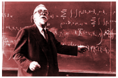
Spatio–spectral limiting on Boolean cubes

Jubilee of Fourier Analysis and Applications,

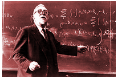
NWC at UMD, 2019

joint work with Jeff Hogan



Overview

1. Review: Time and band limiting: on $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{Z}_N$
2. Spatio-spectral limiting on graphs: definitions
3. Hypercube graphs
4. Results
5. Adjacency maps and invariant subspaces
6. Matrix reduction of spatio–spectral limiting operator
7. Numerical aspects
8. Potential extensions



Time and band limiting on $\mathbb{R}$: The 1960s Bell Labs Theory

Fourier transform: $\widehat{f}(\xi) = \int_{\mathbb{R}} f(t)\,\mathrm{e}^{-2\pi i t\xi}\,dt$

Bandlimiting: $P_\Omega f(x) = (\widehat{f}\,\mathbb{1}_{[-\Omega/2,\Omega/2]})^\vee(x)$

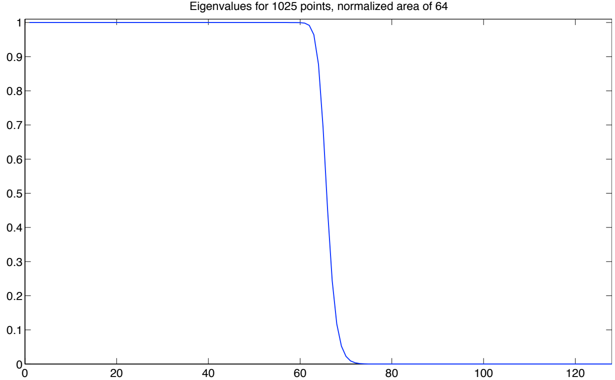
Time limiting: $(Q_T f)(x) = \mathbb{1}_{[-T,T]}(x)\,f(x)$

Bell Labs theory: basic questions

1. What are the eigenfunctions of $P_\Omega Q_T$?
2. What is the distribution of eigenvalues of $P_\Omega Q_T$

Eigenvalue distribution:

Approximately $2\Omega T - O(\log(2\Omega T))$ eigenvalues close to one
Plunge region of width proportional to $2\Omega T$
Exponential decay of remaining eigenvalues



Eigenfunctions: The *lucky accident*[1]

$P_\Omega Q_T$ commutes with

$$\text{(PDO)} \qquad (4T^2 - t^2)\,\frac{d^2}{dt^2} - 2t\,\frac{d}{dt} - \Omega^2 t^2\,.$$

Eigenfunctions: Prolate Spheroidal Wave Functions (PSWFs)

Methods to compute PSWFs based on PDO

[1] S. Slepian, Some comments on Fourier analysis, uncertainty and modeling, SIAM Review, 25, 379–393 1983

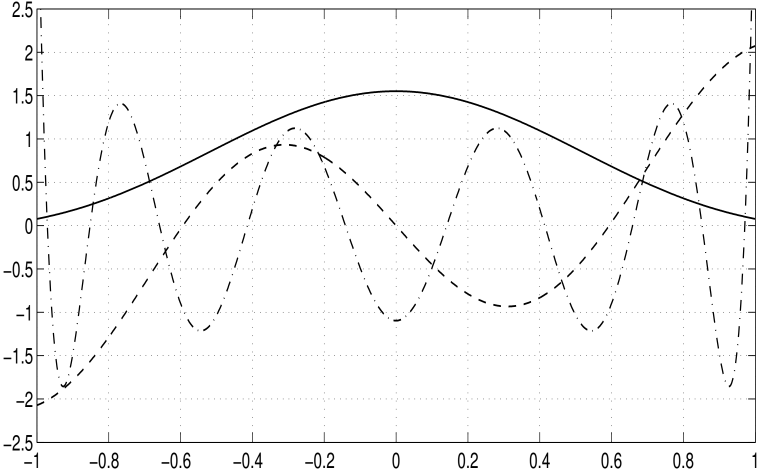
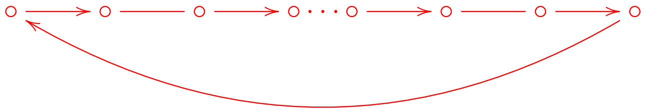


Figure: φ_n, $n = 0, 3, 10$, $c = \pi T\Omega/2 = 5$

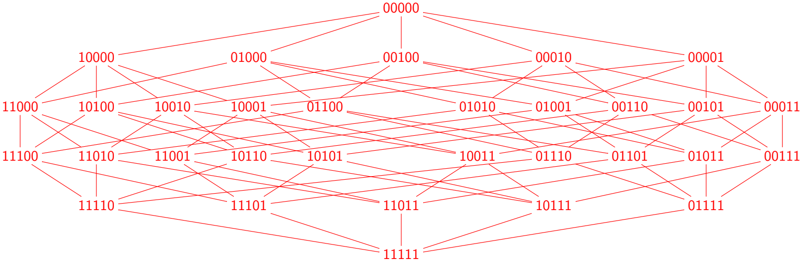
Finite dimensional analogue: cycle



Discrete setting $\mathbb{Z} \leftrightarrow \mathbb{T}$: Slepian, (1978) DPSS
Finite $\mathbb{Z}_N$ setting: Grünbaum (1981), others
Results analogous to continuous setting
Zhu et al 2018: Non-asymptotic bound on plunge region[2]
Many other developments in time and band limiting since 2000

[2] Z. Zhu, S. Karnik, M. A. Davenport, J. Romberg, and M. B. Wakin, The Eigenvalue Distribution of Discrete Periodic Time-Frequency Limiting Operators, IEEE Signal Process. Lett,, **25**, 95–99, 2018.

Hypercubes: $N = 5$



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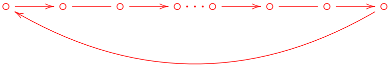


Graphs and Spatio–spectral limiting

Unnormalized Graph Laplacian and Fourier transform $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

$$\begin{aligned} f : \mathcal{V} \to \mathbb{R}, Lf(v) &= \sum_{w \sim v} f(v) - f(w) \\ L &= D - A \end{aligned}$$

D: degree of each vertex
A: adjacency map (undirected)
Graph Fourier transform φ_n: eigenvectors of L.

$$\hat{f}(\lambda_\ell) = \langle f, \varphi_\ell \rangle$$

Analogue of Q_T: truncation to path neighborhood of a vertex
Analogues of P_Ω: truncation to $\mathrm{span}\,\{\varphi_\ell : \lambda_\ell \quad \mathrm{small}\}$

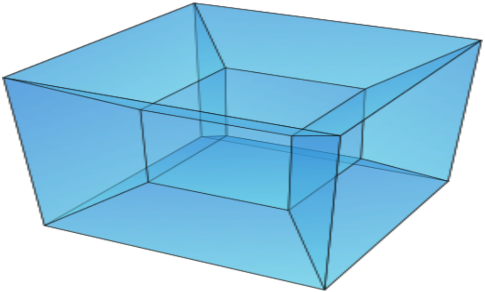
Motivation for GFT (e.g, Sardellitti Barbarossa Di Lorenzo [2016]): identify smooth clusters in vertex data that varies across clusters

Other time–frequency analysis on graphs: Shuman, Ricaud and Vandergheynst [e.g., ACHA 2016], Stanković, Daković and Sedjić [IEEE SP Magazine, 2017]

Our thesis: particular graphs admit concrete analytical expressions

Very particular graphs: Boolean hypercubes

$\mathcal{B}_N = \mathbb{Z}_2^N$
$\mathcal{B}_N$: unweighted metric *Cayley* graph
$v = v_S = (\epsilon_1, \ldots, \epsilon_N)$, $S \subset \{1, \ldots, N\}$: $i \in S \Leftrightarrow \epsilon_i = 1$
$L = D - A$
$D = N\, I_N$
$A_{RS} = 1$ if $R \Delta S$ is a singleton



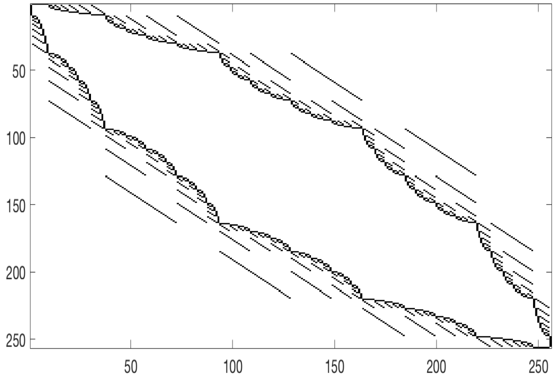
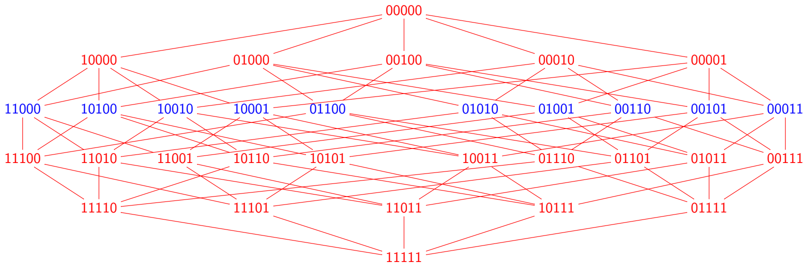


Figure: Adjacency matrix for $N = 8$ in dyadic lexicographic order.

$\mathcal{B}_5$ following dyadic lexicographic order

Σ_r: Hamming sphere of radius r: vertices with r one-bits



Why hypercubes

Historical use: Sampling
Known Fourier transform
Non-Euclidean geometry

Our thesis: particular graphs admit concrete analytical expressions
Accessible generalizations and restrictions: generalized hypercubes, partial cubes

Spatio–spectral limiting: Tsitsvero, Barbarossa, Di Lorenzo [2016]: relate properties of compositions QP and PQ on graphs to (sub)-sampling strategies for recovery of sparse vertex functions.

Sampling of bandlimited vertex functions was developed in the setting of hypercubes by Mansour et al in early 1990s in context of learning (sparse) Boolean functions.

Fourier transform on $\mathcal{B}_N$: Hadamard matrix

Lemma (Boolean Fourier transform)

Let $H_S(R) = (-1)^{|R \cap S|}$ *and* $L = L_{\mathcal{B}_\mathcal{N}}$ *as above. Then* H_S *is an eigenvector of* L *with eigenvalue* $2|S|$.

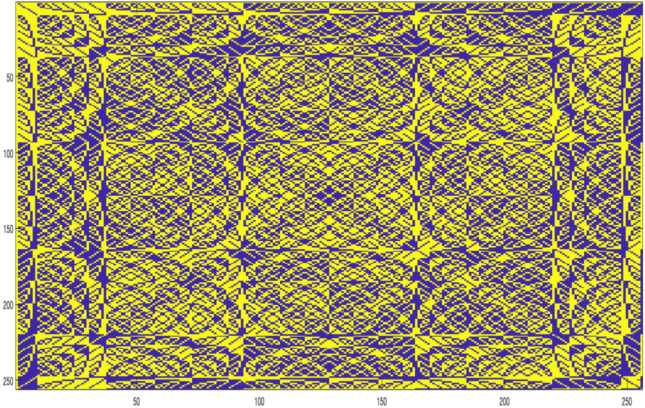


Figure: Hadamard (Fourier) matrix for $N = 8$ in dyadic lexicographic order.

Spatial and spectral limiting on $\mathcal{B}_N$

Space-limiting matrix $Q = Q_K$: $Q_{R,S} = \begin{cases} 1, & R = S \ \& \ |S| \le K \\ 0, & \text{else} \end{cases}$

Spectrum-limiting matrix $P = P_K$ by $P = \bar{H} Q \bar{H}$

Results and approach

Results: identify eigenvectors of spatio–spectral limiting PQP

Approach:

- Work in *spectral domain*: $QPQ = \bar{H}PQP\bar{H}$
- Identify salient *invariant subspaces* of QPQ
- These subspaces *factor*
- Reduce to *small matrix problem* on one of the factors
- *Numerical computation* via almost commuting operator and power method with a weight

Eigenspaces of spatio-spectral limiting on $\mathcal{B}_N$

A: adjacency matrix of $\mathcal{B}_N$ (dyadic lexicographic order)
$A = A_+ + A_-$: $A_- = A_+^T$; A_+: lower triangular
A_+ maps data on Σ_r to data on Σ_{r+1}: *outer adjacency*
A_- maps data on Σ_r to data on Σ_{r-1}: *inner adjacency*

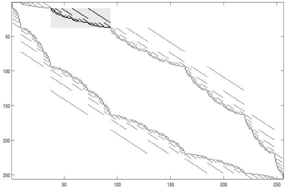


Figure: Highlighted: A_-, $\Sigma_3 \to \Sigma_2$

$$\ell^2(\Sigma_r) = A_+\ell^2(\Sigma_{r-1}) \oplus \mathcal{W}_r$$

$\mathcal{W}_r$: the orthogonal complement of $A_+\ell^2(\Sigma_{r-1})$ inside $\ell^2(\Sigma_r)$.

$$\ell^2(\Sigma_r) = A_+\ell^2(\Sigma_{r-1}) \oplus \mathcal{W}_r = \cdots = A_+^r\mathcal{W}_0 \oplus A_+^{r-1}\mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_r$$

Projection Matrix onto $\mathcal{W}_r$: columns form a Parseval frame

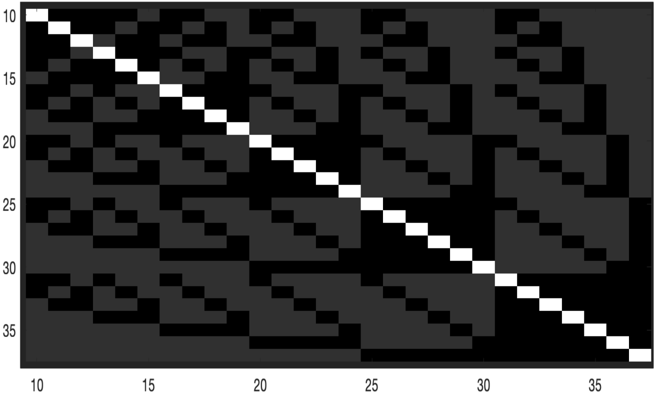


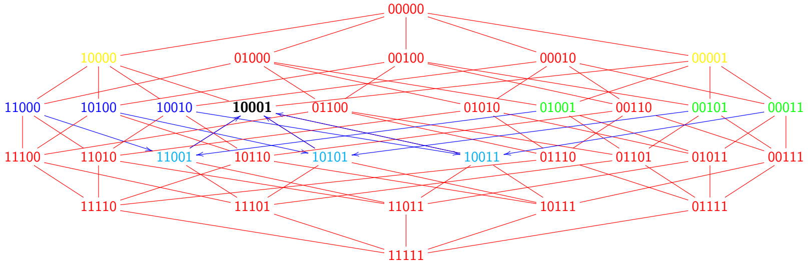
Figure: Matrix of projection onto $\mathcal{W}_r$, $N = 8$, $r = 3$.

Theorem

Let $V \in \mathcal{W}_r$ *and* k *such that* $r + k < N$. *Then*

$$\begin{aligned} A_- A_+^{k+1} V &= [(N-2r) + \cdots + (N - 2(r+k))]\, A_+^k V \\ &= (k+1)(N-2r-k) A_+^k V \\ &\equiv m(r,k) A_+^k V \end{aligned}$$

Base case ($k = 0$, $r = 2$)



Commutators of A_+ and A_-

$C = [A_-, A_+] = A_- A_+ - A_+ A_-$: commutator of A_+ and A_-.

Proposition

C is diagonal with $C_{RR} = N - 2|R|$.

Theorem follows from induction on k

Adjacency invariant subspaces

$V \in \mathcal{V}_r$ if $V = \sum_{k=0}^{N-r} c_k A_+^k W$, $W \in \mathcal{W}_r$

Lemma

A_+ and A_- map $\mathcal{V}_r$ to itself.

Corollary

A maps $\mathcal{V}_r$ to itself. Polynomials $p(A)$ preserve $\mathcal{V}_r$.

Proposition

The spectrum-limiting operator $P = P_K$ can be expressed as a polynomial $p(A)$ of degree N.

Proof.

$$p_k = \prod_{j=0, j\neq k}^{N} \frac{x-(N-2j)}{2(j-k)}; \qquad p(x) = \sum_{k=0}^{K} p_k$$

Then $P = p(A)$ as verified on Hadamard basis. □

P_K factors through $\mathcal{V}_r \simeq \mathcal{W}_r \otimes \mathbb{R}^{N-r+1}$

Matrix of Spectral limiting P_K on $\mathcal{V}_r$

$M^P_{(N,K,r)}$ of size $(N-r+1)$: represents P_K on $\mathcal{V}_r$

$$P(A_+^k W) = \sum_{\ell=0}^{N-r} M^P_{(N,K,r)}(k,\ell) A_+^\ell W, \quad (W \in \mathcal{W}_r)$$

$$PV = \sum_{k=0}^{N-r} d_k A_+^k W = \sum_{k=0}^{N-r} \sum_{\ell=0}^{N-r} M^P_{(N,K,r)}(k,\ell) c_\ell A_+^k W \quad (W \in \mathcal{W}_r)$$

Matrix of QPQ on $\mathcal{V}_r$

$M^{QPQ}_{(N,K,r)}$: $(K-r+1)$-principal minor of $M^{P}_{(N,K,r)}$.

$$QPQV = \sum_{k=0}^{K-r} d_k A_+^k W = \sum_{k=0}^{K-r} \sum_{\ell=0}^{K-r} M^{P}_{(N,K,r)}(k,\ell) c_\ell A_+^k W, \quad (W \in \mathcal{W}_r)$$

Corollary (Coefficient eigenvectors of QPQ)

If $\boldsymbol{c} = [c_0, \ldots c_{K-r}]^T$ is a λ-eigenvector of the principal minor $M^{QPQ}_{(N,K,r)}$ of size $(K - r + 1)$ of the matrix $M^{P}_{(N,K,r)}$ then $V = \sum_{k=0}^{K-r} c_k A_+^k W$, any $W \in \mathcal{W}_r$, is a λ-eigenvector of QPQ and $\bar{H}V$ is a λ-eigenvector of PQP.

Remark (Completeness)

Any eigenvector of QPQ is attached to one of the spaces $\mathcal{V}_r$

$M^P_{(N,K,r)} = p(M_A)$

A_+ as *right shift* on 2nd component of $\mathcal{V}_r \simeq \mathcal{W}_r \times \mathbb{R}^{N+1-r}$:
$A_+ : (c_0 + c_1 A_+ + \dots)W \mapsto (c_0 A_+ + c_1 A_+^2 + \dots)W$
A_- as *multiplicate left shift* on $\mathcal{V}_r$:
Matrices M_{A_+}, M_{A_-} on $\mathbb{R}^{N-r+1}$:

$$M_{A_+} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \quad M_{A_-} = \begin{pmatrix} 0 & m(r,0) & 0 & \cdots & 0 \\ 0 & 0 & m(r,1) & 0 & \cdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \cdots & 0 & m(r, K+1-r) \\ 0 & & \cdots & & 0 \end{pmatrix}$$

$M_A = M_{A_+} + M_{A_-}$

Matrix of $M^P_{(N,K,r)}$ of P by substituting M_A for A in $P = p(A)$

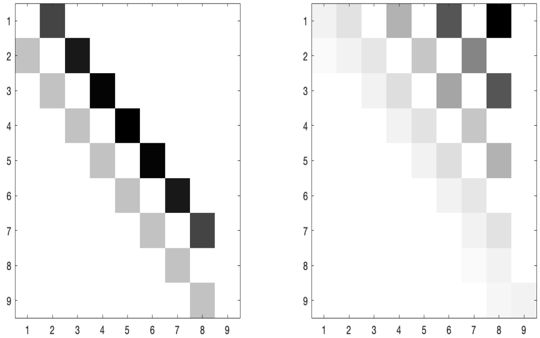


Figure: Matrices M_A and M^P , $N = 9,\ K = 4,\ r = 1.$ (log scale)

Problem: large numbers

Inner product on $\mathcal{V}_r$

$$\begin{aligned} \langle A_+^k W_1,\, A_+^k W_2\rangle &= w(r,k)\langle W_1, W_2\rangle \\ w(r,k) &= \prod_{j=0}^{k-1} m(r,j) \end{aligned}$$

$$\Big\langle \sum_{k=0}^{N-r} c_k A_+^k W_1, \sum_{k=0}^{N-r} d_k A_+^k W_2 \Big\rangle = \langle W_1, W_2\rangle \underbrace{\sum_{k=0}^{N-r} c_k d_k w(r,k)}_{\langle \boldsymbol{c}, \boldsymbol{d}\rangle_{w_r}}$$

Proposition

Coefficient eigenvectors of $M_{(N,K,r)}^{QPQ}$ *are orthogonal wrt weight* $[w(r,0),\ldots,w(r,K+1-r)]$

Boolean analogue of prolate differential operator

$$\text{(BDO)} \qquad D(\alpha I - T^2)D + \beta T^2 \,.$$

T: diagonal, sqrts of eigenvalues of L
$D = \bar{H} T \bar{H}$, $\bar{H} = 2^{-N/2} H$
$D^2 = L$.

Proposition

If $\beta = 2\sqrt{K(K+1)}$ then BDO commutes with P_K. Equivalently, the conjugation of BDO by H commutes with Q_K.

BUT BDO *does not* commute with Q_K

Matrix of HBDO on $\mathcal{V}_r$

$$M^{\mathrm{HBDO}}(k,\ell) = \begin{cases} (2\sqrt{\ell(\ell-1)} - \beta) m(r, \ell-1-r);\ \ k = \ell - 1 \geq r \\ 2\ell(\alpha - N) + \beta N; k = \ell \geq r \\ 2\sqrt{\ell(\ell+1)} - \beta; k = \ell + 1; r \leq \ell < N \\ 0,\ \mathrm{else}\,. \end{cases}$$

If $\alpha = \beta = 2\sqrt{K(K-1)}$:

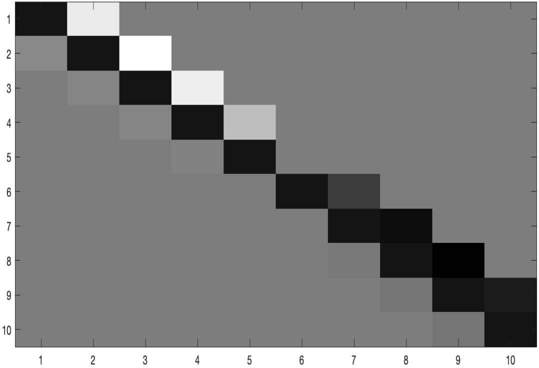


Figure: Matrix M^{HBDO}, $N = 9$, $K = 4$

Theorem ($\mathcal{V}_r$ is HBDO-invariant)

If $V \in \mathcal{V}_r$, $V = \sum_{k=0}^{N-r} c_k A_+^k W$, *then* $\mathrm{HBDO}\, V = \sum_{k=0}^{N-r} d_k A_+^k W$ *where* $\mathbf{d} = M^{\mathrm{HBDO}}\mathbf{c}$ *where* $\mathbf{c} = [c_0, \ldots, c_{N-r}]^T$.

- Entries of M^{QPQ} can exceed `maxint` for moderate sized N
- M^{HBDO} is tridiagonal and eigendecomposition is fine
- HBDO and QPQ *almost commute*
- Eigenvectors of HBDO as seeds for weighted power method

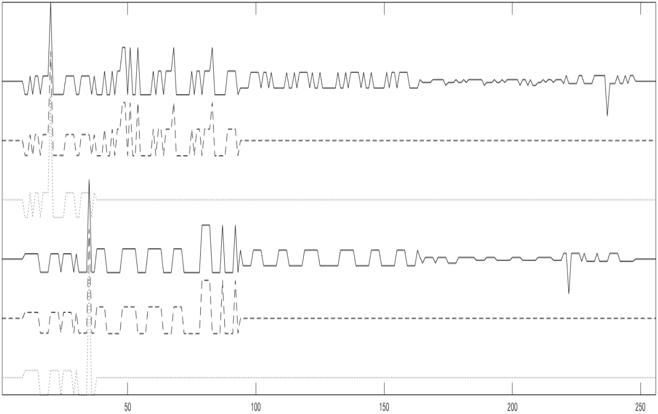


Figure: Eigenvectors of PQP, $N = 8$, $K = 3$, $r = 2$.
Dotted curves: two different elements W of $\mathcal{W}_r$
Dashed curves: corresponding eigenvectors V of QPQ
Solid curves: Eigenvector HV of PQP for eigenvector V of QPQ

Algorithm 1 Adapted power method eigen-decomposition of QPQ

Inputs: N, $K \in \{0, \ldots, N\}$, $r \in \{0, \ldots, K\}$
Compute coefficient matrix $M^{\mathrm{HBDO}}_{(N,K,r)}$ of $2^{-N} H \mathrm{BDO} H$ on $\mathcal{V}_r$
Compute eigenvectors $\boldsymbol{c}$ of $M^{\mathrm{HBDO}}_{(N,K,r)}$
Sort eigenvectors $\boldsymbol{c}^k = [c^k_0, \ldots, c^k_{N-r}]$: $c^k_{K+1} = \cdots = c^k_{N-r} = 0$
Sub M_A for A: Compute $M^{\mathrm{QPQ}}_{(N,K,r)}$: principal minor of $M^P_{(N,K,r)}$
for $k = 0$ to $K - r$ **do**
  while stopping criteria = False **do**
    Apply $M^{\mathrm{QPQ}}_{(N,K,r)}$ *factor-wise* to $\boldsymbol{d}^k$
    Project output onto $(\mathrm{span}\{\boldsymbol{d}^0, \ldots, \boldsymbol{d}^{k-1}\})^\perp$ wrt $\langle \cdot, \cdot \rangle_w$
    Update $\boldsymbol{d}^k$ = normalized projection (wrt $\| \cdot \|_w$)
  end while
end for
Return: approximate coefficient eigenvectors $\boldsymbol{d}^0$, ..., $\boldsymbol{d}^{K-r}$ of M^{QPQ}, the matrix of QPQ acting on $\mathcal{V}_r$.

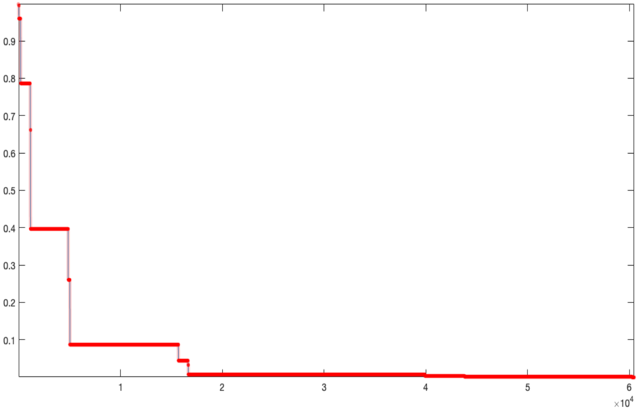


Figure: Eigenvalues of PQP with multiplicity (60460), $N = 20$, $K = 6$.

HAPPY BIRTHDAY JOHNNY!!

https://www.youtube.com/channel/UCKChX5APWWHOLwu4CVestDA/featured?disable_polymer=1