Multivariate Multifractal analysis

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Based on joint works with

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Stéphane Roux, Stéphane Seuret, Herwig Wendt

Jubilee of Fourier Analysis and Applications:

A Conference Celebrating

John Benedetto's 80th Birthday

Norbert Wiener Center University of Maryland September 19-21 2019


IL CIOCCO

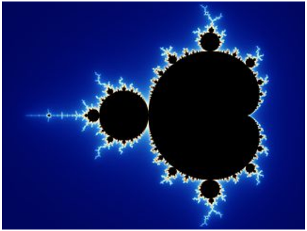

IL CIOCCO

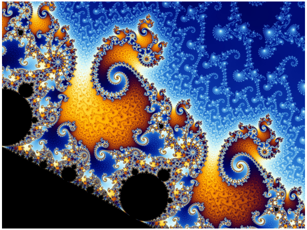

SAN BENEDETTO
OLIGOMINERALE
NATURALE

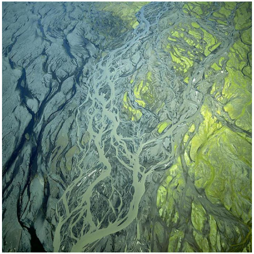
"Don't judge each day by the harvest you reap
but by the seeds that you plant"

Mark Twain

What is fractal geometry ?







Geometry dealing with irregular sets

How can one quantify this irregularity ?

Classification of fractals sets

Box dimension

Let $N(\varepsilon)$ be the minimal number of balls of radius ε needed to cover the set A

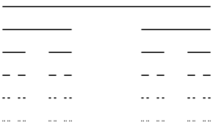
$$N(\varepsilon) \sim \varepsilon^{-dim_B(A)}$$

Advantage :

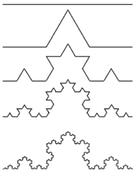
Numerically computable through log-log plot regressions : $\log(N(\varepsilon))$ is plotted as a function of $\log(\varepsilon)$

The slope yields the dimension

$$dim_B = \frac{\log 2}{\log 3}$$



Triadic Cantor set

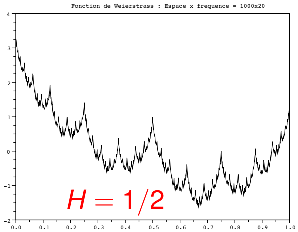


$$dim_B = \frac{\log 4}{\log 3}$$

Van Koch curve

Everywhere irregular functions

The existence of everywhere irregular functions was doubted by mathematicians untill Weierstrass proposed his example in 1872



$$W_H(x) = \sum_{j=0}^{+\infty} 2^{-Hj} \cos(2^j x)$$

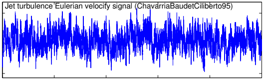
$$0 < H < 1$$

C. Hermite : I turn my back with fright and horror to this appalling wound : Functions that have no derivative

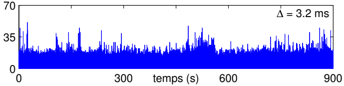
H. Poincaré called such functions “monsters”

Everywhere irregular functions

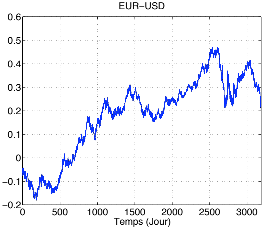
Jean Perrin, in his book, “Les atomes” (1913), stated that irregular (nowhere differentiable) functions, far from being exceptional, are common in natural phenomena



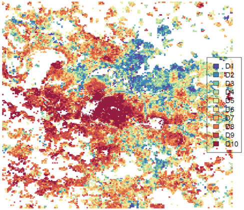
Fully developed turbulence



Internet trafic



Euro vs Dollar (2001-2009)



Density of population

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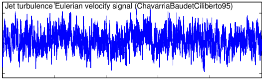

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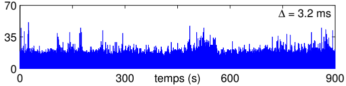

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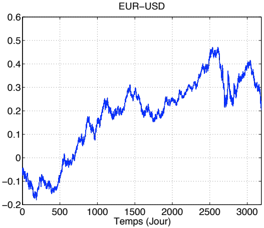

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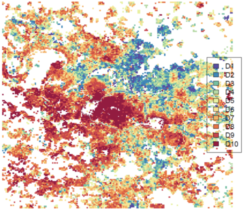


Density of population

Multifractal analysis studies classification parameters for data (functions, measures, signals, images) based on regularity

Orthonormal wavelet bases

An orthonormal wavelet basis on $\mathbb{R}^d$

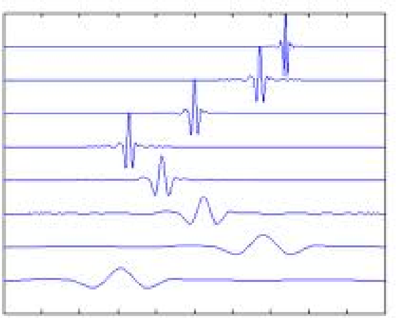
is generated by $2^d - 1$ smooth,

well localized, oscillating

functions ψ^i such that the

$$2^{dj/2}\psi^i(2^j x - k),$$
$$i = 1, \cdots 2^d - 1, \; j, k \in \mathbb{Z}^d$$

form an orthonormal basis of $L^2(\mathbb{R}^d)$



Credit to : http ://www.kfs.oeaw.ac.at/content/blogcategory/0/502/lang,8859-1/

Why use wavelet bases ?

- Fast algorithms
- Sparse representations
- Characterization of regularity (global and pointwise)

$$\int \psi(x)dx = \int x\psi(x)dx = \cdots = \int x^N \psi(x)dx = 0$$

$\Longrightarrow$ Wavelet analysis is blind to superimposed polynomial and (more generally) smooth trends

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Wavelets translate hard problems on functions on (more) simple problems on sequences

Wavelet structure functions

Notations :

Dyadic cubes : $\lambda = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_d}{2^j}, \frac{k_d+1}{2^j}\right)$

Wavelet coefficients : $c_\lambda = 2^{dj} \int_{\mathbb{R}^d} \int f(x,y)\, \psi^i \left(2^j x - k\right)\; dx$

Dyadic cubes at scale j : $\Lambda_j = \{\lambda : \quad |\lambda| = 2^{-j}\}$

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$$\forall p > 0, \qquad S_f(p,j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p$$

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Wavelet scaling function :

$$\forall p > 0, \qquad S_f(p,j) \sim 2^{-\zeta_f(p) j} \quad \text{when} \quad j \to +\infty$$

Wavelet scaling function

$$\Longrightarrow \qquad \zeta_f(p) = p \cdot \sup\{s : f \in L^{p,s}\} = p \cdot \sup\{s : f \in B_p^{s,\infty}\}$$

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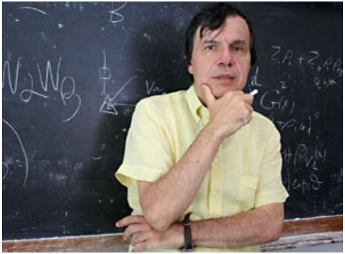
Advantages of using the wavelet scaling function for classification :

- Effectively computable on experimental data through log-log plot regressions with respect to the scale parameter
- Independent of the (smooth enough) wavelet basis
- Invariant under the addition of polynomials or (smooth enough) trends
- “deformation invariant” (i.e. under a smooth change of coordinates)
- deterministic for large classes of stochastic processes

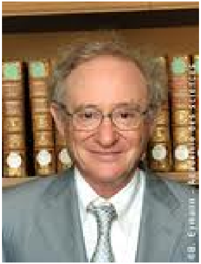
Limitations of wavelet structure functions

Classification only based on structure functions proved insufficient in several occurrences (turbulence, ...)

This motivated new developments based on seminal ideas introduced by Uriel Frisch and Georgio Parisi, and led to the construction of new structure functions



Giorgio Parisi



Uriel Frisch

Pointwise regularity

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$$f(x) = f(x_0) + o(1)$$

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Extension to non-integer orders of derivation :

Let f be a locally bounded function $\mathbb{R}^d \to \mathbb{R}$ and $x_0 \in \mathbb{R}^d$; $f \in C^\alpha(x_0)$ if there exist $C > 0$ and a polynomial P of degree less than α such that

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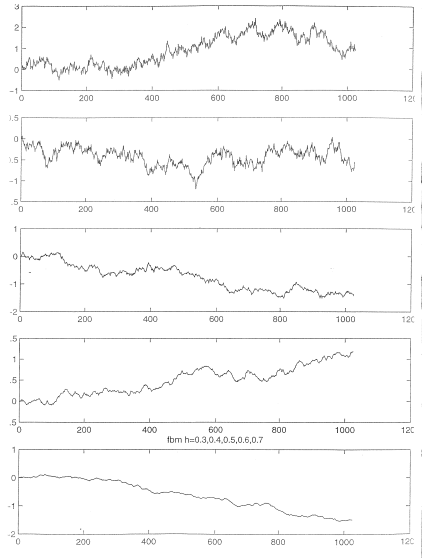
The Hölder exponent of f at x_0 is

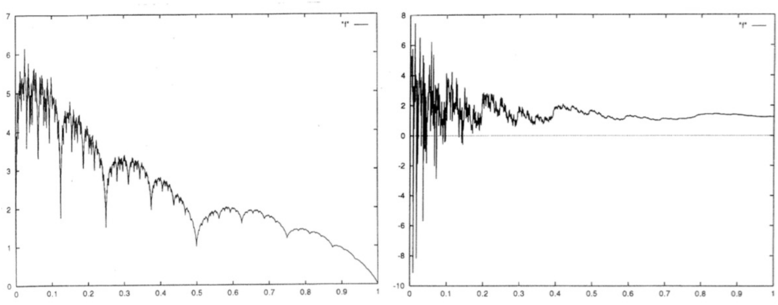
$$h_f(x_0) = \sup\{\alpha : \quad f \in C^\alpha(x_0)\}$$

Functions with constant Hölder exponents



Fractional Brownian motions with Hölder exponents 0.3, 0.4, 0.5, 0.6 and 0.7

Functions with varying Hölder exponent



$$h_f(x) = x$$

Constructions obtained by K. Daoudy, J. Lévy-Véhel and Y. Meyer

Wavelet leaders

Idea : In the scaling function, replace increments by quantities which encapsulate information on pointwise regularity

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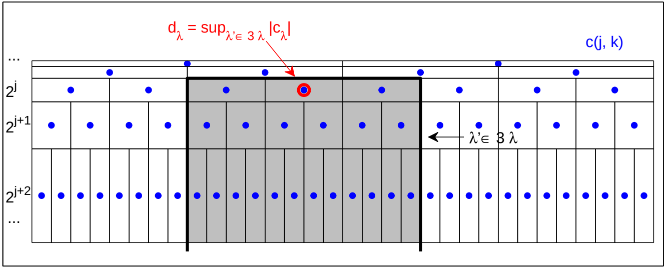
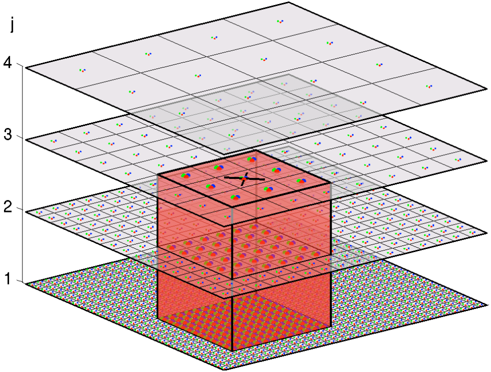
Let λ be a dyadic cube ; 3λ is the cube of same center and three times wider

Let f be a locally bounded function ; the wavelet leaders of f are the quantities

$$d_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|$$



Computation of 2D wavelet leaders



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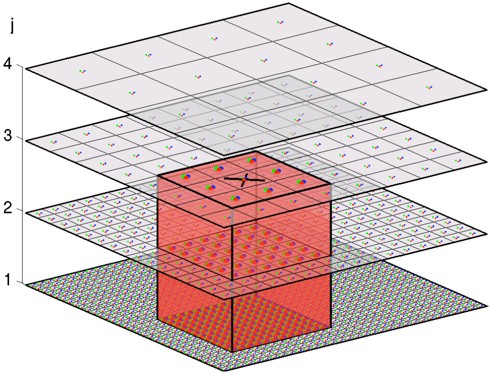


Wavelet leaders allow to estimate pointwise Hölder exponents : Let $\lambda_j(x_0)$ denote the dyadic cube of width 2^{-j} which contains x_0

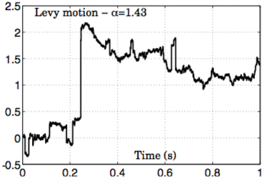
$$d_{\lambda_j(x_0)} = \sup_{\lambda' \subset 3\lambda_j(x_0)} |c_{\lambda'}|$$

Theorem : If $H_{min} > 0$, then $\forall x_0 \in \mathbb{R}^d$: $h_f(x_0) = \liminf_{j \to +\infty} \dfrac{\log(d_{\lambda_j(x_0)})}{\log(2^{-j})}$

Difficulty to use directly the pointwise regularity exponent for classification

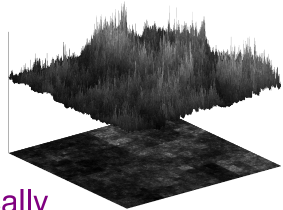
For classical models, such exponents are extremely erratic

Lévy processes



The function *h* is random and everywhere discontinuous

multiplicative cascades



$\Longrightarrow$ Impossible to estimate numerically

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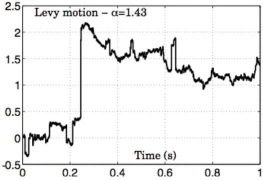

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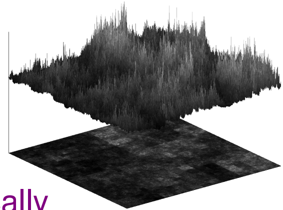

$\Longrightarrow$ Impossible to estimate numerically

Goal : Recover some information on $h(x)$ from averaged quantities which would be :

- numerically computable by log-log plot regressions
- deterministic (independent of the sample path)

Back to scaling functions

"Improve" the scaling function by using quantities that incorporate pointwise regularity information

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Λ_j denotes the set of dyadic cubes of width 2^{-j}

Wavelet scaling function

$$2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p)j}$$

Leader scaling function

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Advantages :

- Same as the wavelet scaling function $+$
- $\eta_f(p)$ is also defined for $p < 0$
- $\eta_f(p)$ encapsulates information on the inter-scales correlations of wavelet coefficients

Heuristic derivation of the multifractal formalism

$E_f(H)$ is the set of points where $h_f(x) = H$
$\mathcal{D}_f(H)$ denotes its Hausdorff dimension (i.e. the multifractal spectrum)

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Leader structure function : $T_f(p,j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p \sim 2^{-\eta_f(p)j}$

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Estimation of the contribution to $T_f(p,j)$ of the cubes of length 2^{-j} containing a point where the Hölder exponent takes the value H :

On such a cube $|d_\lambda| \sim 2^{-Hj}$ and there are $\sim 2^{\mathcal{D}_f(H)j}$ such cubes
Thus the contribution is

$$\sim 2^{-dj} \cdot (2^{-Hj})^p \cdot \left(2^{-j}\right)^{\mathcal{D}_f(H)} = (2^{-j})^{d+Hp-\mathcal{D}_f(H)}$$

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In the limit $j \to +\infty$, the main contribution comes from the smallest exponent, so that : $\eta_f(p) = \inf_H (d + Hp - \mathcal{D}_f(H))$

Thus η_f is expected to be the Legendre transform of $\mathcal{D}_f$

The Leader Legendre Spectrum

If $\mathcal{D}_f$ is concave, it should be recovered from η_f through an inverse Legendre transform

$$\mathcal{D}_f(H) = \inf_{p \in \mathbb{R}} \left(d + Hp - \eta_f(p) \right)$$

The Leader Legendre Spectrum is

$$\mathcal{L}_f(H) = \inf_{p \in \mathbb{R}} \left(d + Hp - \eta_f(p) \right)$$

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Theorem : If $f \in C^\varepsilon(\mathbb{R}^d)$ for an $\varepsilon > 0$ then

$$\forall H \in \mathbb{R}, \ \ \mathcal{D}_f(H) \leq L_f(H)$$

When $\mathcal{D}_f(H) = L_f(H)$ the multifractal formalism is satisfied

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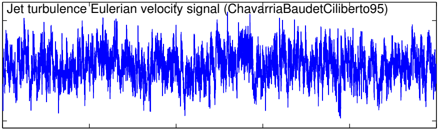
There are two kinds of validity theorems :

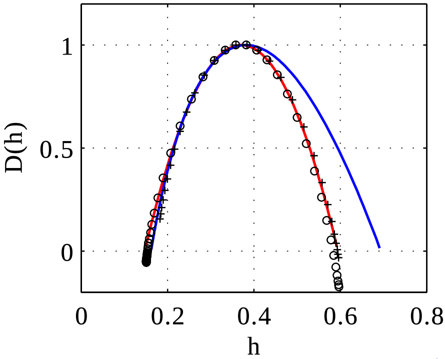
- Generic results (Baire and prevalence)
- Particular models

Model refutation : Fully developed turbulence

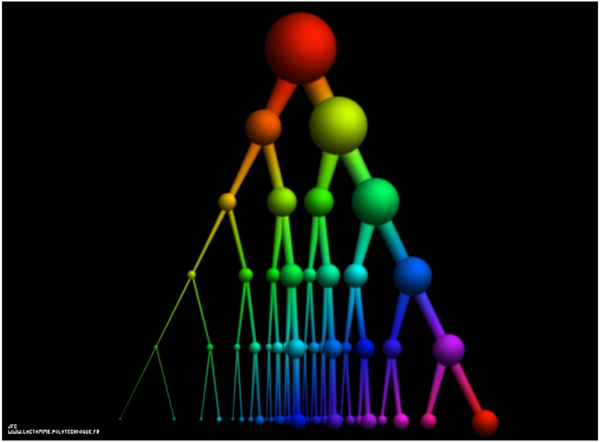
(joint work with Bruno Lashermes)



Log-normal vs. Log-Poisson model



Cascade models : Binomial cascade



Courtesy of Jean-François Colonna, LACTAMME

Construction of a measure μ on the interval $[0, 1]$: A quantity of total mass 1 of sand is poured at the top

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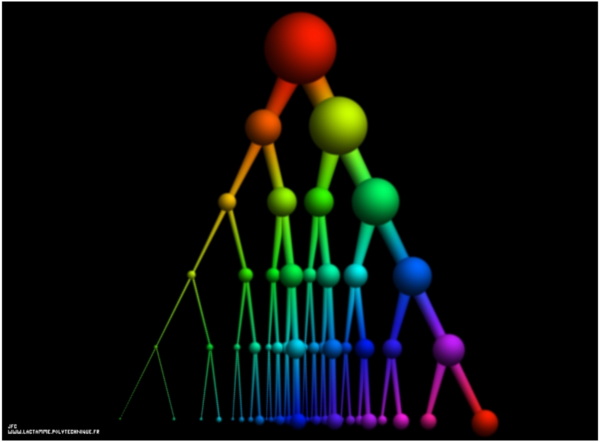


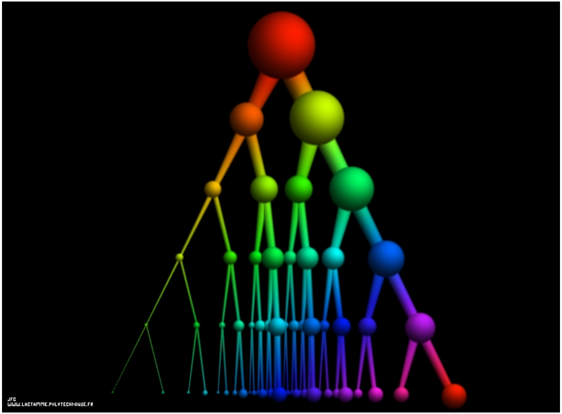
Courtesy of Jean-François Colonna, LACTAMME

Construction of a measure μ on the interval $[0, 1]$: A quantity of total mass 1 of sand is poured at the top At each node,

1/4 of the remaining sand falls on the left and 3/4 on the right

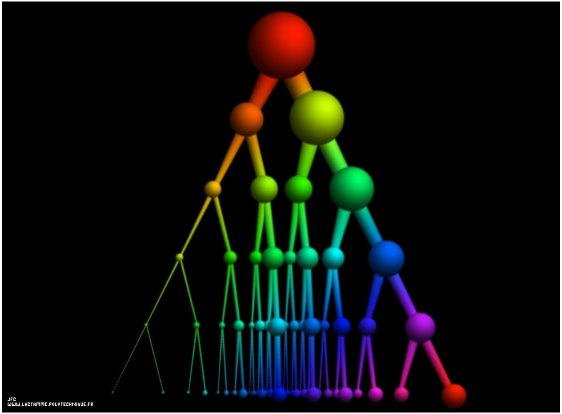
$\mu(I) =$ quantity of sand falling inside I

Cascade models : Binomial cascade



Quantity of sand falling inside intervals of length 2^{-n} :

Cascade models : Binomial cascade



Quantity of sand falling inside intervals of length 2^{-n} :

$$\begin{cases} \text{far left :} & \mu(I) = \left(\frac{1}{4}\right)^n = |I|^2 \\ \text{far right :} & \mu(I) = \left(\frac{3}{4}\right)^n = |I|^{\log(4/3)/\log 2} \\ \text{average :} & \mu(I) = \left(\frac{1}{4}\right)^{n/2}\left(\frac{3}{4}\right)^{n/2} = |I|^{\log(4/\sqrt{3})/\log 2} \end{cases}$$

Exponents fluctuate from point to point

Cascade models : Binomial cascade

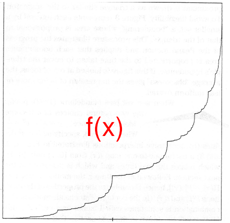
Repartition function

of the measure μ :

$f(x) := \mu([0, x])$

= amount of sand

falling in $[0, x]$



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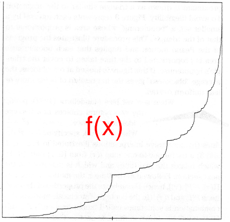

$$f(x+\delta) - f(x) = \mu([x, x+\delta]) \sim \delta^{h(x)}$$

$$h_f(x) \in \left[\frac{\log(4/3)}{\log 2},\ 2\right]$$

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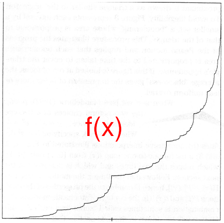

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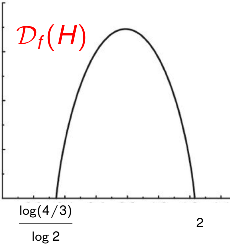
f is a multifractal function



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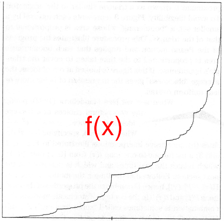

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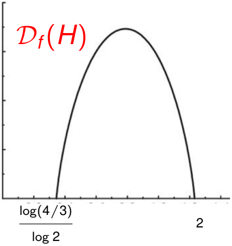
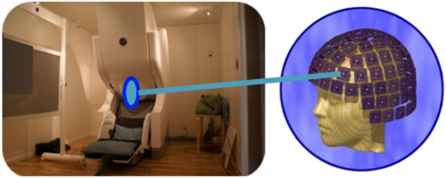


Rule of thumb :

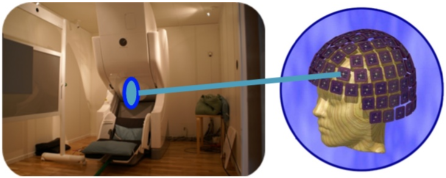
The multifractal formalism holds for “homogeneous data”

Why multivariate analysis ?



MEG recordings

Why multivariate analysis ?



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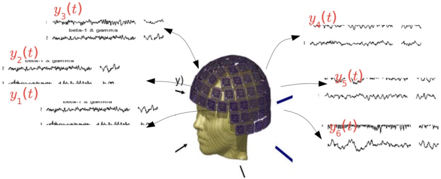

A collection of signals are recorded simultaneously

Multivariate analysis

Regularity exponents $h_1(x), \cdots, h_m(x)$ are associated with each signal $y_i(t)$

Each exponent is associated with a *multiresolution quantity* d^i_λ through the formula

$$\text{pour } i = 1, 2, \quad \forall x_0 \in \mathbb{R}^d \qquad h_i(x_0) = \liminf_{j \to +\infty} \frac{\log\left(d^i_{\lambda_j(x_0)}\right)}{\log(2^{-j})}$$

Let $\quad E(H_1, H_2) = E(H_1) \cap E(H_2)$

$E(H_1, H_2)$ is the set of points where $h_1(x) = H_1$ and $h_2(x) = H_2$

The joint multifractal spectrum is (for $m = 2$)

$$\mathcal{D}(H_1, H_2) = dim\,(E(H_1, H_2))$$

The multivariate multifractal formalism

The multivariate structure function is

$$T_f(p,q,j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} \left(d_\lambda^1\right)^p \left(d_\lambda^2\right)^q$$

The multivariate scaling function is

$$\forall p, q \in \mathbb{R}, \quad \eta(p,q) = \liminf_{j \to +\infty} \frac{\log(T_f(p,q,j))}{\log(2^{-j})}$$

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The multivariate Legendre spectrum is

$$\mathcal{L}(H_1,H_2) = \inf_{(p,q) \in \mathbb{R}^2} \left(d + H_1 p + H_2 q - \eta(p,q)\right)$$

The multivariate multifractal formalism holds if

$$\mathcal{D}(H_1,H_2) = \mathcal{L}(H_1,H_2)$$

Heuristic derivation of the multifractal formalism

$E(H_1, H_2)$ is the set of points where $h_1(x) = H_1$ and $h_2(x) = H_2$

$\mathcal{D}(H_1, H_2)$ is the Hausdorff dimension of $E(H_1, H_2)$

Structure functions : $$T_f(p, q, j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} \left(d_\lambda^1\right)^p \left(d_\lambda^2\right)^q$$

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When $j \to +\infty$, the main contribution is given by the smallest exponent, so that
$$\eta(p,q) = \inf_H \left(d + H_1 p + H_2 q - \mathcal{D}(H_1,H_2)\right)$$

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The bivariate multifractal spectrum is recovered by an inverse Legendre transform

$$\mathcal{D}(H_1,H_2) = \inf_{p,q} \left(d + H_1 p + H_2 q - \eta(p,q)\right)$$

Inspecting the formula

An extreme case : $f_1(x) = f_2(x)$ and $h_1(x) = h_2(x)$

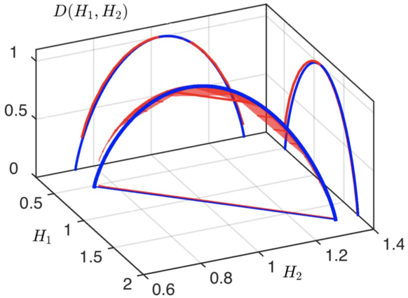
Both spectra are carried by the diagonal

$$\mathcal{D}(H_1, H_2) \begin{array}{ll} = \mathcal{D}(H_1) & \text{if } H_1 = H_2 \\ = -\infty & \text{else} \end{array}$$

$$\mathcal{L}(H_1, H_2) \begin{array}{ll} = \mathcal{L}(H_1) & \text{if } H_1 = H_2 \\ = -\infty & \text{else} \end{array}$$

Red : Theoretical multifractal spectrum

Blue : Estimated Legendre spectrum



Multivariate multifractal analysis of a binomial cascade with itself

Independent processes

Assumption : Wavelet leaders are stationary with short range correlations only

$$T_f(p,q,j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} \left(d_\lambda^1\right)^p \left(d_\lambda^2\right)^q \sim \mathbb{E}\left(\left(d_\lambda^1\right)^p \left(d_\lambda^2\right)^q\right)$$

If the signals are independent, then

$$T_f(p,q,j) = \mathbb{E}\left(\left(d_\lambda^1\right)^p\right) \mathbb{E}\left(\left(d_\lambda^2\right)^q\right)$$

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$$\mathcal{L}(H_1, H_2) = \mathcal{L}(H_1) + \mathcal{L}(H_1) - d$$

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This is similar to the codimension formula for intersections

$$\dim_H(A \cap B) = \dim_H(A) + \dim_H(B) - d$$

codimensions add up

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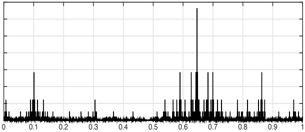
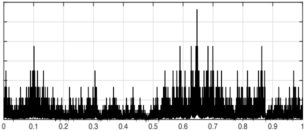
Generically true for fractal sets (P. Mattila et al.) under "reasonable" assumptions

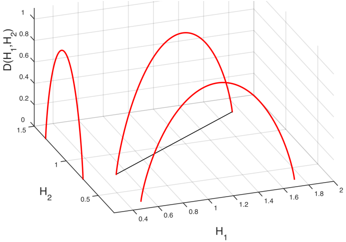
This is usually not true for the sets $E(H_1, H_2) = E(H_1) \cap E(H_2)$

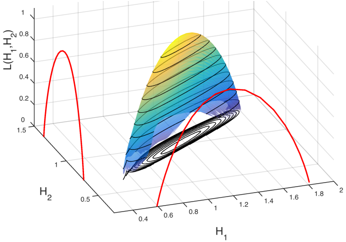
Binomial cascades of parameters p and q

p and q on the same side of $1/2$





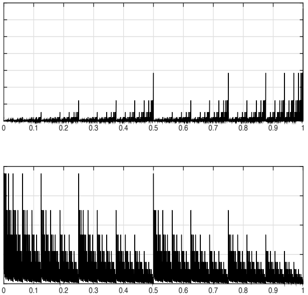


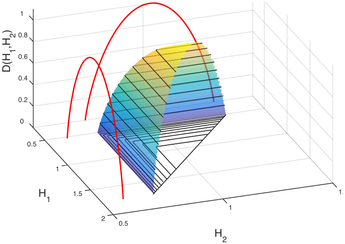


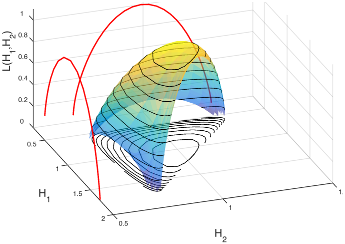
Theoretical Legendre spectrum vs. its estimation on a sample path

Binomial cascades of parameters *p* and *q*

p and *q* are
on opposite sides
of $1/2$



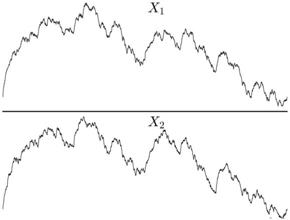


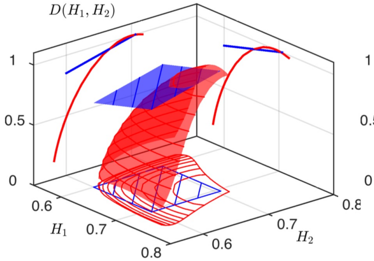


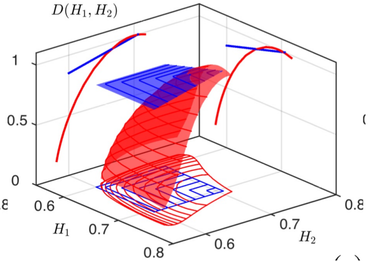
Theoretical Legendre spectrum vs. its estimation on a sample path

Independent lacunary wavelet series

Sample paths of
two independent
lacunary wavelet series







theoretical multifractal spectra
$\mathcal{D}(H)$ in blue
$\mathcal{L}(H)$ in red

computation on a sample path
theoretical $\mathcal{D}(H)$ recalled in blue
$\mathcal{L}(H)$ in red

Why is the multivariate multifractal formalism so wrong ?

Multifractal spectra of many models and processes do not follow the codimension formula

$$\mathcal{D}(H_1, H_2) = \begin{cases} \mathcal{D}(H_1) + \mathcal{D}(H_2) - d & \text{if} \quad \mathcal{D}(H_1) + \mathcal{D}(H_2) - d \geq 0 \\ -\infty & \text{else} \end{cases}$$

but instead the large intersection formula

$$\mathcal{D}(H_1, H_2) = \min(\mathcal{D}(H_1), \mathcal{D}(H_2))$$

Whereas, the Legendre spectrum of independent processes follows the codimension formula

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Whereas, the Legendre spectrum of independent processes follows the codimension formula

Generic results of validity of the multivariate multifractal formalism have been proved in products of function spaces (Mourad Ben Slimane et al.)

Intuitions and questions

- Univariate miracle : The leader structure functions $T_{p,j} = 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p \sim 2^{-\eta_f(p)j}$ simultaneously have a function space and a probabilistic interpretation, whereas the multivariate structure functions $S_{p,q,j} = 2^{-dj} \sum_{\lambda \in \Lambda_j} \left(d^1_\lambda\right)^p \left(d^2_\lambda\right)^q$ have no function space interpretation and have a probabilistic interpretation only in the independent case
- Are there "natural" function spaces which "encode" some correlation between wavelet leaders ?
- The multivariate structure functions do not take into account cross-scale correlations between wavelet leaders
- Which information does the multivariate structure functions yield ?
- Work out more examples to get some intuition !
- What is John staring at ?

A few references

- Multivariate multifractal analysis, *S. Jaffard, S. Seuret, H. Wendt, R. Leonarduzzi, S. Roux, P. Abry,* Applied and Computational Harmonic Analysis, Vol. 46, N. 3, May 2019, pp. 653–663
- Multifractal Characterization for Bivariate Data, *R. Leonarduzzi, P. Abry, S. G. Roux, H. Wendt, S. Jaffard, S. Seuret,*, European Signal Processing Conference (EUSIPCO), Rome, Italy, Sept. 2018.
- Multifractal formalisms for multivariate analysis, *S. Jaffard, S. Seuret, H. Wendt, R. Leonarduzzi, P. Abry,* To appear Proceedings Royal Society A
- A multivariate multifractal analysis of lacunary wavelet series. *P. Abry, R. Leonarduzzi, H. Wendt, S. Jaffard, S. Seuret,* Proc. of CAMSAP 2019

Thank you

for your attention !