Commutative harmonic analysis on noncommutative Lie groups

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Something very old

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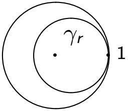
The following is a direct consequence of the Wiener Tauberian Theorem on the real line:

Theorem

Let f be a bounded holomorphic function on the unit disc Δ. For $0 < r < 1$, let $\gamma_r \subset \Delta$ be the circle of radius r tangent to $\partial\Delta$ at 1. If, for some $r \in (0,1)$,

$$\lim_{\substack{z \to 1 \\ z \in \gamma_r}} f(z) = \ell \ ,$$

then the same holds true for every other r.



Proof

Via the Cayley transform $C : z \longmapsto i\frac{1+z}{1-z} = w$, the disc Δ is replaced by the upper half plane $U = \{w = x + iy : y > 0\}$ and γ_r by the horizontal line $y = \frac{1-r}{r}$.

The function $g = f \circ C^{-1}$ is bounded and holomorphic on U. By Fatou's theorem, g is the Poisson integral of a bounded function g_0 on $\partial U = \mathbb{R}$,

$$g(x+iy) = g_0 * p_y(x) \ , \qquad p_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2} \ .$$

The hypothesis implies that, for a fixed $y > 0$, $\lim_{x\to\infty} g_0 * p_y(x) = \ell$. Since $\widehat{p_y}(\xi) = e^{-y|\xi|} \neq 0$ and $\int_{\mathbb{R}} p_{y'} = 1$ for every $y' > 0$, the WTT gives $\lim_{x\to\infty} g_0 * p_{y'}(x) = \ell$ for all y'.

An analogue for the ball in $\mathbb{C}^n$

The same argument can be adapted to the unit ball in $\mathbb{C}^n$, $n \geq 2$ (A. Hulanicki, F. R., Adv. Math. 1980) proceeding as follows:

- interpret the γ_r (resp. the horizontal lines in U) as the *horocycles* pointed at $1 \in \partial\Delta$ (resp. at $\infty \in \partial U$) in the Poincaré metric;

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- transform the ball B_n into the Siegel domain $U_n = \left\{ w = (w_1, w') \in \mathbb{C} \times \mathbb{C}^{n-1} : \operatorname{Im} w_1 - |w'|^2 > 0 \right\}$ via a generalized Cayley transform C;

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- recognize that the horocycles in U_n pointed at infinity are the vertical translates of ∂U_n;
- use the n-dimensional Fatou theorem to express $g = f \circ C^{-1}$ on the hypersurface $\operatorname{Im} w_1 - |w'|^2 = y > 0$ as a convolution $g_0 * p_y^{(n)}$, with $g_0 \in L^\infty(H_{n-1})$ and $p_y^{(n)}$ the generalized Poisson kernels.

Commutativity of L^1

The Wiener Tauberian Theorem (for a locally compact abelian group G) is the statement that $L^1(G)$ has the *Wiener property*, i.e., a closed ideal containing a function φ with $\hat{\varphi} \neq 0$ at all points of $\widehat{G}$ is all of $L^1(G)$.
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The proof in higher dimension makes use of a "rotation invariance" property of the Poisson kernels $p_y^{(n)}$.
This property identifies a closed subalgebra of $L^1(H_{n-1})$ which is commutative and satisfies the Wiener property (with $\widehat{G}$ replaced by its Gelfand spectrum).

z-radial functions on H_d

The Heisenberg group H_d is $\mathbb{C}^d \times \mathbb{R}$ with product

$$(z,t)\cdot(w,u) = \big(z+w, t+u+2\,\mathrm{Im}\langle z,w\rangle\big)\ .$$

Convolution of two functions f, g is defined as

$$f * g(z,t) = \int_{H_d} f\big((z,t)\cdot(w,u)^{-1}\big) g(w,u)\, dw\, du\ .$$

A function f is *z-radial* if it depends on $|z|, t$ only (i.e., it is invariant under unitary transformations in the $\mathbb{C}^d$-component).

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If f, g are both z-radial, then $f * g$ is also z-radial and $f * g = g * f$.

Due to this commutativity property, Fourier analysis of z-radial functions on H_d can be done using the *scalar-valued* Gelfand transform of $L^1_{z\text{-rad}}(H_d)$ rather than the *operator-valued* group Fourier transform.

Commutative pairs

The context described above is the simplest, but nontrivial, example of *commutative pair* [1].

If G is a locally compact group and K is a compact group of automorphisms of G, we say that (G, K) is a commutative pair if the convolution algebra $L^1_K(G)$ of K-invariant functions on G is commutative.

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The spectrum $\Sigma(G, K)$ of $L^1_K(G)$ consists of the *bounded spherical functions* φ, satisfying the equation

$$\varphi(x)\varphi(y) = \int_K \varphi\big(x \cdot (ky)\big)\, dk \ .$$

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- Inversion formula: $f(x) = \int_\Sigma \mathcal{G}f(\varphi)\,\varphi(x)\,d\nu(\varphi)$,
- Multipliers: every bounded operator T on $L^2(G)$ which commutes with left translations and with the automorphisms in K can be expressed as $Tf = \mathcal{G}^{-1}(m\,\mathcal{G}f)$, for a unique $m \in L^\infty(\Sigma,\nu)$.

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It is then possible to obtain interesting models of Σ as closed subsets of some Euclidean space.

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- φ *is* K*-invariant, bounded and* $\varphi(e) = 1$*;*
- φ *is an eigenfunction of every* $D \in \mathbb{D}_K(G)$.

Embeddings

The algebra $\mathbb{D}_K(G)$ is finitely generated. We can then choose a finite system of generators $\mathcal{D} = \{D_1, \dots, D_k\}$ and associate to every $\varphi \in \Sigma$ the k-tuple $\xi_\varphi = (\xi_1, \dots, \xi_k) \in \mathbb{C}^k$ of its eigenvalues,

$$D_j \varphi = \xi_j \varphi \ , \qquad j = 1, \dots, k \ .$$

Theorem (F. Ferrari-Ruffino)

The map $\varphi \longmapsto \xi_\varphi$ is a homeomorphism of Σ onto a closed subset $\Sigma_{\mathcal{D}}$ of $\mathbb{C}^k$.

Example 1 (Hankel transform)

$G = \mathbb{R}^n$, $K = O_n$.

Then $L^1_K(G) = L^1_{\text{rad}}(\mathbb{R}^n)$ and $\mathbb{D}_K(G) = \mathbb{C}[\Delta]$.

The bounded spherical functions are expressed in terms of Bessel functions:

$$\varphi_r(x) = \fint_{|\xi|=\sqrt{r}} e^{i\,\xi\cdot x}\, d\xi = c_n\big(\sqrt{r}|x|\big)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}\big(\sqrt{r}|x|\big)\ , \qquad r \geq 0\ .$$

With $\mathcal{D} = \{-\Delta\}$, we have $\Sigma_{\mathcal{D}} = [0, +\infty)$ and $\xi_{\varphi_r} = r$.

Example 2

$G = H_d$, $K = U_d$.

Then $L^1_K(G) = L^1_{z\text{-rad}}(H_d)$ and $\mathbb{D}_K(G) = \mathbb{C}[L, T]$, where $T = \partial_t$ and

$$L = \sum_{j=1}^{d} \left((\partial_{x_j} - y_j\partial_t)^2 + (\partial_{y_j} + x_j\partial_t)^2\right)$$

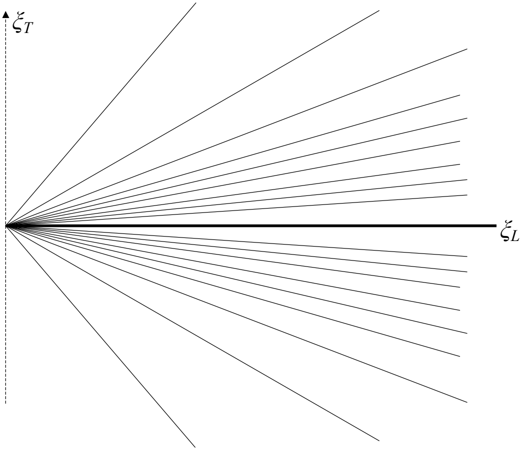
$$= \Delta_z + 2\partial_t \sum_{j=1}^{d} (x_j\partial_{y_j} - y_j\partial_{x_j}) + |z|^2\partial_t^2$$

Take $\mathcal{D} = \{-L, -iT\}$. For a given eigenvalue $\xi_2 = \lambda \in \mathbb{R}$ of $-iT$, the spherical functions are

$$\begin{aligned} &\text{if } \lambda = 0\,, \quad \varphi_{r,0}(z,t) &&= c_{2d}\left(\sqrt{r}|z|\right)^{-(d-1)} J_{d-1}\left(\sqrt{r}|z|\right)\,, \quad r \geq 0\,, \\ &\text{if } \lambda \neq 0\,, \quad \varphi_{|\lambda|(2j+d),\lambda}(z,t) &&= c'_{d,j} e^{i\lambda t} e^{-|\lambda||z|^2} L_j^{(d-1)}\left(2|\lambda||z^2|\right)\,, \quad j \in \mathbb{N}\,, \end{aligned}$$

where $L_j^{(d-1)}$ are the Laguerre polynomials.

Picture of $\Sigma_{\mathcal{D}}$ (the Heisenberg fan)



(Faraut-Harzallah, 1987)

Fourier transforms of Schwartz functions

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In his Ph.D. thesis of 1977 under the direction of E. Stein, D. Geller described the image of the $\mathcal{S}(H_d)$ under the group Fourier transform. They are characterized by continuity and rapid decay of iterates of any order of certain combinations of derivatives in the parameter λ identifying the Schrödinger representation and difference operators in the discrete parameters identifying the matrix entries at each representation.

Spherical transforms of z-radial Schwartz functions

In 1998 C. Benson, J. Jenkins and G. Ratcliff obtained a similar description of spherical transforms of K-invariant Schwartz functions for general commutative pairs (H_d, K).

The conditions are of the same nature as those of Geller, and the two overlap for z-radial functions, i.e., for $K = U_n$.

For instance, if $K = U_n$, the difference-differential operators to be iteratively applied to $u(\lambda, j) = \mathcal{G}f(\varphi_{|\lambda|(2j+d),\lambda})$ are

$$u \longmapsto \partial_\lambda u(\lambda, j) - \frac{j}{\lambda}\big(u(\lambda, j) - u(\lambda, j-1)\big) \qquad (\text{if } \lambda > 0) \ ,$$

$$u \longmapsto \partial_\lambda u(\lambda, j) - \frac{d+j}{\lambda}\big(u(\lambda, j+1) - u(\lambda, j)\big) \qquad (\text{if } \lambda < 0) \ .$$

The spectral perspective

It has been mentioned before that, for general commutative pairs (G, K) of Lie groups, each bounded operator T on $L^2(G)$ which commutes with left translations and with the automorphisms in K is identified by a spherical multiplier $m_T \in L^\infty(\Sigma_{\mathcal{D}}, \nu)$ through the formula

$$Tf = \mathcal{G}^{-1}(m_T\,\mathcal{G}f)\ .$$

Conversely, given $m \in L^\infty(\Sigma_{\mathcal{D}}, \nu)$, the corresponding operator T_m is bounded on $L^2(G)$.

Choosing $\mathcal{D}$ consisting of self-adjoint operators (which is always possible), this is strictly related to the following fact:

The support of the Plancherel measure ν in $\Sigma_{\mathcal{D}}$ is the joint L^2-spectrum of the operators in $\mathcal{D}$ and, for every T as above, $T = m_T(D_1, \dots, D_k)$.

Multiplier theorems

The following statement, concerning spectral multipliers of a single differential operator on a Lie group, is the consequence of a series of intermediate (and sharper) results, due to A. Hulanicki and J. Jenkins, A. Hulanicki, G. Alexopoulos.

Theorem

Let G a Lie group with polynomial volume growth and D a self-adjoint, hypoelliptic, left-invariant differential operator on G. If $m \in \mathcal{S}(\mathbb{R})$, then $m(D)f = f * K_m$ with $K_m \in \mathcal{S}(G)$.

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(The situation is quite different for groups with exponential volume growth: for instance, if G is a noncompact semisimple group, the condition $K_m \in L^1(G)$ already implies that m extends analytically to some an open neighborhood of $\sigma(D)$ in $\mathbb{C}$.)

Multivariate multipliers

This result extends to commuting families of self-adjoint differential operators $D_1, \ldots, D_k$ such that some polynomial in $D_1, \ldots, D_k$ is hypoelliptic:

$$m \in \mathcal{S}(\mathbb{R}^k) \implies m(D_1, \ldots, D_k)f = f * K_m \text{ with } K \in \mathcal{S}(G) \ .$$

For commutative pairs (G, K) with G of polynomial growth, one has the following consequence.

Corollary

Let $\mathcal{D} = \{D_1, \ldots, D_k\}$ be a system of self-adjoint generators of $\mathbb{D}_K(G)$. If $m \in \mathcal{S}(\mathbb{R}^k)$, then $\mathcal{G}^{-1}(m_{|_{\Sigma_\mathcal{D}}}) \in \mathcal{S}_K(G)$.

Condition (S)

The inverse implication is

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- G nilpotent with $G/[G, G]$ irreducible w.r. to K (V. Fischer, F. R., O. Yakimova);

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- G nilpotent with $G/[G, G]$ irreducible w.r. to K (V. Fischer, F. R., O. Yakimova);
- $G = U_n \ltimes \mathbb{C}^n$, $n \leq 2$, $K = \mathrm{Int}(U_n)$ (F. Astengo, B. Di Blasio, F. R.).

Bootstrapping

Level 0.0 $G = \mathbb{R}^n$, $K = O_n$ (same for $G = \mathbb{C}^d$, $K = U_d$)

$\mathcal{D} = \{-\Delta\}$, $\Sigma_{\mathcal{D}} = [0, +\infty)$.

$$f \in \mathcal{S}_{rad}(\mathbb{R}^n) \implies \widehat{f} \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n) \overset{\text{Whitney}}{\Longrightarrow} \hat{f}(\xi) = g\big(|\xi|^2\big) \ , \quad g \in \mathcal{S}(\mathbb{R})$$

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Level 0.1 $G = \mathbb{R}^n$, $K \subset O_n$.

$\mathcal{D} = \{p_1(-i\partial), \dots, p_k(-i\partial)\}$, with $P = (p_1, \dots, p_k)$ real Hilbert basis of $\mathcal{P}_K(\mathbb{R}^n)$. $\Sigma_{\mathcal{D}} = P(\mathbb{R}^n) \subset \mathbb{R}^k$.

$$f \in \mathcal{S}_K(\mathbb{R}^n) \implies \widehat{f} \in \mathcal{S}_K(\mathbb{R}^n) \overset{\text{G. Schwartz}}{\implies} \hat{f}(\xi) = g\big(P(\xi)\big) , \quad g \in \mathcal{S}(\mathbb{R}^k)$$

$$\mathcal{G}f = g_{|_{\Sigma_{\mathcal{D}}}}$$

Level 1

$G = H_d$, $K = U_d$, $\quad \mathcal{D} = \{-L, -iT\}$, $\quad \Sigma_{\mathcal{D}} =$Heisenberg fan $\subset \mathbb{R}^2$.

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Step 1. It is not hard to prove that property (S) holds for $f \in \mathcal{S}_K(H_d)$ with $\int_{\mathbb{R}} t^m f(z,t)\, dz\, dt = 0$ for every $m \geq 0$, i.e., when $\mathcal{G}f$ vanishes of infinite order on the singular half-line.

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Step 2. For general $f \in \mathcal{S}_K(H_d)$, on the singular half-line $\mathcal{G}f$ coincides with the spherical transform for $(\mathbb{C}^d, U_d)$ of $f^\flat(z) = \int_{\mathbb{R}} f(z,t)\,dt$. Hence $\mathcal{G}f(\cdot, 0)$ admits a Schwartz extension to the full horizontal line.

Continuation

Step 3. (Geller) There exists $f_1 \in \mathcal{S}_K(H_d)$ such that

$$\mathcal{G}f_1(\xi_1, \xi_2) = \frac{\mathcal{G}f(\xi_1, \xi_2) - \mathcal{G}f(\xi_1, 0)}{\xi_2} \ .$$

Iterating one obtains $f_2, \dots, f_m, \dots, \in \mathcal{S}_K(H_d)$ such that, for every m,

$$\begin{aligned} \mathcal{G}f(\xi_1, \xi_2) = \mathcal{G}f(\xi_1, 0) + \xi_2 \mathcal{G}f_1(\xi_1, 0) + \cdots + \frac{\xi_2^m}{m!} \mathcal{G}f_m(\xi_1, 0) \\ + \frac{\xi_2^{m+1}}{(m+1)!} \mathcal{G}f_{m+1}(\xi_1, \xi_2) \ . \end{aligned}$$

End of the proof

Using property (S) at level 0.0, we can extend each $\mathcal{G}f_m(\cdot,0) = \mathcal{G}_{\mathbb{C}^d} f_m^\flat$ to a Schwartz function ψ_m on the real line.

The family $\{\psi_m : m \geq 0\}$ is a Whitney jet on $\{\xi_2 = 0\}$ and we can construct a function $\psi \in \mathcal{S}(\mathbb{R}^2)$ such that

$$\partial_{\xi_2}^m \psi(\xi_1, 0) = \psi_m(\xi_1) \ .$$

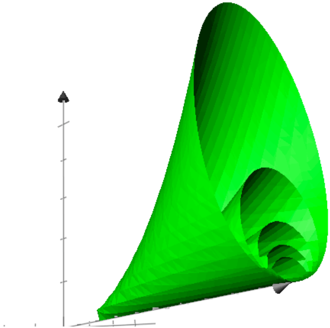
By Corollary, $\psi = \mathcal{G}g$ with $g \in \mathcal{S}_K(H_d)$. Then $\mathcal{G}(f - g)$ vanishes of infinite order on the singular half line, hence it extends to a Schwartz function $\eta \in \mathcal{S}(\mathbb{R}^2)$. Finally

$$\mathcal{G}f = (\psi + \eta)_{|\Sigma_\mathcal{D}} \ .$$

Level 2, an example

$G = \mathbb{R}^3 \times \mathbb{R}^3$, $K = SO_3$.

$$(x, y) \cdot (x', y') = (x + x', y + y' + x \wedge x')$$



Spherical transforms of distributions

Property (S) is the equality

$$\mathcal{G}\big(\mathcal{S}_K(G)\big) = \mathcal{S}(\Sigma_{\mathcal{D}}) \stackrel{\mathsf{def}}{=} \mathcal{S}(\mathbb{R}^k)/\{\psi : \psi_{|\Sigma_{\mathcal{D}}} = 0\} \ .$$

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This allows to define $\mathcal{G}\Phi$ for $\Phi \in \mathcal{S}'_K(G)$ by duality as a "synthesisable" tempered ditribution supported on $\Sigma_{\mathcal{D}}$:

$$\mathcal{G}\big(\mathcal{S}'_K(G)\big) = \big\{\Psi \in \mathcal{S}'(\mathbb{R}^k) : \langle \Psi, g\rangle = 0 \ \forall\, g = 0 \text{ on } \Sigma_{\mathcal{D}}\big\} \ .$$

Happy Birthday, John