Discrete Translates in Function Spaces

Alexander Olevskii

The talk is based on joint work with Alexander Ulanovskii

Introduction

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$$\{f(t-\lambda), \lambda \in \mathbb{R}\}.$$

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Let $f \in L^1(\mathbb{R})$.

Theorem (Wiener). *The set of translates $\{f(t-\lambda), \lambda \in \mathbb{R}\}$ spans the whole space $L^1(\mathbb{R})$ if and only if $\hat{f}$ has no zeros on $\mathbb{R}$.*

Consider the zero set of $\hat{f}$:

$$Z(\hat{f}) := \{w : \hat{f}(w) = 0\}.$$

Wiener expected that similar characterizations hold for the spaces $L^p(\mathbb{R})$ in terms of "smallness" of $Z(\hat{f})$.

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Theorem (N.Lev, A.O., Annals 2011). *For every $p, 1 < p < 2$, there are two functions $f_1, f_2 \in (L^1 \cap L^p)(\mathbb{R})$ such that*

(i) $Z(\hat{f}_1) = Z(\hat{f}_2)$;
(ii) The set of translates of f_1 spans $L^p(\mathbb{R})$, while the set of translates of f_2 does not.

Discrete Translates

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Two examples:

$$\Lambda_1 := \{\sqrt{n}, n \in \mathbb{Z}_+\}, \quad \Lambda_2 := \mathbb{Z}.$$

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SIZE VERSUS ARITHMETICS!

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$$\Lambda := \{n + \gamma(n), 0 < |\gamma(n)| = o(1)\}.$$

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Theorem (A.0., 1997). *For any almost integer set of translates there is a generator.*

The construction is based on "small denominators" argument.

L^p-generators

The case $p > 2$:

Theorem (A.Atzmon, A.O., Journal of Approximation Theory, 1996). *For every $p > 2$ there is a smooth function $f \in (L^p \cap L^2)(\mathbb{R})$ such that the family $\{f(t-n), n \in \mathbb{Z}\}$ is complete and minimal in $L^p(\mathbb{R})$.*

Hence, $\Lambda = \mathbb{Z}$ admits an L^p-generator for every $p > 2$ (and it does not for $p \leq 2$).

L^1-generators

No u.d. set Λ may admit an L^1-generator.

Theorem (J.Bruna, A.O., A.Ulanovskii, Rev. Mat. Iberoam., 2006) Λ *admits an L^1-generator iff it has infinite Beurling-Malliavin density.*

For $1 < p < 2$ the problem remained open.

Discrete Translates in Function Spaces

Which function spaces can be spanned by a uniformly discrete set of translates of a single function?

All results below are from A.O., A.Ulanovskii:

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We also assume

(II) Conditions $g \in X^*$ and $\operatorname{spec} g \subset \mathbb{Z}$ imply $g = 0$.

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Theorem 1. *There exist a smooth function f and a uniformly discrete set Λ of translates such that the family $\{f(t-\lambda), \lambda \in \Lambda\}$ spans X.*

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Below we present an explicit construction of f and Λ in this result.

Examples

Theorem 1 is applicable to

1. $L^p(\mathbb{R}), p > 1.$
2. Separable symmetric spaces (like Orlitz, Marzienkevich). The only exception is $L^1(\mathbb{R})$.
3. Sobolev spaces $W^{l,p}(\mathbb{R}), p > 1.$
4. Weighted spaces $L^1(w; \mathbb{R})$, where the weight is bounded and vanishes at infinity.

Construction

Definition. $F \in S(R)$ is said to have a deep zero at point t if

$$|F(t+h)| < Ce^{-1/|h|}, \quad |h| < \frac{1}{2}.$$

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GENERATOR: Take an even real function F with deep zeros at all integers (with the same constant) and at infinity, and which has no other zeros. Consider its Fourier transform $f := \hat{F}$.

TRANSLATES: Now define the translates as exponentially small perturbation of integers:

$$\Lambda := \{n + e^{-|n|}, n \in \mathbb{Z}\}.$$

Theorem 1'. *The set of translates $\{f(t-\lambda), \lambda \in \Lambda\}$ is complete in every X satisfying (I) and (II).*

Universality!

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Main Lemma. *Let F and Λ be as above, $g \in S'$. If the convolution $\hat{F} * g$ vanishes on Λ then it is zero.*

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Proof. $\hat{F}$ is analytic in a strip. Denote

$$H(t) := \sum_{k \in \mathbb{Z}} F(t+k).$$

By the Poisson formula,

$$H(t) = \sum_{n \in \mathbb{Z}} \hat{F}(n) e^{2\pi i n t}.$$

Since $\hat{F}(n)$ is exponentially small, then H is analytic on the circle. And it has a deep zero, so that $H = 0$. Hence, $\hat{F}|_{\mathbb{Z}} = 0$.

Iterate the argument above for $tF, t^2F, \ldots$ to get $\hat{F}^{(k)}|_{\mathbb{Z}} = 0, k = 1, 2, \ldots,$ so that $\hat{F} = 0$.

Proof of Theorem 1'

Suppose the translates $\{f(t-\lambda), \lambda \in \Lambda\}$ are not complete in X. Then there is a functional g "orthogonal" to them, which means

$$g * f|_\Lambda = 0.$$

By the Main Lemma, $g * f = 0$. That is $\hat{g}F = 0$. So, $\hat{g} = 0$ on $\mathbb{R} \setminus \mathbb{Z}$. This means $\operatorname{Spec} g \subset \mathbb{Z}$. Applying Property (II), we get $g = 0$.

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Open Problem. Does there exist a set of translates of a single function, which is complete and minimal in $L^2(\mathbb{R})$?

Two generators

Theorem 2. *There are $f_1, f_2 \in S(\mathbb{R})$ such that the Λ-translates of them span every space X, satisfying property (I) only.*

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THANKS!