(x, \omega) \equiv \lim_{n\to\infty} \frac{G(x, \omega_n)}{G(o, \omega_n)}.$$

For every $\omega \in \Omega$, $K(\cdot, \omega)$ is harmonic on $T$, and integral over $\Omega$ with integral 1. It is an approximate identity on $\Omega$:

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# Poisson integral representation

**Proposition (Explicit expression of Poisson kernel)**

*If $v = v_o(x, \omega)$ join of rays from $o$ to $\omega$ and from $x$ to $\omega$, then*

- *horocycle index $h(x, \omega) :=$ number of edges from $o$ to $v$ minus number of edges from $x$ to $v$;*
- $K(x, \omega)(= K_o(x, \omega)) = q^{h(x,\omega)}$ .

The Poisson integral of a function $h$ in $L^1(\Omega)$ is defined as

$$\mathbf{K}h(x) = \int_{\Omega} h(\omega) K(x, \omega) d\nu \,.$$

# Radial functions on free groups and **homogeneous** trees

## Definition

$G = \mathbb{F}_r$ free group with $r$ generators: its Cayley graph is the homogeneous tree $T$ with valency $q+1 = 2r$. *Radial functions*: functions on $G$ that depend only on the length $|w|$ of the words $w$, that is on the distance from $w$ to the identity vertex $e \in T$. The convolution algebra $\mathcal{R}$ of finitely supported radial function is maximal abelian.
For functions $f, g$:

$$\langle f, g \rangle := \sum_v f(v)\, g(v)\,.$$

Let $\mu_n$ be the equidistributed probability on words of length $n$.
Then: *radialization* $\mathcal{E}$ given by

$$\mathcal{E} f(v) = \langle f,\, \mu_{|v|} \rangle\,.$$

# Radial convolutions and the Poisson kernel on a homogeneous tree

- Look at: a free group $\mathbb{F}_r$, that is, a homogeneous tree $T_q$ with forward growth $q = 2r - 1$, (valency $q + 1 = 2r$)
- isotropic nearest neighbour transition operator $p(u, v) = \frac{1}{2r} = \frac{1}{q+1}$ if $u \sim v$. T

That is, $P(u, v) = \mu_1(v^{-1}u)$, or more precisely, *regarded as a convolution operator on* $\mathcal{R}$*,* $P$ *is given by convolution by the function* $\mu_1$.

**Lemma**

*For* $n > 1$, $\mu_1 * \mu_n = \mu_{n-1} + (q - 1)\, \mu_{n+1}$.

**Corollary**

*The radial algebra* $\mathcal{R}$ *is generated by* $\mu_1$.

- The Martin kernel becomes

$$K(v, w) = q^{2N(v,w)-|v|}$$

where $N(v, w)$ is the number of common vertices in the paths from $e$ to $v$ and $w$, respectively.

- If the vertex $w$ moves to infinity without returns backwards, it follows a geodesic $\omega$, that is, converges to the endpoint $\omega \in \Omega$. Then $K(v, w)$ converges pointwise to the Poisson kernel $K(v, \omega)$, because $K(v, w)$ is actually locally constant: it is constant on all vertices $w$ in the sector past the bifurcation point of the paths $v$ and $w$.
- We denote again by $N(v, \omega)$ the number of common vertices between the finite path $e \to v$ and the infinite geodesic $e \to \omega$. Then the Poisson kernel is

$$K(v, \omega) = q^{2N(v,\omega)-|v|}.$$

# Spherical functions on free groups

## Definition

A function $\phi$ on $\mathbb{F}_r \approx T$ is *spherical* if it is radial and is a joint convolution eigenvector of all functions in $\mathcal{R}$, normalized by $\phi(e) = 1$.

## Lemma

1. *A non-zero function $\phi$ is spherical if and only if it is radial and the functional $Lf = \langle f, \phi \rangle$ is multiplicative on $\mathcal{R}$.*
2. *As a consequence, spherical functions, regarded as functions of the length of words, that is on $\mathbb{N}$, must satisfy the following difference equation of the second order:*

$$\phi(n-1) + q\,\phi(n+1) = (q+1)\,\phi(1)\,\phi(n)\,,$$

   *i.e., they are uniquely determined by their value on words of length 1.*

3. *As a consequence, the eigenvalue of the multiplicative functional associated to the spherical function $\phi$ is $\phi(1)$, and every radial convolution eigenfunction is a multiple of a spherical function.*

# Poisson representation of spherical functions

## Remark

*The free group acts (by automorphisms) on $T$, hence also on $\Omega$, hence on measures on $\Omega$. So $x \in \mathbb{F}$ maps the harmonic measure $\nu$ to another boundary measure $\nu_x$, and $K(v,\omega)$ is precisely the Radon–Nikodim derivative $d\nu_x/d\nu$.*

## Theorem

$\forall z \in \mathbb{C}$, *let*

$$\phi_z(v) = \int_\Omega K^z(v,\omega)\, d\nu(\omega)\,.$$

*Then $\phi_z$ is the spherical function with eigenvalue $\gamma(z) = \phi_z(1) := \frac{q^z+q^{1-z}}{q+1}$. Conversely, every spherical function arises this way, and $\phi_z(v) = \mathcal{E} K^z(v,\cdot)$.*

Now, by computing $\mathcal{E}K^z$, we obtain a decomposition of spherical functions as sums of pure exponentials that on $SL(2,\mathbb{R})$ would only be an asymptotic expansion:

**Corollary**

*$\forall z \in \mathbb{C}$, let*

$$h_z(v) = q^{-z|v|}.$$

*Then, if $q^{2z-1} \neq 1$,*

$$\phi_z = c(z)h_z + c(1-z)h_{1-z}\,,$$

*while, if $q^{2z-1} = 1$,*

$$\phi_z(v) = \left(1 + \frac{q-1}{q+1}\,|v|\right) h_z(v)\,.$$

# $\ell^p$ behaviour of spherical functions

## Corollary

*The spherical function $\phi_z$ is bounded if and only if $0 \leqslant \operatorname{Re} z \leqslant 1$, that in turn corresponds to the condition*

$$\operatorname{Re}\gamma(z)^2 + \frac{q-1}{q+1}\operatorname{Im}\gamma(z)^2 \leqslant 1\,.$$

*For $2 < p < \infty$, $\phi_z \in \ell^p(T)$ if and only if $\frac{1}{p} < \operatorname{Re} z < 1 - \frac{1}{p} = \frac{1}{q}$. In particular, $\phi_z$ does never belong to $\ell^2$.*

## Remark

*$\mathbb{F}$ is a discrete subgroup of the automorphism group $\mathcal{G}$ of $T$. If $\mathcal{K}$ is the (left) stabilizer in $\mathcal{G}$ of the reference vertex $e$, then $\mathcal{G} = \mathbb{F}\,\mathcal{K}$ and $\mathbb{F} \sim \mathcal{G}/\mathcal{K}$. So, spherical functions lift to functions of $\mathcal{G}$ that are right invariant under $\mathcal{K}$. The theory of spherical functions that we have outlined is taken from III−[3, 4], but for the lifting to $\mathcal{G}$ it was previously known: it amounts to the statement that $(\mathcal{G}, \mathcal{K})$ is a Gelfand pair III−[5].*

# The Gelfand spectrum of the Banach algebra $\ell^1_\sharp$

Denote by $\ell^1_\sharp$ the $\ell^1$ completion of the radial algebra $\mathcal{R}$: it is a commutative Banach algebra with identity, generated by $P = \mu_1$. So:

**Remark**

*The spectrum of $P = \mu_1$ on $\ell^1$ is the Gelfand spectrum of $\ell^1_\sharp$ (= space of continuous multiplicative functionals) = set of eigenvalues of those spherical functions that are continuous functionals on $\ell^1_\sharp$ = set of eigenvalues of bounded spherical functions.*

The next theorem on the spectrum of a homogeneous tree is taken from *III*−[3, 4]. Many previous results were known, although not elegantly formulated in terms of spherical functions: see *II*−[5], *III*−[6, 1, 5], *IV*−[2]. Some results were already known to P. Gerl and W. Woess: see the later paper *IV*−[5]. Another elegant proof in *IV*−[4].
Via the spectrum and spherical functions, Figà-Talamanca and Picardello built a powerful theory of unitary representations of $\mathbb{F}$ *III*−[3, 4].

# The spectrum of radial functions (and homogeneous trees)

A previous theory, not based on spherical functions and not as far-reaching, had been obtained in $III-[2]$; later results are in $III-[3]$.

**Theorem**

1. *Spectrum of $\mu_1$ in $\ell^1$ = image of the eigenvalue map $\gamma$: ellipse* $\mathrm{Re}\,\gamma(z)^2 + \frac{q-1}{q+1}\,\mathrm{Im}\,\gamma(z)^2 \leqslant 1 = \gamma(\{z : 0 \leqslant \mathrm{Re}\, z \leqslant 1\})$.
2. *For $1 < p \leqslant 2$, $q = 1 - \frac{1}{p}$, spectrum of $\mu_1$ in $\ell^p$ = spectrum of $\mu_1$ in $\ell^q$ = sub-ellipse $\gamma\{z : \frac{1}{p} \leqslant \mathrm{Re}\, z \leqslant \frac{1}{q}\|$. For $p = 2$, this is the real segment $[-\rho, \rho]$, where rho $= \gamma(1/2) = \sqrt{2r-1}r = \frac{2\sqrt{q}}{q+1}$.*
3. *Spectrum of $\mu_1$ in the full group $C^*-$algebra $= [-1, 1]$.*
4. *Spectral measure $m$ of $\mu_1$ on $\ell^2$: absolutely continuous with respect to Lebesgue measure, and precisely, on the interval $\{\frac{1}{2} + it : 0 \leqslant t \leqslant \pi / \ln q\} = \gamma^{-1}([-rho, \rho])$, one has $dm(t) = |c(\frac{1}{2} + it)|^{-2} dt$ up to normalization.*

We also note that similar results were originally proved by Kesten ($II-$[1], 1959; see also $II-$[2], 1961), and many other authors, including Gerl and Woess, and Cohen $II-$[7] and Cohen and de Michele $III-$[1]. The $\ell^p$-spectrum of an *non-isotropic* nearest neighbour teansition operator on homogeneous trees has been computed, for every $p$, by Figà-Talamanca and Steger $IV-$[7]; see also Aomoto $IV-$[6].

# Invariant transition operators with bounded jumps

Y. Derriennic (1975, *VII*−[1]) considers transition operators on trees that are invariant under the free group but not nearest neighbours: bounded jumps and characterizes the Martin boundary of this walk by means of a combinatorial-probabilistic approach. From the viewpoint of the allowed transitions, the tree becomes a homogeneous graph (call it the *augmented tree*): all loops have bounded range, and, seen from a distance, this augmented tree looks like the original tree (that is, the two graphs are quasi-isometric).

*Recall*: if $X_n$ is the random vertex at time $n$ of the random walk generated by the transition operator $P$,

$$F(u,v) := \Pr[\,\exists n \geqslant o : X_n = v | X_0 = u\,] \,.$$

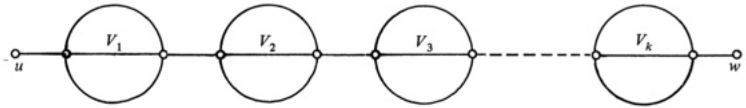
Strong Markov property:

$$G(u,v) = F(u,v)G(v,v)$$

and for every vertex $w$ intermediate between $u$ and $v$,

$$F(u,w)F(w,v) \leqslant F(u,v) \,.$$

The latter inequality is an identity if the walk is forced to visit $w$ while moving from $u$ to $v$, for instance for nearest neighbour operators. But the jumps are bounded, and so there are *intermediate sets* $W$ that must be visited with probability 1: each provides a stopping time $S_W$.




We now outline Derriennic's proof by means of a more general argument.

# Intermediate matrices

For every finite subset $V \subset T$, denote by $Q_V$ the substochastic transition operator

$$Q_V f(v) = \sum_{v \in V} \mathrm{Pr}_v[X_{S_V} = v] f(v)\,.$$

It is clear that, if $\mathrm{dist}(u, V)$ exceeds the maximum length of the jumps, then one cannot reach $v$ from $u$ in a single jump and so the random walk must visit the intermediate set $V$ with probability 1. So the Strong Markov Property yields that $Q_V f$ is harmonic at $u$, for every function $f$ on $T$, and

$$F(u, w) = Q_V F(u, w)\,.$$

Denote by $\{V_j,\, j = 1, \ldots, n\}$ a sequence of intermediate sets between $u$ and $v$, and by $v_k^{(i)}$ the vertices in $V_i$, by $F_1^u$ the vector $(F_1^u)_i = F(u, v_i^{(1)})$, and by $F_n^v$ the vector $(F_n^v)_l = F(v_l^{(n)}, v)$. Thanks to the multiplicativity of the $F$-kernel and the fact that the walk must hit every intermediate set $V_j$ from $u$ to $w$, for each pair of consecutive intermediate sets $V_j$, $V_{j+1}$ we have a stochastic renewal matrix $A_{V_j, V_{j+1}}$, defined by

$$A_{V_j,\, V_{j+1}}(w_1, w_2) = (Q_{V_{j+1}}\delta_{w_2})(w_1)$$

for $v_1 \in V_j, v_2 \in V_{j+1}$.
These matrices verify

$$F(u, w) = \langle F_0,\, A_1 A_2 \dots A_n F_n \rangle\,.$$

Since $K(u, w) = G(u, w)/G(e, w)$, this reduces the problem of the existence of radial limits $\lim_{v \to \omega} K(u, w)$ to the fact that stochastic matrices operate on the positive quadrant as projective contractions (towards the eigenvector with highest eigenvalue: Perron–Frobenius theorem). This characterizes the Martin boundary as the space of all ends of the augmented tree, that is clearly the same as the space of ends of the original tree,
The main tool: due to non-amenability of the free group, the Green function is *square-summable*.

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# Abandoning group invariance: transition operators on graphs

Picardello and Woess (TAMS 1987, $VIII-[1]$) characterized the Martin boundary of a tree as the space of edges when there is no group-invariance: on non-homogeneous trees.
The main tool of Derriennic, square-summability of the Green kernel $H$, is not available here. The authors were able to replace this group-theorethical tool with assumptions that relate naturally to the geometry of the trees and transition operators and are automatically satisfied in the group-invariant case (therefore, this argument is more general even for the group-invariant case):

- the transition operator has bounded jumps
- the transition probabilities are bounded below
- the time of first visit to any neighbour is bounded above

By making edges correspond to jumps of the transition operator $P$, the tree is replaced by an augmented graph, now not homogeneous: the last assumption is equivalent to the condition that this graph be quasi isometric with the tree.

## Theorem

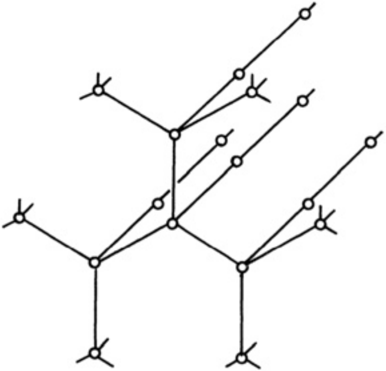
*For trees and transition operators as above,*

- *the Martin boundary is the set of ends of the augmented graph*
- *there is a Poisson-Martin representation theorem of positive harmonic functions as boundary integrals of Borel measures*
- *there is a Fatou' theorem of almost everywhere convergence to the boundary of positive harmonic functions*
- *more generally, all this is true for random walks on groups admitting a uniformly spanning tree.*

# Example of a pathological graph

## Example

The following figure shows a graph that, equipd with the simple adjacency transition operater, has Green kernel not in $\ell^2$, and Martin boundary smaller than the space of ends, and the transition operator is invariant under a non-transitive, non-amenable free product $(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2)$.

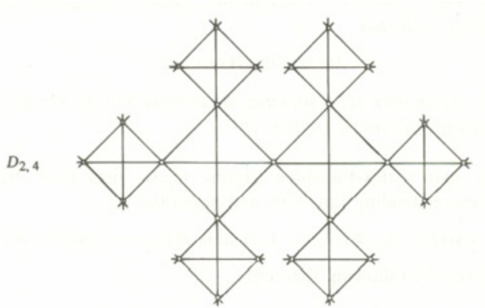


$\bar{T}$

# Other examples of spectra of graphs and groups

We present some examples from the survey paper *VIII*−[2] (Mohar and Woess, 1989).

- Free products and amalgamated products: vaguely similar to free groups, see [4, 5, 6].
- Distance-regular graphs: graphs that admit a function $f : \mathbb{N}^3 \to \mathbb{N}$ such that the set $\{w \in V(G) : \mathrm{dist}(u,w) = j,\, \mathrm{dist}(v,w) = k\}$ has cardinality $f(j,k,\mathrm{dist}(u,v))$ depending only on distances. They are tree-like graphs with a semi-homogeneous tree as spanning tree.



$\ell^2$-spectrum of the distance-regular graph $D_{m,s}$: the interval

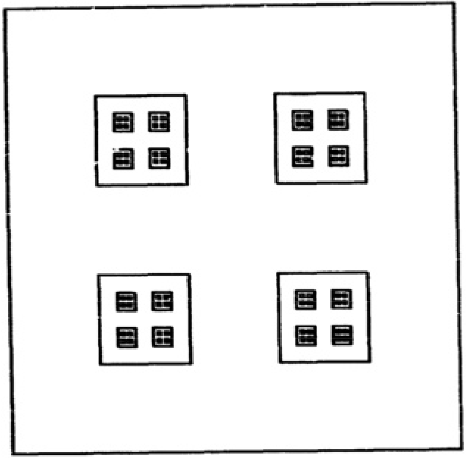
$$I_{m,s} = [s - 2 - 2\sqrt{(m-1)(s-1)},\, s - 2 + 2\sqrt{(m-1)(s-1)}]$$

if $m \geqslant s$, and $I_{m,s} \cup \{-m\}$ if $m < s$.

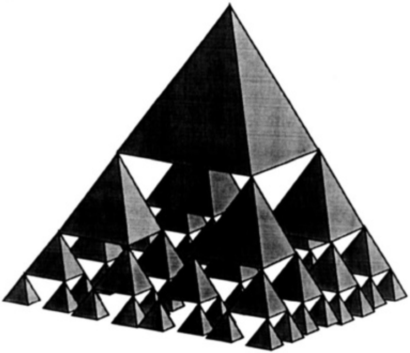
- Semi-homogeneous trees, symmetric graphs, amalgamated products: Cohen $V$–[1], Cohen and Trenholme $V$–[2], Iozzi and Picardello $V$–[3, 4], Faraut and Picardello $V$–[5], Iozzi $V$–[6], Picardello and Woess $VI$–[1]

# Harmonic measure of Brownian motion on a planar Cantor set

Planar Cantor set: recursively cutting away the first, middle and last fifth in each dimension.



The geometric boundaries of the squares form, generation after generation, a sequence of barriers that give rise to stopping times of the Brownian motion. Moving from each barrier to the next, or the previous, gives rise to a non-planar graph that admits a uniformly spanning tree:



The harmonic measure has Hausdorff dimension less than 1 but capacity 1 (Carleson; Makarov and Volberg). It can be studied by means of the generating functions of the random walk on the graph (Picardello, Taibleson and Woess, Discrete Math. 1992 [5]).

# Transition operators with unbounded jumps

Kaimanovich and Woess $IX-[6]$ consider random walks on infinite graphs which are not necessarily invariant under some transitive group action and whose transition probabilities may have infinite range. Assumptions:

- the underlying graph $G$ satisfies a strong isoperimetric inequality
- the transition operator $P$ is strongly reversible, uniformly irreducible and satisfies a uniform first moment condition.

### Theorem

*Under these hypotheses the random walk converges almost surely to a random end of $G$ and that the Dirichlet problem for $P$-harmonic functions is solvable with respect to the end compactification. If in addition the graph as a metric space is hyperbolic in the sense of Gromov, then the same conclusions also hold for the hyperbolic compactification in the place of the end compactification.*

Their main tool is the exponential decay of the transition probabilities implied by the strong isoperimetric inequality. They also apply the same technique to Brownian motion to obtain analogous results for Riemannian manifolds satisfying Cheeger's isoperimetric inequality. In particular, in this general context they give new (and simpler) proofs of well known results on the Dirichlet problem for negatively curved manifolds.

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**Conclusion: the Poisson boundary of stochastic transition operators on most trees and graphs is the space of ends.**

# Minimal Martin boundary of a cartesian product

The minimal Martin boundary of the hyperbolic bi-disk was computed in groundbreaking papers of Karpelevic (1965, $XI-$[1]) and Guivarc'h (1984, $XI-$[4]): it is the smallest boundary that reproduces all minimal positive harmonic functions with respect to the Laplace–Beltrami operator on the bi-disk (and, for other eigenvalues $t$, the minimal positive $t$-eigenfunctions: it may change with $t$). See also $XI-$[6] (Bull. London Math. Soc. 1990) and $XI-$[7] (J. Diff. Geom. 1991) for the cartesian product of Riemannian manifolds.

The minimal Martin boundary of the cartesian product of two homogeneous trees with isotropic transition operators has been computed by Picardello and Sjögreen ($XII-$[5], Canberra 1988), and extended to cartesian products of Markov chains (for instance, non-homogeneous trees) by Picardello and Woess ($XII-$[2], Nagoya Math.J. 1992). Here is its characterization.

Let $P$, $Q$ be stochastic transition operators on discrete spaces $X$, $Y$ respectively.
On $Z = X \times Y$, for $0 < \alpha < 1$, let $R_\alpha := \alpha P + (1 - \alpha)Q$. Then

**Theorem**

*(i)* *Minimal t-harmonic functions for $R_\alpha$ on $Z$ split as products of minimal r-harmonic functions for $P$ on $X$ and minimal s-harmonic for $Q$ on $Y$ with $\alpha r + (1 - \alpha)s = t$.*

*(ii)* *Conversely, all these products give rise to minimal t-harmonic functions for $R_\alpha$ on $Z$.*

This theorem mixes boundaries for varying eigenvalues: in most cases the boundaries are stable under this change.
Part $(i)$ was proved differently by Molchanov ($XI-$[2], Th.Prob.Appl. 1967; $XI-$[3], Siber.J. 1970) for the *direct product* $P \otimes Q$: our result immediately implies his, but the converse is not true. Besides, the simple random walk on the product does not arise as a tensor product but as a cartesian product. Conclusion: minimal Martin boundary of product = product of the boundaries; hence same for Poisson boundary.

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# Full Martin boundary of a cartesian product

The problem of describing the full Martin boundary of products of discs (that reproduces all positive harmonic functions) has been raised by Guivarc'h and Taylor ($XII-$[1], Coll.Math. 1990), where the description is obtained only for the minimal eigenvalue (the bottom of the spectrum of the Laplacian).

Some hints of a full description for products of homogeneous trees are in Picardello and Sjögren ($XIII-$[2], J.Reine Angew.Math. 1992): they describe the asymptotic directions along which the Poisson -Martin kernels have limits at infinity.

The general solution for the product of two homogeneous trees $T_1$ and $T_2$ with boundaries $\Omega_1$ and $\Omega_2$ is given by Picardello and Woess ($XIII-$[3], Ann.Prob. 1994), as follows.

**Theorem**

*Full Martin boundary:*
$\Omega_1 \times \Omega_2 \times \{0 < t < 1\}) \quad \cup \quad \Omega_1 \times T_2 \quad \cup \quad T_1 \times \Omega_2$, *with the natural topologies.*

In particular, the full Martin boundary is much larger than the minimal one, and the latter is not dense. The parameter $t \in (0, 1)$ is the *asymptotic escape angle*: it measures the ratio of speeds of escape in the two components.

*Main tool: a beautiful uniform estimate of powers of the transition operator on a tree ("local limit theorem") given by Lalley (XII−[1], J.Theor.Prob. 1991).*

The same result was later adapted to the product of hyperbolic spaces by Giulini and Woess.

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