On the conditioning of subensembles

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The Ohio State University

Jubilee of Fourier Analysis and Applications

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Conditioning

Notions of conditioning for vectors $(f_i)_{i \in I}$ in a Hilbert space H:

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- Linearly independent.
  Quantitative version: **Riesz sequence**

$$c\sum_{i\in I}|a_i|^2 \le \|\sum_{i\in I} a_i f_i\|^2 \le C\sum_{i\in I}|a_i|^2 \qquad \forall a\in \ell^2(I)$$

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$$c = C \quad \iff \quad \text{equal norm and orthogonal}$$

- Spanning.
  Quantitative version: **Frame**

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$c = C \quad \Longleftrightarrow \quad$ tight

$C_{\mathrm{Riesz}} = C_{\mathrm{Frame}}$ "how uniformly the energy of $(f_i)_{i\in I}$ is spread"

Meta-question

Various applications point to the same meta-question:

Given an ensemble of vectors,
what can you say about
the conditioning of subensembles?

This talk: Important instances, open problems

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- **Symmetric type.**
  Subensembles of **symmetric** ensembles (cf. clique number of Paley graph)

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  Conditions for **existence** of a well-conditioned subensemble
  (cf. Ramsey's theorem)
- **Symmetric type.**
  Subensembles of **symmetric** ensembles
  (cf. clique number of Paley graph)
- **Design type.**
  Explicit ensembles with **all** well-conditioned subensembles
  (cf. explicit Ramsey graphs)

Part I

Ramsey type

Conditions for **existence** of a well-conditioned subensemble

A warm up

$\mathcal{B} =$ subsets (*blocks*) of $\mathcal{V}$ of size k such that
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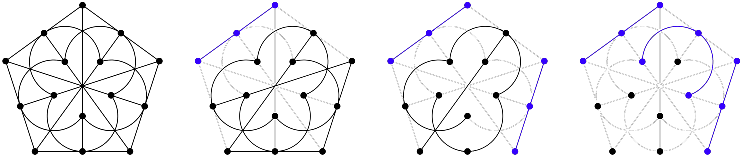
Lemma. There exist disjoint $\mathcal{D} \subseteq \mathcal{B}$ such that $|\mathcal{D}| \geq \dfrac{|\mathcal{B}|}{(r-1)k+1}$.

Proof: Iteratively select a block and discard intersecting blocks. $\square$



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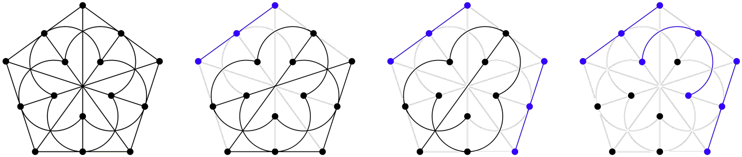

- $(\frac{1}{\sqrt{k}}\mathbf{1}_B)_{B\in\mathcal{D}}$ is orthonormal (well conditioned)
- r small $\implies \mathcal{D}$ large

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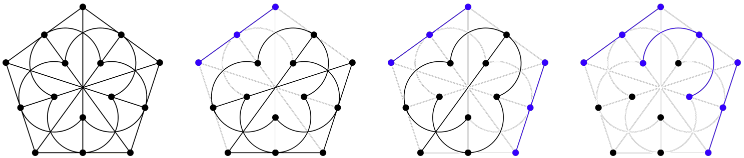

- $(\frac{1}{\sqrt{k}}\mathbf{1}_B)_{B\in\mathcal{D}}$ is orthonormal (well conditioned)
- r small $\implies$ $\mathcal{D}$ large
- $(\frac{1}{\sqrt{k}}\mathbf{1}_B)_{B\in\mathcal{B}}$ has upper Riesz bound r

Restricted invertibility

More general phenomenon:

Well-spread unit vectors enjoy a Riesz subensemble.

Bourgain, Tzafriri, Israel J. Math., 1987

Spielman, Srivastava, Israel J. Math., 2012

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Well-spread unit vectors enjoy a Riesz subensemble.

Theorem

Given n unit vectors with upper Riesz bound C, there exists a subensemble of $\geq \epsilon^2 n/C$ vectors with lower Riesz bound $(1-\epsilon)^2$.

Proof gives $O(n^4)$ time algorithm to find subensemble

Numerical analysis: column subset selection problem

Historically, this served as a stepping stone to the next result

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Kadison–Singer

Question

Does every pure state on $\{$ bounded diagonal operators on ℓ^2 $\}$ extend uniquely to a pure state on $\{$ bounded operators on ℓ^2 $\}$?

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Casazza, Fickus, Tremain, Weber, Operator Theory, Operator Algebras, and Applications, 2006

Bownik, Contemp. Math., 2018

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Equivalent:

- Paving conjecture
- Weaver's conjecture
- Bourgain–Tzafriri conjecture
- Feichtinger conjecture
- R_ϵ conjecture

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Answer: Yes (!)

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Unit norm tight frames partition into two frames.

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$\mathrm{KS}_2(\eta)$: $\exists \theta > 0$, $\forall$ finite-dim H, $\forall \eta$-tight frame $(f_i)_{i \in I}$, $\|f_i\| \leq 1$,
$\exists$ partition $I_1 \sqcup I_2 = I$ such that

$$\theta \|x\|^2 \leq \sum_{i \in I_j} |\langle f_i, x \rangle|^2 \leq (\eta - \theta)\|x\|^2 \quad \forall x \in H,\ j \in \{1, 2\}$$

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Theorem

- $\mathrm{KS}_2(\eta)$ does not hold for $\eta = 2$.

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- $\mathrm{KS}_2(\eta)$ does not hold for $\eta = 2$.
- $\mathrm{KS}_2(\eta)$ holds for $\eta > 4$.

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Bownik, Casazza, Marcus, Speegle, J. Reine Angew. Math., 2019

Part II

Symmetric type

Subensembles of **symmetric** ensembles

HRT

Conjecture

For every $0 \neq f \in L^2(\mathbf{R})$ and every finite $\Lambda \subseteq \mathbf{R}^2$, the ensemble $(e^{2\pi i b x} f(x-a))_{(a,b)\in\Lambda}$ is linearly independent.

Heil, Ramanathan, Topiwala, Proc. Am. Math. Soc., 1996

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Solved instances (among several):

- $\Lambda \subseteq$ lattice
- $\Lambda = 4$ points, 2 on each of 2 parallel lines
- f satisfies $e^{cx \log x}|f(x)| \to 0$ as $x \to \infty$ for each $c > 0$

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Open instances (among several):

- $\Lambda \subseteq \mathbf{Z} \times \mathbf{R}$
- $\Lambda = \{(0,0),(1,0),(0,1),(\sqrt{2},\sqrt{2})\}$
- functions with faster-than-exponential decay

Heil, Ramanathan, Topiwala, Proc. Am. Math. Soc., 1996

Demeter, Zaharescu, J. Math. Anal. Appl., 2012

Bownik, Speegle, Bull. Lond. Math. Soc., 2013

Fuglede

Problem

Classify **spectral sets**, that is, domains $\Omega \subseteq \mathbf{R}^d$ for which $L^2(\Omega)$ admits an orthogonal basis of exponentials.

- example: segment in $\mathbf{R} \longleftrightarrow$ Fourier series
- conjecture: a set is spectral iff it tiles by translates

Fuglede, J. Func. Anal., 1974

Tao, Math. Res. Lett., 2004

Kolounzakis, Matolcsi, Collect. Math., 2006

Farkas, Matolcsi, Móra, J. Fourier Anal. Appl., 2006

Lev, Matolcsi, `arXiv:1904.12262`

Iosevich, Mayeli, Pakianathan, Anal. PDE, 2017

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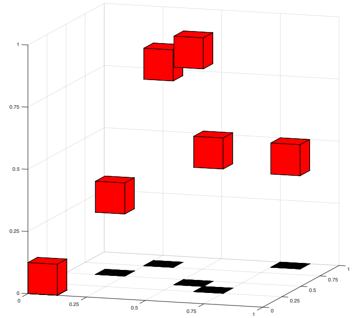
A spectral set in $\mathbf{R}^3$ that does not tile:



$$+ \quad \{0, \ldots, 2^{10}\}^3$$

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Uniform uncertainty principles

Dual problem:

size-$|\Omega|$ *subensembles of* $(\chi|_\Omega)_{\chi\in\hat{G}}$ *are linearly independent*

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For every finite abelian group G, classify subsets $\Omega \subseteq G$ such that every $0 \neq f \in \ell^2(G)$ with $\|\hat{f}\|_0 \leq |\Omega|$ satisfies $f|_\Omega \neq 0$.

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- Chebotarëv: For p prime, every $\Omega \subseteq \mathbf{Z}_p$
- For q prime power, every $\Omega \subseteq \mathbf{Z}_q$ such that

$$|\Omega \cap (H+x)| \in \left\{ \left\lfloor \frac{|\Omega|}{|\mathbf{Z}_q : H|} \right\rfloor, \left\lceil \frac{|\Omega|}{|\mathbf{Z}_q : H|} \right\rceil \right\} \quad \forall H \leq \mathbf{Z}_q,\ x \in \mathbf{Z}_q$$

- Otherwise, open (!)

Stevenhagen, Lenstra, Math. Intelligencer, 1996

Alexeev, Cahill, M., J. Fourier Anal. Appl., 2012

Uniform uncertainty principles

Dual problem: (harder)

size-$|\Omega|/r$ subensembles of $(\chi|_\Omega)_{\chi\in\hat{G}}$ are uniformly Riesz

Problem (quantitative version)

For every finite ab gp G, classify subsets $\Omega \subseteq G$ such that every $f \in \ell^2(G)$ with $\|\hat{f}\|_0 \leq |\Omega|/r$ satisfies $c\|f\|^2 \leq \|f|_\Omega\|^2 \leq C\|f\|^2$.

Application: Compressed sensing

Haviv, Oded, Geometric Aspects of Functional Analysis, 2017

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Błasiok, Lopatto, Luh, Marcinek, Rao, `arXiv:1903.12135`

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Application: Compressed sensing

Subproblem: Smallest r for which **random** Ω satisfies $C < 2c$ whp

- $r \lesssim \log^2|\Omega| \cdot \log|G|$
- $r \gtrsim \log|G|$
- $r \gtrsim \log|\Omega| \cdot \log(|G|/|\Omega|)$ if $G = \mathbf{Z}_2^n$

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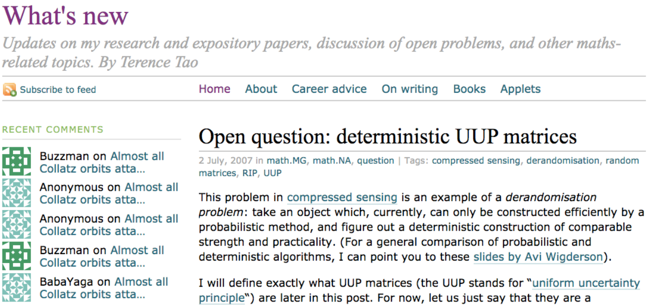
Uniform uncertainty principles

Removing randomness is hard, even without the Fourier structure



Problem

Find an explicit k-**restricted isometry** $(f_i)_{i\in I}$, meaning all size-k subensembles are uniformly (c, C)-Riesz for some $C < 2c$

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terrytao.wordpress.com/2007/07/02/open-question-deterministic-uup-matrices/

Part III

Design type

Explicit ensembles with **all** well-conditioned subensembles

Projective codes

Find n unit vectors in $\mathbf{C}^d$ that minimize C/c with $k = 2$

Strohmer, Heath, Appl. Comput. Harmon. Anal., 2003

Iverson, M., arXiv:1806.09037, arXiv:1905.06859

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Find n points in $\mathbf{CP}^{d-1}$ that maximize the minimum distance.

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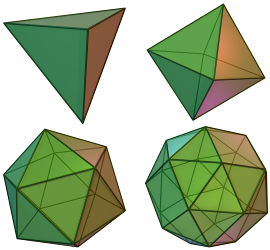
Problem

Find n points in $\mathbf{CP}^{d-1}$ that maximize the minimum distance.

- generalization of Tammes problem
- applications in communication
- doubly transitive lines
- equiangular tight frames
- $n = d^2$: Zauner $\leftrightarrow$ Stark conjectures
- online competition: Game of Sloanes



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Game of Sloanes

Leader Board Open Problems Code Papers

Game of Sloanes

A game to find putatively optimal packings in complex projective space with the goal of proving optimality of as many packings as possible.

This website concerns the problem of packing n lines, represented by unit vectors, through the origin of $\mathbb{C}^d$ such that the angles between the lines are as large as possible. The problem has applications in fields such as compressed sensing, digital fingerprinting, quantum state tomography, and multiple description coding.

Let $\Phi = (\varphi_j)_{j=1}^n$ be a set of unit norm vectors. The *coherence* μ of Φ is defined to be

$$\mu(\Phi) = \max_{1 \le j < k \le n} |\langle \varphi_j, \varphi_k \rangle|.$$

We call Φ a *Grassmannian frame* or *(optimal) projective packing* if

www.math.colostate.edu/~king/GameofSloanes.html

Explicit restricted isometries

Problem

What is the largest $k = k(d)$ for which there exists an explicit k-restricted isometry of $(1 + \Omega(1)) \cdot d$ vectors in $\mathbf{C}^d$?

- nonexplicit: $k \asymp d$
- Gershgorin: $k \asymp d^{1/2}$ "square-root bottleneck"
- BDFKK: $k \asymp d^{1/2+\epsilon}$ with $\epsilon = 10^{-16}$
- conjectured cancellations in Legendre symbol $\Rightarrow$ larger ϵ

Bandeira, Fickus, M., Wong, J. Fourier Anal. Appl., 2013
Bourgain, Dilworth, Ford, Konyagin, Kutzarova, STOC 2011
Bandeira, M., Moreira, Int. Math. Res. Not., 2017
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Can we close the remaining gap for **most** subensembles?

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Random subensembles

Problem

Given an equiangular tight frame in $\mathbf{C}^d$, what is the largest k for which 99% of the size-k subensembles are Riesz?

- Tropp: $k \asymp d/\log d$
- HZG: data suggests $k \asymp d$ is possible

Tropp, Appl. Comput. Harmon. Anal., 2008

Haikin, Zamir, Gavish, Proc. Natl. Acad. Sci. U.S.A., 2017

Random subensembles

Problem

Given an equiangular tight frame in $\mathbf{C}^d$, what is the largest k for which 99% of the size-k subensembles are Riesz?

- Tropp: $k \asymp d/\log d$
- HZG: data suggests $k \asymp d$ is possible

Experiment.

- fix $p = 10^6 + 33, \quad d = \frac{p+1}{2}$
- $(f_i)_{i\in I}$ = Paley ETF in $\mathbb{R}^d$
- draw random subensemble of size 10^3
- record Riesz bounds

Tropp, Appl. Comput. Harmon. Anal., 2008

Haikin, Zamir, Gavish, Proc. Natl. Acad. Sci. U.S.A., 2017

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0.9356	1.0643

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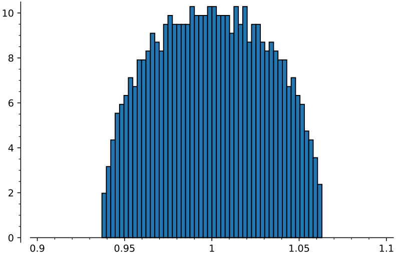
0.9374	1.0624
0.9349	1.0652
0.9367	1.0627

Tropp, Appl. Comput. Harmon. Anal., 2008

Haikin, Zamir, Gavish, Proc. Natl. Acad. Sci. U.S.A., 2017

Random subensembles

Full spectrum of Gram matrix of subensemble:



Haikin, Zamir, Gavish, Proc. Natl. Acad. Sci. U.S.A., 2017

Magsino, M., Parshall, `arXiv:1905.04360`

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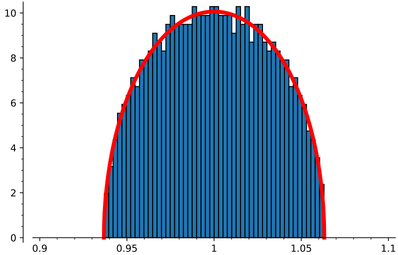

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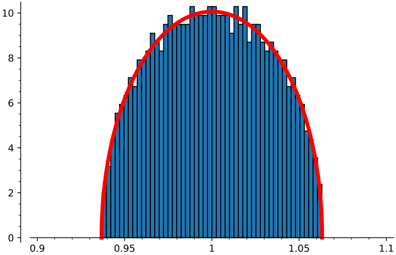

- **HZG conjecture:** ETF subensembles obey a Wachter law

Haikin, Zamir, Gavish, Proc. Natl. Acad. Sci. U.S.A., 2017

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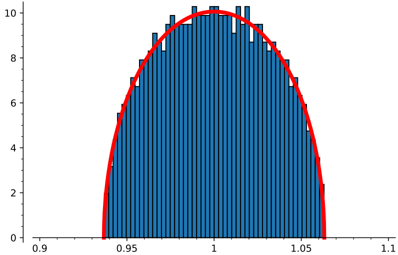


- **HZG conjecture:** ETF subensembles obey a Wachter law
- true for ETFs of $2d$ vectors in $\mathbf{R}^d$
- edge vs. bulk

Haikin, Zamir, Gavish, Proc. Natl. Acad. Sci. U.S.A., 2017

Magsino, M., Parshall, `arXiv:1905.04360`

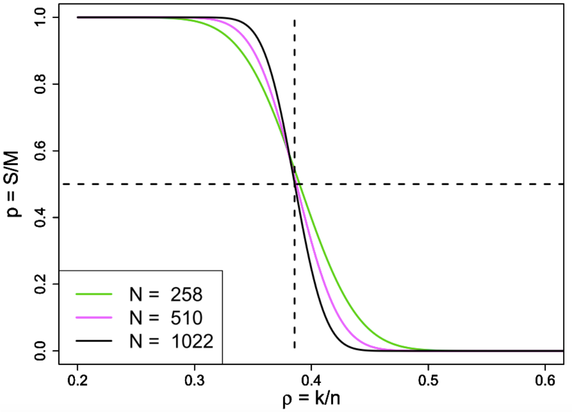
Consequences for compressed sensing?

Phase transitions:	fixed signal random sensor	$\sim$	random signal fixed sensor	(!)

Monajemi, Jafarpour, Gavish, Stat 330/CME 362, Donoho, Proc. Natl. Acad. Sci. U.S.A., 2013

Consequences for compressed sensing?

Phase transitions: fixed signal / random sensor $\sim$ random signal / fixed sensor (!)



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Open problems

- Kadison–Singer gap
- HRT
- Fuglede in $\mathbf{R}$ and $\mathbf{R}^2$
- non-quantitative uniform uncertainty principle
- quantitative uniform uncertainty principle
- Zauner's conjecture
- Game of Sloanes
- explicit restricted isometries
- King's problem
- HZG conjectures
- MJGD phase transition

Happy birthday, John!

Questions?

Google **short fat matrices** for my research blog.