Optimal group invariant subspaces

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JJB - Jubilee 2019

Happy birthday John!

Collaborators:

Early work:

- **Crystal groups:**

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- **Approximation in SIS:**

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Current Work:

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Problem: Given $\mathcal{F} = \{f_1, \ldots, f_m\}$, $f_i \in \mathcal{H}$, $m \gg$ find a small subspace $V \subseteq \mathcal{H}$ that is "close" to $\mathcal{F}$. Meaning that

$$\mathcal{E}(V, \mathcal{F}) := \sum_{j=1}^{m} \| f_j - P_V f_j \|^2$$

is as small as possible among all subspaces with certain properties.

How can I chose an adequate V?

MSAP: We say that a class $\mathcal{C}$ of subspaces of $\mathcal{H}$ has the Minimal Subspace Approximation Property if for any $\mathcal{F} = \{f_1, \ldots, f_m\}$, there exists $V_0 \in \mathcal{C}$ / $\mathcal{E}(V_0, \mathcal{F}) \leq \mathcal{E}(V, \mathcal{F})$

$\forall\, V \in \mathcal{C}$.

Aldroubi + Tessera (2011 - FoCM) gave necessary and sufficient conditions for $\mathcal{C}$ in order to have MSAP

Example 1: $\mathcal{H} = \mathbb{R}^N$, N huge (1024^2)

$\mathcal{C}_\ell = \{$ subspaces of dimension ℓ $\}$ $\ell \ll N$

Given $\{f_1, \dots, f_m\} \subseteq \mathbb{R}^N$, find the subspace of dimension $\leq \ell$ that best fits the data f, or minimizes

$$\sum \| f_i - \mathrm{proj}_V f_i \|^2$$

over all $V \in \mathcal{C}_\ell$.

Solution :

in german!

* Eckard - Young (1936) or Schmidt 1907

Singular Value Decomposition :

$F = [f_1 | \dots | f_m] \in \mathbb{R}^{N \times m}$. $G_F = F^* F \ (m \times m)$

$G_F = U \Delta U^*$ with $\Delta = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix}$ $\lambda_i \geq \lambda_{i+1}$

$U = [u_1 | \dots | u_m]$ unitary in $\mathbb{R}^{m \times m}$

If $\varphi_j := \sigma_j \sum_{i=1}^{m} u_j(i) f_i$, $\sigma_j = \begin{cases} \lambda_j^{-1/2} & \lambda_j > 0 \\ 0 & \text{else} \end{cases}$

Then $V_0 = \text{span}\{\varphi_1, \dots, \varphi_\ell\}$ "orthonormal"

V_0 satisfies

$$\sum \|f_i - \mathrm{proj}_{V_0} f_i\|^2 \leq \sum \|f_i - \mathrm{proj}_V\|^2$$

for any V, subspace of dimension $\leq l$ and furthermore

$\{\varphi_1, \ldots, \varphi_l\}$ are an orthonormal* basis.

In general: $\{f_1,\dots,f_m\} \subseteq \mathcal{H}$, $l<m$

$G_F := [\langle f_i, f_j \rangle]_{i,j=1}^{m}$ is a selfadjoint positive semidefinite matrix in $\mathbb{C}^{m\times m}$

$G_F = U \Delta U^*$ with $\Delta = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix}$ $\lambda_i \geq \lambda_{i+1}$

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Example: $\ell_2(\mathbb{Z})$ $\mathcal{F}=\{a^1,\dots,a^m\}$ sequences.

The "best" subspace of dimension $\leq \ell$ is given by $V_0 = \text{span}\{\varphi^1,\dots,\varphi^\ell\}$ with $\varphi^j = \sigma_j \sum_{i=1}^{m} u_j(i)\, a^i$ with

$$\sigma_j = \begin{cases} \lambda_j^{-1/2} & \lambda_j > 0 \\ 0 & \text{else} \end{cases}$$

and

$U = [u_1 | \cdots | u_m]$ is such that $G_F = U \Lambda U^*$

and $\Lambda = \text{diag}(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m)$

$H = L^2(\mathbb{R}^N)$ $\quad \mathcal{F} = \{ f_1, \dots, f_m \}$

$\mathcal{C}_\ell := \{ V \subseteq L^2(\mathbb{R}^N) : V$ is shift invariant and has at most ℓ generators$\}$

A Shift Invariant Space (SIS) in $L^2(\mathbb{R}^d)$ is a closed subspace $V \subseteq L^2(\mathbb{R}^d)$ with the property that $f \in V \Leftrightarrow \tau_k \circ f \in V$, $k \in \mathbb{Z}$ where $\tau_k \circ f(x) = f(x-k)$.

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$\{\varphi_j\}_{j \in J}$ is a set of generators of V if

$$V = \overline{\text{span}}\{\tau_k \varphi_j : k \in \mathbb{Z}^d, j \in J\}$$

The length of V, is defined as

$$\ell(V) = \min\{\ell \in \mathbb{N} : \exists\, \varphi_1, \dots, \varphi_\ell \text{ generators for } V\}$$

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If the length is $\not\infty$ we say V is finitely generated

$H = L^2(\mathbb{R}^N)$ $\quad \mathcal{F} = \{ f_1, \dots, f_m \}$

$\mathcal{C}_\ell := \{ V \subseteq L^2(\mathbb{R}^N) : V$ is shift invariant and has at most ℓ generators$\}$

Does $\mathcal{C}_\ell$ have the MSAP?

We have:

$\mathcal{C}_\ell$

Theorem *Let $\mathcal{F} = \{f_1, \dots, f_m\}$ be a set of functions in $L^2(\mathbb{R}^d)$. Then*

(1) *There exists $V \in \mathcal{V}_n$ such that*

$$\sum_{i=1}^{m} \|f_i - P_V f_i\|^2 \leq \sum_{i=1}^{m} \|f_i - P_{V'} f_i\|^2, \quad \forall\, V' \in \mathcal{V}_n \tag{2.1}$$

(2) *The optimal shift-invariant space V in (2.1) can be chosen such that $V \subset \mathcal{S}(\mathcal{F})$.*

A. Aldroubi, C. A. Cabrelli, D. Hardin, and U. M. Molter. Optimal shift invariant spaces and their parseval frame generators. *Applied and Computational Harmonic Analysis*, 23(2):273–283, 2007.

and hence $\mathcal{C}_\ell$ has the MSAP

Solution: Use the range function

$$\mathcal{T} : V \to L^2(\Omega, \ell_2(\mathbb{Z})) \qquad \Omega = \mathbb{R}^d / \mathbb{Z}^d$$

$$\mathcal{T}[f](\omega) = \{\hat{f}(\omega + k)\}_{k \in \mathbb{Z}}$$

$$V = \overline{\operatorname{span}}\{\tau_k \circ \phi_i : i \in J,\ k \in \mathbb{Z}\} \Rightarrow$$

$$\mathcal{T}(V) = \overline{\operatorname{span}}\{e^{-2\pi i k}\, \mathcal{T}(\phi_i) : i \in J,\ k \in \mathbb{Z}\}$$

$$f \in V \iff \mathcal{T}(f)(\omega) \in \overline{\operatorname{span}}^{\,\ell_2(\mathbb{Z})}\{\mathcal{T}[\phi_i](\omega) : i \in J\}$$

$$\text{a.e. } \omega \in \Omega$$

This is now finite dimensional!

Define $J(\omega) := \overline{\text{span}} \{ \tilde{T}[\phi_i](\omega) : i \in J \} \subseteq l^2(\mathbb{Z})$

$\omega \in [0,1)$

$\omega \longmapsto J(\omega)$

$[0,1) \to$ subspace of $l^2(\mathbb{Z})$ is called

<u>range function</u> $V \simeq (J(\omega))_{\omega \in [0,1)}$

$f \in V \iff \tilde{T}[f](\omega) \in J(\omega)$ a.e. ω

(Helson 1962 _ Bownik 20?)

We use $\widehat{\tilde{f}}(\omega) = \{ \tilde{T}[f_1](\omega), \ldots, \tilde{T}[f_m](\omega) \}$

and solve as before:

$$\hat{\varphi}_j(\omega + k) = \sigma_j(\omega) \sum_{i=1}^{m} u_j(i)(\omega) \, \hat{f}_j(\omega + k) \qquad j = 1, \ldots l$$

with $\sigma_j(\omega) = \begin{cases} (\lambda_j(\omega))^{-1/2} & \lambda_j(\omega) \neq 0 \\ 0 & \text{else} \end{cases}$

and $G_{\tilde{f}}(\omega) = U(\omega) \Lambda(\omega) U^*(\omega)$

$$[G_{\tilde{f}}(\omega)]_{s,t} = \langle \tilde{T}[f_s](\omega), \tilde{T}[f_t](\omega) \rangle_{l_2(\mathbb{Z})}$$

$V_0 = \overline{\text{span}}\,\{\varphi_1, \dots, \varphi_\ell\}$ is SIS such that:

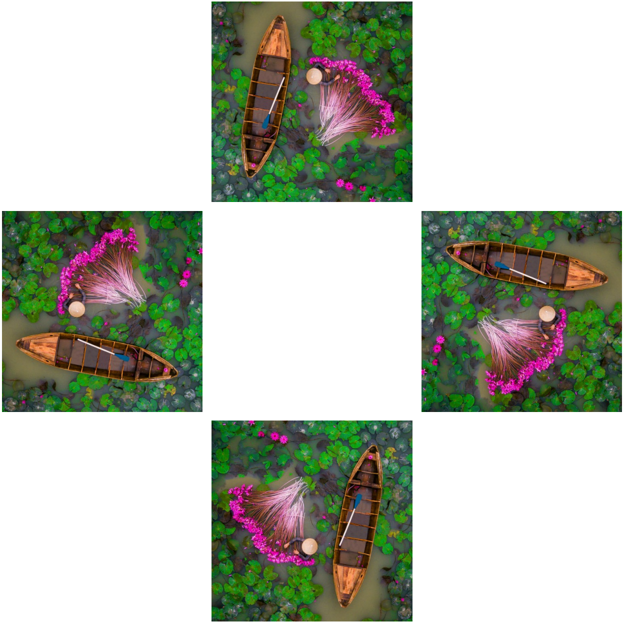
* V_0 is minimizer in $\mathcal{C}_\ell$

* $\{\varphi_1, \dots, \varphi_\ell\}$ is Parseval frame for V_0

* $$\mathcal{E}(\mathcal{F}, \mathcal{C}_\ell) = \sum_{k=\ell+1}^{\infty} \int_\Omega (\lambda_k(\omega)) \, d\omega$$

SIS are good models for images with "abelian" symmetries

symmetries in data - non abelian



Can we find a better model than SIS?

Non abelian group-invariant spaces:

For example: crystal groups

Crystal groups (or space groups) are groups of isometries of $\mathbb{R}^d$, that generalize the notion of translations, to allow for different (rigid) movements in $\mathbb{R}^d$.

Definition

A ***crystal group*** *is a discrete subgroup* $\Gamma \subseteq \mathrm{Isom}(\mathbb{R}^d)$ *having a (bounded) fundamental domain, i.e. a bounded closed set* P *such that*

$$\bigcup_{\gamma \in \Gamma} \gamma(P) = \mathbb{R}^d \text{ and } \gamma(P^\circ) \cap \gamma\prime(P^\circ) \neq \emptyset \Rightarrow \gamma = \gamma\prime$$

where P° *is the interior of* P*.*

Theorem [Bieberbach]

The theorem of Bieberbach yields the following: Let Γ be a crystal subgroup of $\mathrm{Isom}(\mathbb{R}^d)$. Then

1. $\Lambda = \Gamma \cap \mathrm{Trans}(\mathbb{R}^d)$ is a finitely generated abelian group of rank d which spans $\mathrm{Trans}(\mathbb{R}^d)$, and
2. G, the *point group* of Γ is finite. The pointgroup stands for the linear parts of the symmetries of Γ and satisfies $G \cong \Gamma/\Lambda$.

Definition

Γ *is called a* ***splitting crystal group*** *if it is the semidirect product of the subgroups* G *and* Λ*, i.e.* $\Gamma = \Lambda \rtimes G$.

We consider: $\mathcal{R}$ $(\mathbb{R}^d)$ second countable LCA group

G a non-commutative group of automorphisms of $\mathcal{R}$ (rotations)

Λ a discrete uniform lattice of $\mathcal{R}$ (i.e. $\mathcal{R}/\Lambda$ compact) $(\mathbb{Z}^d)$

$\Gamma = \Lambda \rtimes G$ the semidirect product

Γ acts on $\mathcal{R}$ by $\gamma x = gx + k$ $\quad (\gamma = (k, g) \in \Gamma)$

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Γ acts on $\mathcal{R}$ by $\gamma x = gx + k$ $(\gamma = (k, g) \in \Gamma)$

Also: i) $g\Lambda = \Lambda$ $\forall g \in G$

ii) $\exists\, \Omega_0 \subseteq \hat{\mathcal{R}}/_{(\Lambda^\perp \rtimes G)}$ "Borel section"

Example: Splitting Crystal Groups.

Γ invariant subspaces of $\mathcal{L}^2(\mathbb{R})$

$S \subseteq \mathcal{L}^2(\mathbb{R})$ is Γ-invariant if S is closed and $T_k R_g f \in S$ $\forall f \in S$ $(k, g) \in \Gamma$.

Consider $\Phi := \{\phi_1, \ldots, \phi_n\} \subseteq \mathcal{L}^2(\mathbb{R})$ Define

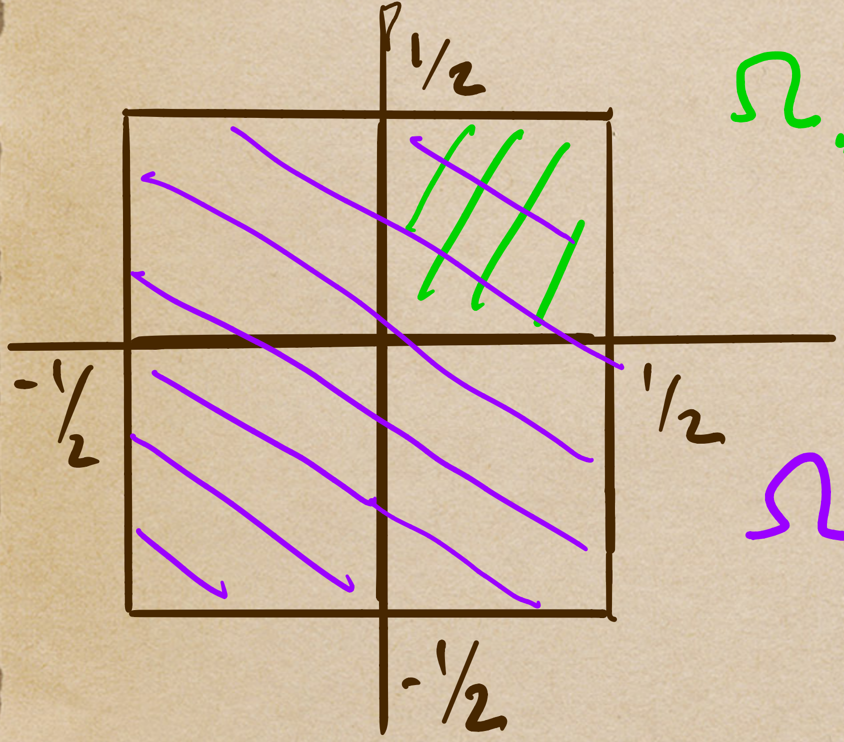
$$S_\Gamma(\Phi) = \overline{\operatorname{span}}\{T_k R_g \phi_i : \phi_i \in \Phi, (k, g) \in \Gamma\}$$

$S_\Gamma(\Phi)$ is Γ invariant (finitely generated)

$\ell(S_\Gamma)$ minimum number of generators.

Toy example $R = \mathbb{R}^2$, $\Lambda = \mathbb{Z}^2$

$G = \{\text{rotations by } \pi/2,\ \pi,\ 3\pi/2,\ I\}$



$\Omega_0 = \mathbb{R}^2 / \mathbb{Z}^2 \rtimes G$

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$\Omega = \bigcup_{g \in G} g\,\Omega_0$

Note: $S_\Gamma(\Phi) = \text{span}\{T_k R_g \phi_i : \phi_i \in \Phi,\ k \in \Lambda,\ g \in G\}$

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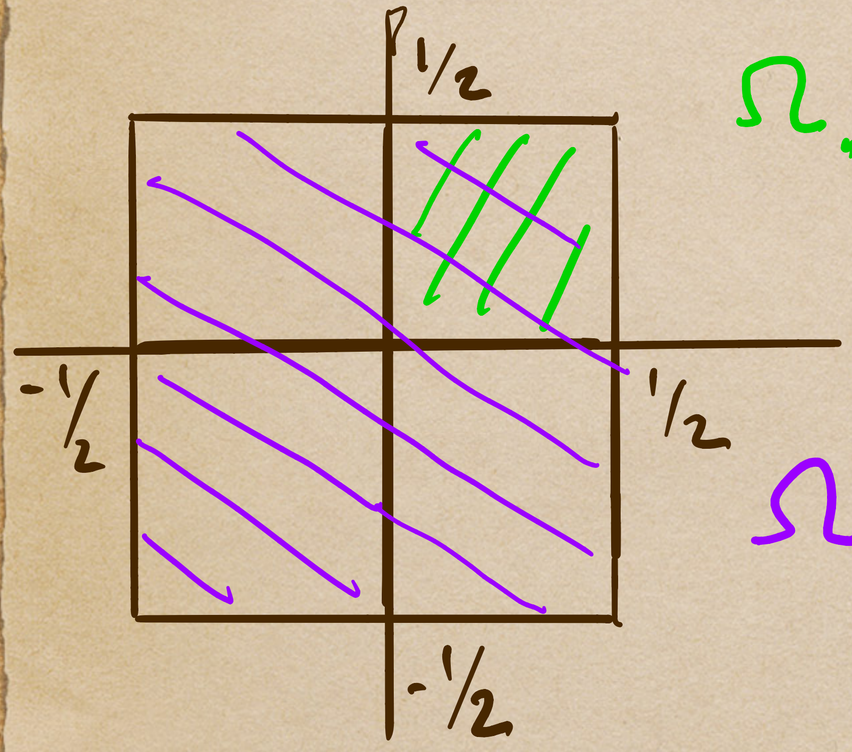


$\Omega_0 = \mathbb{R}^2 / \mathbb{Z}^2 \rtimes G$

$\Omega = \bigcup_{g \in G} g\,\Omega_0$

$\Omega = \mathbb{R}^2 / \mathbb{Z}^2$

Note: $S_\Gamma(\Phi) = \text{span}\{T_k(R_g \phi_i) : \phi_i \in \Phi, k \in \Lambda, g \in G\}$

$= SIS(R_g \phi_i : \phi_i \in \Phi, g \in G)$!!

So S_Γ is also Λ invariant!

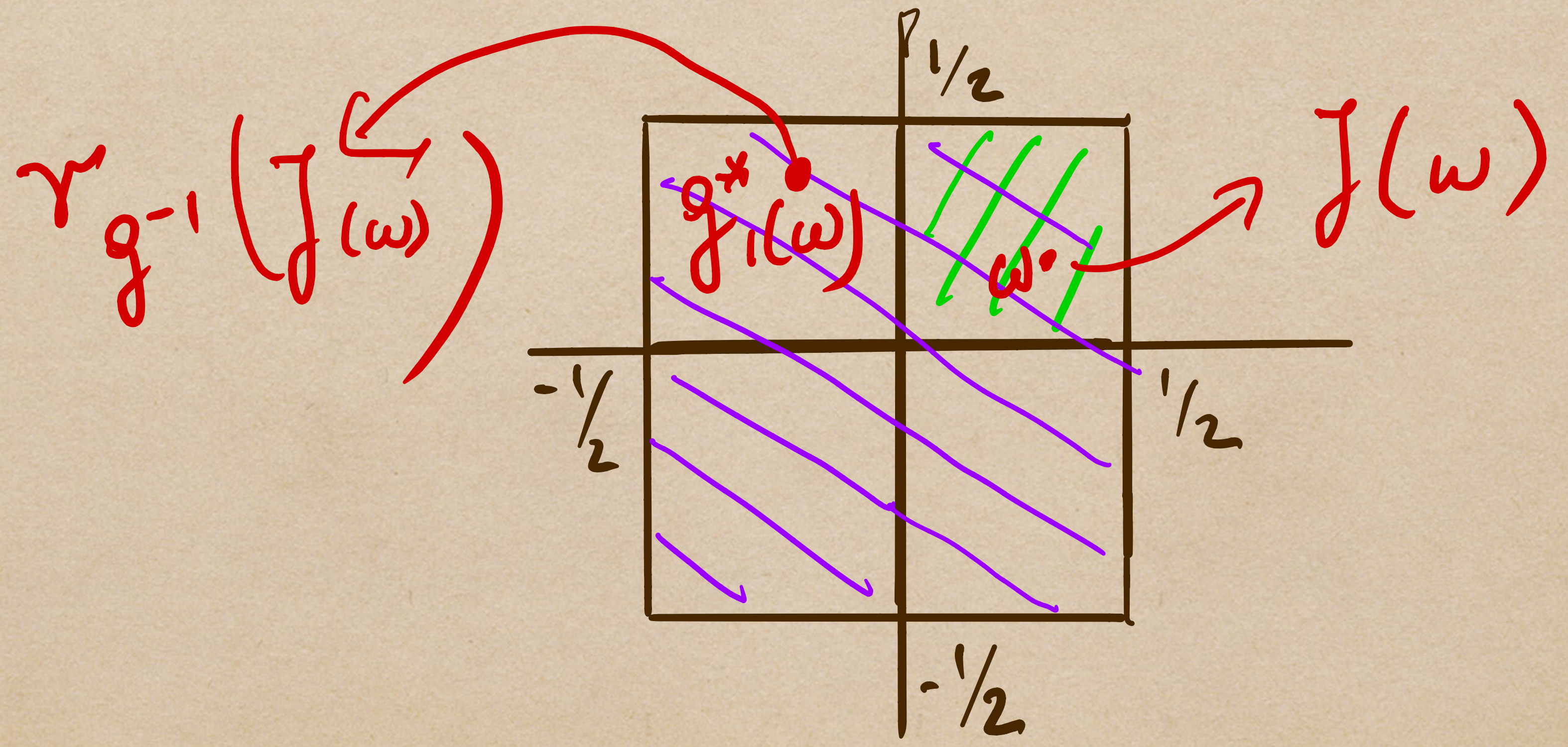
Among all Λ-invariant spaces which are the ones that are Γ invariant?

Thm $V \subseteq L^2(\mathbb{R})$ is Γ-invariant if and only if it is Λ-invariant and its range function satisfies

$$r_g J(g^* \omega) = J(\omega)$$

or $\quad J(g^*\omega) = r_{g^{-1}} J(\omega)$

$$\big(r_g(a)\big)(k) = a(g^* k)$$



Define : $\mathcal{C}_\Gamma^\ell = \{ S_\Gamma \subseteq L^2(\mathbb{R}) : \ell(S_\Gamma) \leq \ell \}$

We ask : Does $\mathcal{C}_\Gamma^\ell$ have the MSAP ?

Define : $\mathcal{C}_\Gamma^\ell = \{ S_\Gamma \subseteq L^2(\mathbb{R}) : \ell(S_\Gamma) \leq \ell \}$

We ask : Does $\mathcal{C}_\Gamma^\ell$ have the MSAP?

Theorem: Given data $\mathcal{F} = \{ f_1, \dots, f_m \} \subseteq L^2(\mathbb{R})$ and $\ell > 0$, there exists $V \in \mathcal{C}_\Gamma^\ell$ such that $V = S_\Gamma(\Psi)$ with $\Psi = \{ \psi_1, \dots, \psi_\ell \}$ a Parseval frame generator and $\mathcal{E}(S_\Gamma(\Psi), \mathcal{F}) \leq \mathcal{E}(S_\Gamma(\Phi), \mathcal{F}) \; \forall \Phi \in \mathcal{C}_\Gamma^\ell$.

Barbieri, Cabrelli, Hernández, Molter, *Approximation by group invariant subspaces*. arxiv.org/abs/1907.08300 (2019)

Hint of proof: $\mathcal{F} = \{f_1, \dots, f_m\}$

$\tilde{\mathcal{F}} = \{R_g f_i\}_{i=1, g \in G}^{m} \to \mathcal{J}(\tilde{\mathcal{F}})$

approximate for $\boxed{\omega \text{ in } \bar{\Omega}_0}$.

Then force the generators to

satisfy $\mathcal{J}(g^* \omega) = r_{g^{-1}} \mathcal{J}(\omega)$

$\Rightarrow$ we have group invariant gener

Examples by Davide Barbieri

Original images from

image-net . 2000 images

$d = 345 \times 345$ pixels $(345 = 23 \times 15)$

% of adm. values

average P(pixel)

Error: $\frac{100}{255} \left(\frac{1}{d^2} \sum_{n \in \mathbb{Z}_d \times \mathbb{Z}_d} \left| f_i(n) - P_{S(\Phi)} f_i(n) \right|^2 \right)^{1/2}$

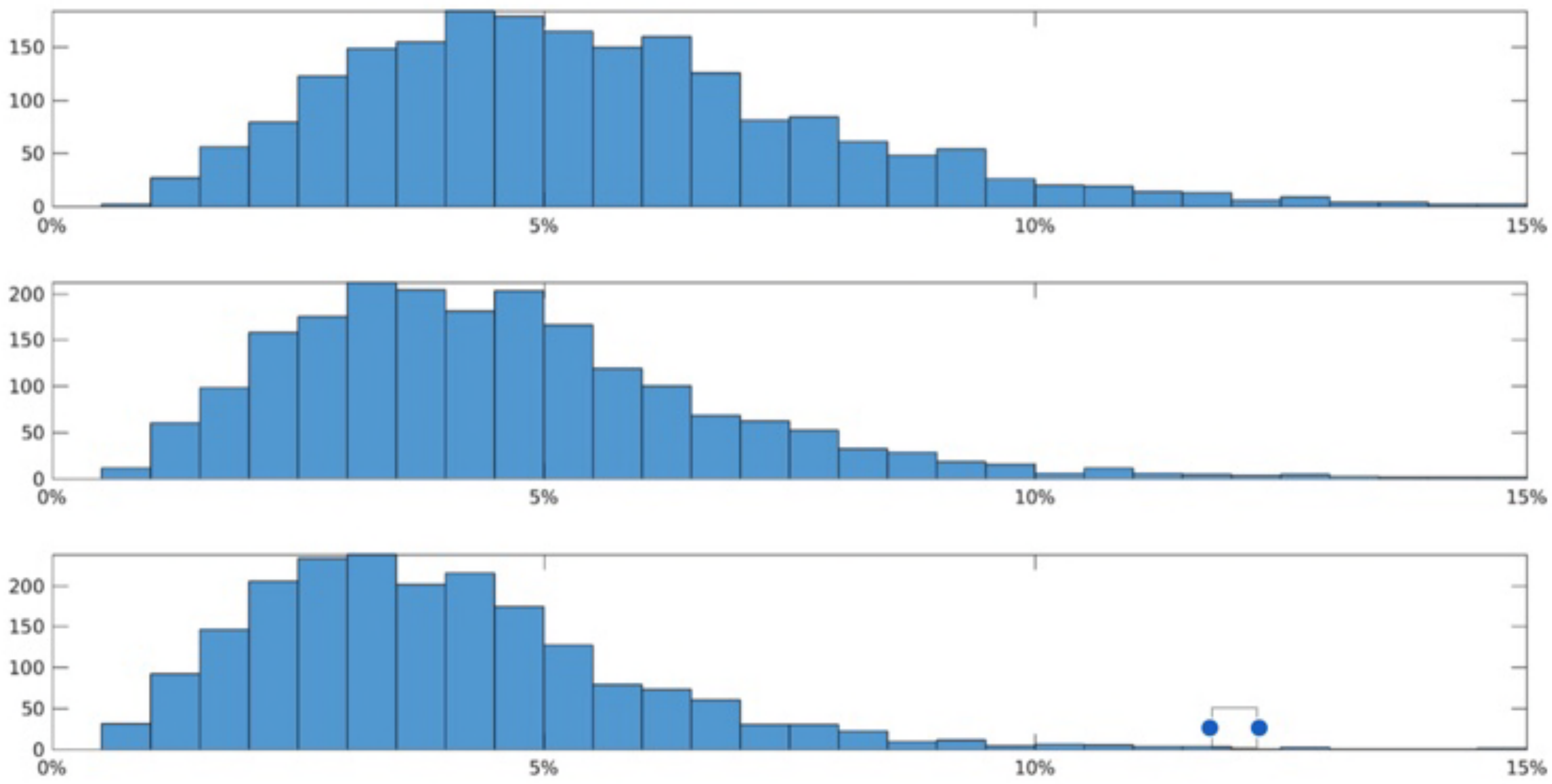


Figure 2. Occurrences of errors for the approximation of the dataset with $\kappa = 8, 14, 19$. On the horizontal axis: the error by pixel (5.1). On the vertical axis: the corresponding number of images for the error.

7.1%

6.3%

5.7%

2.6%

2.0%

1.7%

3.1%

2.3%

1.9%

2.6%

2.2%

2%

orig.

$\ell = 8$

13,9%

$\ell = 14$

12.1%

$\ell = 19$

10.7%

Thank you!

and

Happy birthday John!!

lama!

References

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`for` $\omega \in \Omega_0 \cup \{0\}$

  `for` $i \in \{1, \ldots, m\}, g \in \{0, 1, 2, 3\}$

    `compute` $x^{i,g} = \mathcal{T}_\Gamma[f_i](\omega)^g$ `using` (3.2)

  `end`

  `organize` $\{x^{i,g} : i \in \{1, \ldots, m\}, g \in \{0, 1, 2, 3\}\}$ `in a matrix` X `as in` (4.3)

  `compute the first` 4κ `columns of` U `in the SVD of` $X = U\Sigma V^*$

  `re-organize them into elements` $\{U^s\}_{s=1}^{4\kappa}$ `of` $\ell_2(L)$

  `for` $j \in \{1, \ldots, \kappa\}, g \in \{0, 1, 2, 3\}$

    `store` $\varphi_j(\omega)^g = U^{4(j-1)+g+1}$

  `end`

`end`

`for` $j \in \{1, \ldots, \kappa\}$

  `compute` $\phi_j = \mathcal{T}_\Gamma^{-1}[\varphi_j]$ `using` (3.3)

`end`