Optimal group invariant subspaces

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Happy birthday John!
Collaborators:

Early work:
- Crystal groups:
  - María del Carmen Moure (UNMdP)
  - Alejandro Quintero (UNMdP)
- Approximation in SIS:
  - Akram Aldroubi (Vanderbilt University)
  - Carlos Cabrelli (IMAS - UBA/CONICET)
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Current Work:
- Davide Barbieri (UAM, España)
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Problem: Given $F = \{f_1, \ldots, f_m\}, f_i \in \mathcal{H}(X, m)$ find a small subspace $V \subseteq \mathcal{H}$ that is "close" to $F$. Meaning that

$$E(V, F) := \sum_{j=1}^{m} \|f_j - \rho_V f_j\|^2$$

is as small as possible among all subspaces with certain properties.

How can I choose an adequate $V$?
MSAP: We say that a class $\mathcal{E}$ of subspaces of $H$ has the Minimal Subspace Approximation Property if for any $F = \{f_1, \ldots, f_m\}$ there exists $V_0 \in \mathcal{E}$ such that $E(V_0, F) \leq E(V_i, F)$ for all $V_i \in \mathcal{E}$.

Aldroubi + Tessera gave necessary and sufficient conditions for $\mathcal{E}$ in order to have MSAP.
Example 1: \( \mathcal{H} = \mathbb{R}^N \), \( N \) huge \( (1024^2) \)

\( \mathcal{V}_l = \{ \text{subspaces of dimension } \ell \} \) \( \ell \ll N \)

given \( \{ \mathbf{f}_1, \ldots, \mathbf{f}_m \} \subseteq \mathbb{R}^N \), find the subspace of dimension \( \leq \ell \) that best fits the data \( \mathbf{f} \), or minimizes

\[ \sum \| \mathbf{f}_i - \text{proj}_{\mathcal{V}_l} \mathbf{f}_i \|_2^2 \]

over all \( \mathcal{V} \in \mathcal{V}_l \)
Solution:

* Eckard-Young (1936) or Schmidt 1907

Singular Value Decomposition:

\[ F = [f_1, \ldots, f_m] \in \mathbb{R}^{N \times m}, \quad G_F = F^*F \in \mathbb{R}^{m \times m} \]

\[ G_F = U \Lambda U^* \text{ with } \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ \vdots & \vdots \\ 0 & \lambda_m \end{bmatrix} \lambda_i \geq \lambda_{i+1} \]

\[ U = [u_1, \ldots, u_m] \text{ unitary in } \mathbb{R}^{m \times m} \]

If \( f_j = \sigma_j \sum_{i=1}^{m} u_j(i) f_i \), \( \sigma_j = \{ \lambda_j^{1/2}, \lambda_i > 0 \} \)

Then \( V_\theta = \text{span} \{ f_1, \ldots, f_\ell \} \text{ is orthonormal} \)
$V_0$ satisfies

\[ \sum \| f_i - \text{proj}_V f_i \|^2 \leq \sum \| f_i - \text{proj}_V f_i \|^2 \]

for any $V$, subspace of dimension $\leq l$ and furthermore

\[ \{ p_1, \ldots, p_l \} \text{ are an orthonormal basis.} \]
In general: \{f_1, \ldots, f_m\} \subseteq \mathcal{H}, \quad l \leq m

\[ G_f := [\langle f_i, f_j \rangle]_{i,j=1}^m \] is a selfadjoint positive semi-definite matrix in \( \mathbb{C}^{m \times m} \)

\[ G_f = U \Lambda U^* \] with \( \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ \vdots & \vdots \\ 0 & \lambda_m \end{bmatrix} \quad \lambda_i \geq \lambda_{i+1} \)

\[ U = [u_1 \ldots u_m] \] unitary in \( \mathbb{R}^{m \times m} \)

If \( \tau_j := \sigma_j \sum_{i=1}^m \mu_{j}(i) f_i \), \( \sigma_j = \begin{cases} 2^{-j/2} & \lambda_i > 0 \\ 0 & \text{else} \end{cases} \)

Then \( V_0 = \text{span} \{f_1, \ldots, f_l\} \quad \text{"orthonormal"} \)
Example: $l_2(\mathbb{Z}) \subset \{a_1, \ldots, a_m\}$ sequences.

The "best" subspace of dimension $\leq l$ is given by $V_0 = \text{span}\{\varphi^i, \ldots, \varphi^l\}$ with $\varphi^i = \sigma_i \sum_{j=1}^m \mu_{ij}(i) \mathbf{e}_j$ with

$$
\sigma_i = \begin{cases} 
\lambda_i^{-\frac{1}{2}} & \lambda_i > 0 \\
0 & \text{else}
\end{cases}
$$

and $V = [u_1 \mid \ldots \mid u_m]$ is such that $G_x = U \Lambda U^*$ and $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$.
\[ H = L^2(\mathbb{R}^n) \quad \mathcal{F} = \{ f_1, \ldots, f_m \} \]

\[ \mathcal{P}_e : = \{ V \in L^2(\mathbb{R}^n) : V \text{ is shift invariant and has at most } n \text{ generators} \} \]
A Shift Invariant Space (SIS) in $L^2(\mathbb{R}^d)$ is a closed subspace $V \subseteq L^2(\mathbb{R}^d)$ with the property that $f \in V \iff z_k f \in V$, $k \in \mathbb{Z}$, where $z_k f(x) = f(x - k)$. 
A Shift Invariant Space (SIS) in $L^2(\mathbb{R}^d)$ is a closed subspace $V \subseteq L^2(\mathbb{R}^d)$ with the property that $f \in V \Rightarrow z_k f \in V, \ k \in \mathbb{Z}^d$ where $z_k f(x) = f(x-k)$.

$\{ f_j \}_{j \in J}$ is a set of generators of $V$ if $V = \text{span} \{ z_k f_j : k \in \mathbb{Z}^d, j \in J \}$

The length of $V$, is defined as $l(V) = \min \{ l \in \mathbb{N} : \exists f_1, \ldots, f_l \text{ generators for } V \}$
A Shift Invariant Space (SIS) in $L^2(\mathbb{R}^d)$ is a closed subspace $V \subseteq L^2(\mathbb{R}^d)$ with the property that $f \in V \Rightarrow z_k f \in V$, $k \in \mathbb{Z}^d$ where $z_k f(x) = f(x-k)$.

$\{\phi_j\}_{j \in J}$ is a set of generators of $V$ if

$V = \operatorname{span} \{z_k \phi_j : k \in \mathbb{Z}^d, j \in J\}$

The length of $V$, is defined as

$l(V) = \min \{l \in \mathbb{N} : \exists \phi_1, \ldots, \phi_l \text{ generators for } V\}$

If the length is $\infty$ we say $V$ is finitely generated.
\[ H = L^2(\mathbb{R}^n) \]
\[ \mathcal{F} = \{ f_1, \ldots, f_m \} \]
\[ \mathcal{F}_\chi = \{ \psi \in L^2(\mathbb{R}^n) : \psi \text{ is shift invariant and has at most } \ell \text{ generators} \} \]

Does \( \mathcal{F}_\chi \) have the MSAP?
We have:

**Theorem** Let \( \mathcal{F} = \{f_1, \ldots, f_m\} \) be a set of functions in \( L^2(\mathbb{R}^d) \). Then

1. There exists \( V \in \mathcal{V}_n \) such that

   \[
   \sum_{i=1}^{m} \| f_i - P_V f_i \|^2 \leq \sum_{i=1}^{m} \| f_i - P_{V'} f_i \|^2, \quad \forall \ V' \in \mathcal{V}_n
   \]  
   (2.1)

2. The optimal shift-invariant space \( V \) in (2.1) can be chosen such that \( V \subset S(\mathcal{F}) \).


and hence \( \mathcal{F}_1 \) has the MSAP
Solution: Use the range function
\[ \mathcal{J} : \mathcal{V} \to L^2(\Omega, \ell^2(\mathbb{Z})) \quad \Omega = \mathbb{R} / \mathbb{Z} \]
\[ \mathcal{J}[f](\omega) = \{ \hat{f}(\omega + k) \}_{k \in \mathbb{Z}} \]
\[ \mathcal{V} = \overline{\text{span}} \{ e^{2\pi i k} \phi_i : i \in \mathbb{N}, k \in \mathbb{Z} \} \]
\[ \mathcal{J}(\mathcal{V}) = \overline{\text{span}} \{ e^{-2\pi i k} \mathcal{J}(\phi_i) : i \in \mathbb{N}, k \in \mathbb{Z} \} \]
\[ f \in \mathcal{V} \iff \mathcal{J}(f)(\omega) \in \overline{\text{span}} \{ \mathcal{J}[\phi_i](\omega) : i \in \mathbb{N} \} \quad \text{a.e., } \omega \in \Omega \]

This is now finite dimensional!
Define $J(w) := \overline{\text{span}} \{ \tilde{T}[\varphi_i](w) : i \in J \} \subseteq l^2(\mathbb{Z})$

$\omega \mapsto J(\omega)$

$(0,1) \to \text{subspace of } l^2(\mathbb{Z})$ is called range function $V = (J(\omega))_{\omega \in [0,1)}$

$f \in V \iff \tilde{T}[f](\omega) \in J(\omega)$ a.e. $\omega$

(Helson 1962 - Bochnik 20?)
We use $F_j(w) = \{ \vec{f}_1(j)(w), \ldots, \vec{f}_m(j)(w) \}$ and solve as before:

$$F_j(w) = \sum_{i=1}^{m} \lambda_j(i)(w) \vec{f}_i(w+k) \quad j = 1, \ldots, l$$

with $\sigma_j(w) = \begin{cases} (\langle \vec{f}_j(w) \rangle)^{-\frac{1}{2}} & \lambda_j(w) \neq 0 \\ 0 & \text{else} \end{cases}$

and

$$G_{\vec{f}}(w) = U(w) \wedge (w) U^*(w)$$

$$[G_{\vec{f}}(w)]_{s,t} = \left< \vec{f}_s(j)(w), \vec{f}_t(j)(w) \right>_{\ell_2(Z)}$$
\[ V_0 = \overline{\text{span}} \{ \phi, \ldots, \phi \} \text{ is SIS such that:} \]

* \( V_0 \) is minimizer in \( \ell \)

* \( \{ \phi, \ldots, \phi \} \) is Parseval frame for \( V_0 \)

* \[ E(\tilde{F}, \ell) = \sum_{k=1}^{\infty} \int (\lambda_k(\omega))^{1/2} \, d\omega \]
SIS are good models for images with "abelian" symmetries.
symmetries in data - non abelian
Can we find a better model than S15?

Non-abelian group-invariant spaces:

For example: crystal groups

Crystal groups (or space groups) are groups of isometries of $\mathbb{R}^d$, that generalize the notion of translations, to allow for different (rigid) movements in $\mathbb{R}^d$.

**Definition**

A **crystal group** is a discrete subgroup $\Gamma \subseteq \text{Isom}(\mathbb{R}^d)$ having a (bounded) fundamental domain, i.e. a bounded closed set $P$ such that

$$\bigcup_{\gamma \in \Gamma} \gamma(P) = \mathbb{R}^d \quad \text{and} \quad \gamma(P^\circ) \cap \gamma'(P^\circ) \neq \emptyset \Rightarrow \gamma = \gamma'$$

where $P^\circ$ is the interior of $P$. 
Theorem [Bieberbach]

The theorem of Bieberbach yields the following: Let $\Gamma$ be a crystal subgroup of $\text{Isom}(\mathbb{R}^d)$. Then

1. $\Lambda = \Gamma \cap \text{Trans}(\mathbb{R}^d)$ is a finitely generated abelian group of rank $d$ which spans $\text{Trans}(\mathbb{R}^d)$, and

2. $G$, the point group of $\Gamma$ is finite. The pointgroup stands for the linear parts of the symmetries of $\Gamma$ and satisfies $G \cong \Gamma/\Lambda$.

Definition

$\Gamma$ is called a splitting crystal group if it is the semidirect product of the subgroups $G$ and $\Lambda$, i.e. $\Gamma = \Lambda \rtimes G$. 
We consider: $\mathbb{R}$ second countable LCA group $G$ a non-commutative group of automorphisms of $\mathbb{R}$ (notations) $\Lambda$ a discrete uniform lattice of $\mathbb{R}$ (i.e. $\mathbb{R}/\Lambda$ compact) ($\mathbb{Z}^d$)

$\Gamma = \Lambda \times G$ the semidirect product

$\Gamma$ acts on $\mathbb{R}$ by $x \cdot \gamma = g x + k$ ($\gamma = (k, g) \in \Gamma$)
We consider: \( \mathbb{R}^{(\mathbb{R}^d)} \) a second countable LCA group

\[ G \text{ a non-commutative group of automorphisms of } \mathbb{R} \text{ (notations)} \]

\( \Lambda \text{ a discrete uniform lattice of } \mathbb{R} \)

(i.e. \( \mathbb{R}/\Lambda \text{ compact} \) (\( \mathbb{Z}^d \))

\[ \Gamma = \Lambda \times G \text{ the semidirect product} \]

\( \Gamma \) acts on \( \mathbb{R} \) by \( yx = gx + k \) \( (y = (k, g) \in \Gamma) \)

Also: i) \( g \Lambda = \Lambda \not\forall g \in G \)

ii) \( J \Omega_0 \leq \hat{\mathbb{R}}/(\Lambda^* \times G) \) "Boel section"

Example: Splitting Crystal Groups.
\[ \Gamma \text{ invariant subspaces of } L^2(\mathbb{R}) \]

\[ S \subseteq L^2(\mathbb{R}) \text{ is } \Gamma \text{-invariant if } S \]

\[ \text{is closed and } T_k R_g f \in S \niff \exists f \in S \]

\[ (k, g) \in \Gamma. \]

Consider \( \Phi := \{ \phi_1, \ldots, \phi_n \} \subseteq L^2(\mathbb{R}) \). Define

\[ S_\Gamma (\Phi) = \overline{\text{span}} \{ T_k R_g \phi_i : \phi_i \in \Phi, (k, g) \in \Gamma \} \]

\[ S_\Gamma (\Phi) \text{ is } \Gamma \text{ invariant (finitely generated)} \]

\[ \ell (S_\Gamma) \text{ minimum number of generators}. \]
Toy example \( R = \mathbb{R}^2, \Lambda = \mathbb{Z}^2 \)

\( G = \{ \text{rotations by } \frac{\pi}{2}, \pi, 3\frac{\pi}{2}, I \} \)

\[ \Omega_{x} = \mathbb{R}^2 / \mathbb{Z}^2 \times G \]

\[ \Omega = \bigcup_{g \in G} g \Omega \]

Note: \( S_{\Gamma} (\Phi) = \text{span} \{ T_{k} R_{g} \phi_{i} : \phi_{i} \in \Phi, k \in \Lambda, g \in G \} \)
Toy example $R = \mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$

$G = \{ \text{rotations by } \frac{\pi}{2}, \pi, 3\frac{\pi}{2}, \text{I} \}$

$\Omega_0 = \mathbb{R}^2$ \hspace{1cm} $\Omega = \mathbb{R}^2 / \mathbb{Z}^2 \times G$

$\Omega = \bigcup_{g \in G} g \cdot \Omega_0$

Note: $S_n(\Phi) = \sup \{ T_k(R_g \phi_i) : \phi_i \in \Phi, k \in \Lambda, g \in G \}$

$= S \text{IS} (R_g \phi_i : \phi_i \in \Phi, g \in G)$
So \( S_r \) is also \( \Delta \) invariant!

Among all \( \Delta \)-invariant spaces which are the ones that are \( \Gamma \) invariant?

Then \( V \subseteq \mathbb{L}^2(\mathbb{R}) \) is \( \Gamma \)-invariant if and only if it is \( \Delta \)-invariant and its range function satisfies
\[ r_g \mathcal{F} (g^* \omega) = \mathcal{F} (\omega) \]
\[ \sigma \quad \mathcal{F}(g^* \omega) = \mathcal{F}(\rho_{g^{-1}} \mathcal{F}(\omega)) \]

\[ (r_g(\alpha))(k) = \alpha (g^* k) \]
Define: \( \mathcal{E}_n^l = \{ S_n \in L^2(\mathbb{R}) : l(S_n) \leq l \} \)

We ask: Does \( \mathcal{E}_n^l \) have the MSAP?
Define: $\mathcal{C}_L^\ell = \{ S_\ell \in L^2(\mathbb{R}) : l(S_\ell) \leq \ell \}$

We ask: Does $\mathcal{C}_L^\ell$ have the MSAP?

Theorem: Given data $F = \{ f_1, \ldots, f_m \}$ in $L^2(\mathbb{R})$ and $\varepsilon > 0$, there exists $V \in \mathcal{C}_L^\ell$ such that

$$V = S_\ell(\Psi)$$

with $\Psi = \{ \psi_1, \ldots, \psi_l \}$ a Parseval frame generator and

$$\exists \langle S_\ell(\phi), F \rangle \leq \varepsilon \langle S_\ell(\phi), F \rangle + \phi \in \mathcal{C}_L^\ell.$$

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Hint of proof: \( F = \{ f_1, \ldots, f_m \} \)
\( \tilde{F} = \{ R_g f_i \}_{i=1}^m \), \( g \in G \rightarrow \tilde{F}(\tilde{F}) \)

Then force the generators to approximate for \( w \) in \( \Omega \).

Then we have group invariant gener
Examples by Davide Barbieri

Original images from image-net. 2000 images

$d = 345 \times 345$ pixels \((345 = 23 \times 15)\)

Error: \(\frac{100}{255} \left( \frac{1}{d^2} \sum_{m \in \mathbb{Z}_d \times \mathbb{Z}_d} \left| f_i(m) - P_{\Phi} f_i(m) \right|^2 \right)^{1/2} \)
Figure 2. Occurrences of errors for the approximation of the dataset with $\kappa = 8, 14, 19$. On the horizontal axis: the error by pixel (5.1). On the vertical axis: the corresponding number of images for the error.
orig. $l = 8$
$13.9\%$

$l = 14$
$12.1\%$

$l = 19$
$10.7\%$
Thank you!

and

Happy birthday John!!

lama!
References


for \( \omega \in \Omega_0 \cup \{0\} \)

for \( i \in \{1, \ldots, m\}, g \in \{0,1,2,3\} \)

compute \( x^{i,g} = T_T[f_i](\omega)^g \) using (3.2)

end

organize \( \{x^{i,g} : i \in \{1, \ldots, m\}, g \in \{0,1,2,3\}\} \) in a matrix \( X \) as in (4.3)

compute the first \( 4\kappa \) columns of \( U \) in the SVD of \( X = U\Sigma V^* \)

re-organize them into elements \( \{U^s\}_{s=1}^{4\kappa} \) of \( \ell_2(L) \)

for \( j \in \{1, \ldots, \kappa\}, g \in \{0,1,2,3\} \)

store \( \varphi_j(\omega)^g = U^{4(j-1)+g+1} \)

end

end

for \( j \in \{1, \ldots, \kappa\} \)

compute \( \phi_j = T_T^{-1}[\varphi_j] \) using (3.3)

end