Finite Frames and Optimal Subspace Packings

Matthew Fickus

Department of Mathematics and Statistics
Air Force Institute of Technology
Wright-Patterson Air Force Base, Ohio

September 20, 2019

Party like it's 1999...

A Research Problem

In the fall of 2000, inspired by a talk by Ed Saff at a conference in Bommerholz and a follow-up question by Hans Feichtinger, John asked me the following question (paraphrased):

> *How is the problem of equally-distributing points on a sphere related to finite unit norm tight frames?*

This talk is the 2019 progress update.

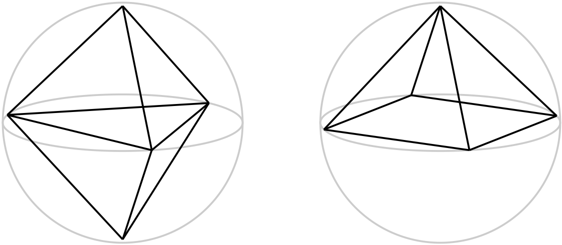
Optimal Packings
on
Spheres

Spherical equidistribution: Thomson vs. Tammes

Over all sets of N unit vectors $\{\mathbf{x}_n\}_{n=1}^N$ in $\mathbb{R}^D$, we can try to:

- minimize $\displaystyle\sum_{n=1}^{N}\sum_{\substack{n'=1\\n'\neq n}}^{N}\frac{1}{\|\mathbf{x}_n-\mathbf{x}_{n'}\|}$ (**Thomson, 1904**)
- maximize $\displaystyle\min_{n\neq n'}\|\mathbf{x}_n-\mathbf{x}_{n'}\|$ (**Tammes, 1930**)

For example, when $N=5$, $D=3$:



Solving Tammes in the Simplest Case

Theorem: [Rankin 55] When $N \leq D+1$, every solution to Tammes problem is a N-vector regular simplex.

Proof: For any unit vectors $\{\mathbf{x}_n\}_{n=1}^N$ in $\mathbb{R}^D$,

$$\|\mathbf{x}_n - \mathbf{x}_{n'}\|^2 = 2(1 - \langle \mathbf{x}_n, \mathbf{x}_{n'} \rangle).$$

Thus, $\underset{\{\mathbf{x}_n\}}{\operatorname{argmax}} \min_{n \neq n'} \|\mathbf{x}_n - \mathbf{x}_{n'}\| = \underset{\{\mathbf{x}_n\}}{\operatorname{argmin}} \max_{n \neq n'} \langle \mathbf{x}_n, \mathbf{x}_{n'} \rangle$. Also,

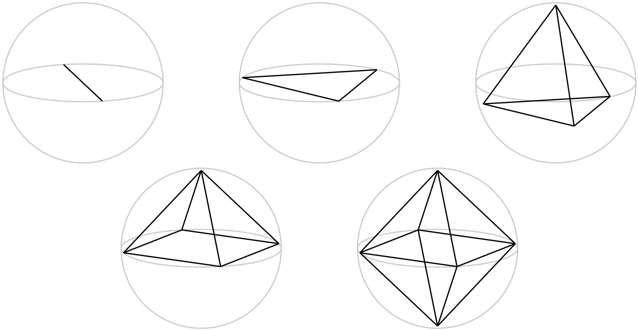
$$0 \leq \left\| \sum_{n=1}^N \mathbf{x}_n \right\|^2 = \sum_{n=1}^N \sum_{n'=1}^N \langle \mathbf{x}_n, \mathbf{x}_{n'} \rangle \leq N + N(N-1) \max_{n \neq n'} \langle \mathbf{x}_n, \mathbf{x}_{n'} \rangle.$$

Equality only holds $\Leftrightarrow \sum_{n=1}^N \mathbf{x}_n = \mathbf{0}$ and $\langle \mathbf{x}_n, \mathbf{x}_{n'} \rangle$ is constant over all $n \neq n'$.

Solving Tammes in the Next Simplest Case

Theorem: [Rankin 55] $\max_{n \neq n'} \langle \mathbf{x}_n, \mathbf{x}_{n'} \rangle \geq 0$ when $N \geq D + 2$.
Moreover, for $N \leq 2D$, this bound can be achieved.

Example: $D = 3$, $N = 2, 3, 4, 5, 6$:



Finite Unit-Norm Tight Frames

Notation

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. We usually regard N vectors $\{\varphi_n\}_{n=1}^N$ in $\mathbb{F}^D$ as the columns of a $D \times N$ matrix

$$\mathbf{\Phi} = \left[\, \varphi_1 \, \cdots \, \varphi_N \,\right].$$

Multiplying $\mathbf{\Phi}$ by its $N \times D$ conjugate-transpose $\mathbf{\Phi}^*$ gives its

- $N \times N$ **Gram matrix** $\mathbf{\Phi}^*\mathbf{\Phi} = \begin{bmatrix} \langle \varphi_1, \varphi_1 \rangle & \cdots & \langle \varphi_1, \varphi_N \rangle \\ \vdots & \ddots & \vdots \\ \langle \varphi_N, \varphi_1 \rangle & \cdots & \langle \varphi_N, \varphi_N \rangle \end{bmatrix}$
- $D \times D$ **frame operator** $\mathbf{\Phi}\mathbf{\Phi}^* = \displaystyle\sum_{n=1}^{N} \varphi_n \varphi_n^*$

In this talk, every φ_n is unit-norm, meaning the diagonal of $\mathbf{\Phi}^*\mathbf{\Phi}$ is all ones and $\mathbf{\Phi}\mathbf{\Phi}^*$ is a sum of rank-one projections.

Orthonormal Bases (ONBs)

Fact: If $\{\varphi_n\}_{n=1}^N$ is an ONB for $\mathbb{F}^N$ then $\mathbf{\Phi}$ is square and satisfies $\mathbf{\Phi}^*\mathbf{\Phi} = \mathbf{I}$. Thus, $\mathbf{\Phi}^* = \mathbf{\Phi}^{-1}$ and so we also have

$$\mathbf{\Phi}\mathbf{\Phi}^* = \mathbf{I}, \quad \text{i.e.,} \quad \mathbf{x} = \mathbf{\Phi}\mathbf{\Phi}^*\mathbf{x} = \sum_{n=1}^{N} \langle \varphi_n, \mathbf{x} \rangle \varphi_n, \ \forall \mathbf{x} \in \mathbb{F}^N.$$

Example: $\mathbf{\Phi} = \frac{1}{\sqrt{7}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 \\ 1 & \omega^3 & \omega^6 & \omega^2 & \omega^5 & \omega^1 & \omega^4 \\ 1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 \\ 1 & \omega^5 & \omega^3 & \omega & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix}, \quad \omega = \exp(\frac{2\pi \mathrm{i}}{7}).$

Finite Unit-Norm Tight Frames (FUNTFs)

Definition: Unit vectors $\{\varphi_n\}_{n=1}^N$ in $\mathbb{F}^D$ form a **FUNTF** for $\mathbb{F}^D$ if there exists $C > 0$ such that

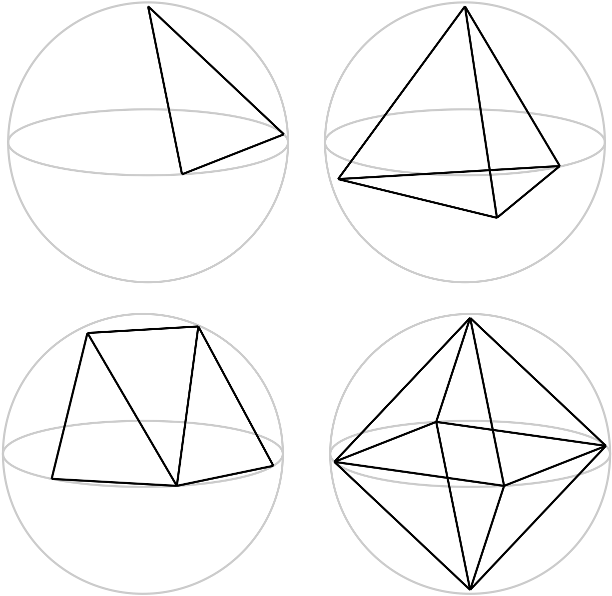
$$\mathbf{\Phi}\mathbf{\Phi}^* = C\mathbf{I}, \quad \text{i.e.,} \quad C\mathbf{x} = \mathbf{\Phi}\mathbf{\Phi}^*\mathbf{x} = \sum_{n=1}^{N} \langle \varphi_n, \mathbf{x} \rangle \varphi_n, \ \forall \mathbf{x} \in \mathbb{F}^N.$$

Here, $C = \frac{N}{D}$ since $CD = \mathrm{Tr}(\mathbf{\Phi}\mathbf{\Phi}^*) = \mathrm{Tr}(\mathbf{\Phi}^*\mathbf{\Phi}) = N$.

Example: Scaling the any three rows of the previous matrix gives a complex FUNTF$(3,7)$. For example, for rows $\{1,2,4\}$,

$$\mathbf{\Phi} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 \\ 1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 \end{bmatrix}, \quad \omega = \exp(\tfrac{2\pi \mathrm{i}}{7}).$$

Some real FUNTFs for $\mathbb{R}^3$ with $N = 3, 4, 5, 6$



Relating FUNTFs to the Tammes Problem

A Big Idea from Conway, Hardin, Sloane 96

A unit vector φ **lifts** to a rank-one projection $\varphi\varphi^*$. The set

$$\{\varphi\varphi^* : \varphi \in \mathbb{F}^D,\ \|\varphi\| = 1\}$$

is a **projective space** and lies in the real space of all $D \times D$ self-adjoint operators, which is a Hilbert space under the Frobenius inner product $\langle \mathbf{A}, \mathbf{B} \rangle_{\mathrm{Fro}} := \mathrm{Tr}(\mathbf{A}^*\mathbf{B})$.

Moreover, for unit vectors $\{\varphi_n\}_{n=1}^N$ and any n, n',

$$\langle \varphi_n\varphi_n^*, \varphi_{n'}\varphi_{n'}^* \rangle_{\mathrm{Fro}} = \mathrm{Tr}(\varphi_n\varphi_n^*\varphi_{n'}\varphi_{n'}^*) = |\langle \varphi_n, \varphi_{n'} \rangle|^2,$$

and so the squared-distance between two such projections is:

$$\|\varphi_n\varphi_n^* - \varphi_{n'}\varphi_{n'}^*\|_{\mathrm{Fro}}^2 = \mathrm{Tr}[(\varphi_n\varphi_n^* - \varphi_{n'}\varphi_{n'}^*)^2] = 2(1 - |\langle \varphi_n, \varphi_{n'} \rangle|^2).$$

Applying a Trivial Bound in Projective Space

Theorem: [Rankin 56] For any unit vectors $\{\varphi_n\}_{n=1}^N$ in $\mathbb{F}^D$,

$$\frac{N^2}{D} \leq \sum_{n=1}^{N}\sum_{n'=1}^{N} |\langle \varphi_n, \varphi_{n'} \rangle|^2$$

where equality holds if and only if $\{\varphi_n\}_{n=1}^N$ is a FUNTF for $\mathbb{F}^D$.

Proof:

$$0 \leq \Big\| \sum_{n=1}^{N} (\varphi_n \varphi_n^* - \tfrac{1}{D}\mathbf{I}) \Big\|_{\mathrm{Fro}}^2 = \mathrm{Tr}[(\boldsymbol{\Phi}\boldsymbol{\Phi}^* - \tfrac{N}{D}\mathbf{I})^2]$$

$$= \sum_{n=1}^{N}\sum_{n'=1}^{N} |\langle \varphi_n, \varphi_{n'} \rangle|^2 - \frac{N^2}{D}.$$

FUNTF Characterization and Construction

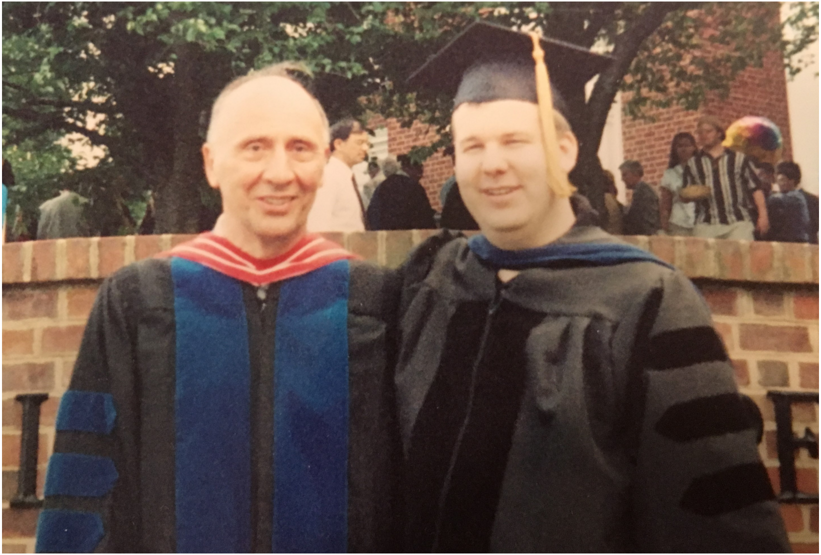
Theorem: [Benedetto, F 03] When $N \geq D$, every *local* minimizer of the **frame potential**

$$\sum_{n=1}^{N} \sum_{n'=1}^{N} |\langle \varphi_n, \varphi_{n'} \rangle|^2$$

is a FUNTF (and so is necessarily a global minimizer).

Theorem: [Cahill, F, Mixon, Poteet, Strawn 13] *Every* FUNTF can be explicitly constructed from **eigensteps**.

Born Again



Equiangular Tight Frames (ETFs)

Theorem: [Strohmer, Heath 03]

Any unit vectors $\{\varphi_n\}_{n=1}^N$ in $\mathbb{F}^D$ satisfy the **Welch bound**:

$$\max_{n\neq n'}|\langle\varphi_n,\varphi_{n'}\rangle|\geq\sqrt{\tfrac{N-D}{D(N-1)}},$$

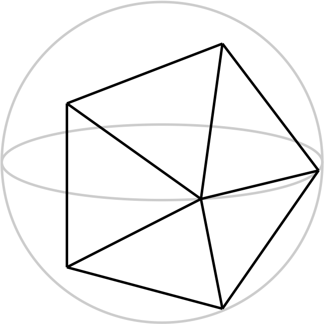
and achieve equality $\Leftrightarrow$ $\{\varphi_n\}_{n=1}^N$ is an ETF for $\mathbb{F}^D$, namely a FUNTF where $|\langle\varphi_n,\varphi_{n'}\rangle|$ is constant over all $n\neq n'$.

Proof: Apply Rankin's simplex bound to $\{\varphi_n\varphi_n^*-\frac{1}{D}\mathbf{I}\}_{n=1}^N$:

$$\tfrac{N^2}{D}\leq\sum_{n=1}^N\sum_{n'=1}^N|\langle\varphi_n,\varphi_{n'}\rangle|^2\leq N+N(N-1)\max_{n\neq n'}|\langle\varphi_n,\varphi_{n'}\rangle|^2.$$

See also: Rankin 56; Welch 74; Conway, Hardin, Sloane 96].

Example: A 6-vector ETF for $\mathbb{R}^3$



$$\mathbf{\Phi}\mathbf{\Phi}^* = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{\Phi}^*\mathbf{\Phi} = \mathbf{I} + \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{bmatrix}$$

Some Remarks on the (Rankin-)Welch Bound

- Following [Rankin 55], Rankin studied packing *antipodal pairs* of points of spheres and discovered the Welch bound about two decades before Welch [Rankin 56].

- The Welch bound is equivalent to

$$\max_{n \neq n'} \|\boldsymbol{\varphi}_n \boldsymbol{\varphi}_n^* - \boldsymbol{\varphi}_{n'} \boldsymbol{\varphi}_{n'}^*\|_{\mathrm{Fro}}^2 \leq \tfrac{2N(D-1)}{D(N-1)}. \tag{1}$$

  In particular, if an $\mathrm{ETF}(D, N)$ exists, then every optimal packing of N lines in $\mathbb{F}^D$ is necessarily tight.

- [Conway, Hardin, Sloane 96] calls (1) the **simplex bound** since it's achieved $\Leftrightarrow \{\boldsymbol{\varphi}_n \boldsymbol{\varphi}_n^* - \frac{1}{D}\mathbf{I}\}_{n=1}^N$ is a simplex. They also consider subspaces of dimension > 1.

More Remarks on the (Rankin-)Welch Bound

- **(Gerzon)** If $\{\varphi_n\varphi_n^* - \frac{1}{D}\mathbf{I}\}_{n=1}^N$ is a simplex, then

$$N \leq \frac{D(D+1)}{2} \text{ when } \mathbb{F} = \mathbb{R}, \quad N \leq D^2 \text{ when } \mathbb{F} = \mathbb{C}.$$

- For larger N, applying Rankin's other bound to$\{\varphi_n\varphi_n^* - \frac{1}{D}\mathbf{I}\}_{n=1}^N$ gives the **orthoplex bound**:

$$\max_{n \neq n'} |\langle \varphi_n, \varphi_{n'} \rangle| \geq \frac{1}{\sqrt{D}}.$$

- An ETF with $N = D^2$ is a **SIC-POVM**. Zauner has conjectured that these exist for all D [Zauner 99].

- ETFs arise in algebraic coding theory [Grey 62], quantum information theory [Zauner 99], wireless communication [Strohmer, Heath 03], and compressed sensing.

Equiangular Tight Frames

Harmonic ETFs: Difference Sets

Definition: Extracting rows from the character table of a finite abelian group $\mathcal{G}$ yields a **harmonic frame**.

Example: $\mathcal{G} = \mathbb{Z}_7, \quad \mathcal{D} = \{1, 2, 4\}$,

$$\mathbf{\Phi} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 \\ 1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 \end{bmatrix}, \quad \omega = \exp(\tfrac{2\pi \mathrm{i}}{3}).$$

Theorem: [Turyn 65] The harmonic ETF arising from $\mathcal{D} \subseteq \mathcal{G}$ is an ETF for $\mathbb{C}^{\mathcal{D}} \Leftrightarrow \mathcal{D}$ is a **difference set** for $\mathcal{G}$.

Idea:

$$\boldsymbol{\varphi}_n \boldsymbol{\varphi}_n^* = \frac{1}{3} \begin{bmatrix} \omega^n \\ \omega^{2n} \\ \omega^{4n} \end{bmatrix} \begin{bmatrix} \omega^{-n} & \omega^{-2n} & \omega^{-4n} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & \omega^{6n} & \omega^{4n} \\ \omega^n & 1 & \omega^{5n} \\ \omega^{3n} & \omega^{2n} & 1 \end{bmatrix}.$$

Steiner ETFs

Theorem: [Goethals, Seidel 70] Every **balanced incomplete block design (BIBD)** with $\Lambda = 1$ yields an ETF.

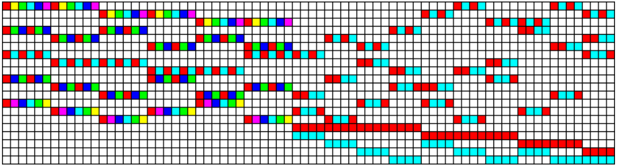
Example: Combine $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$ to form

$$\mathbf{\Phi} = \frac{1}{\sqrt{3}} \left[\begin{array}{cccc|cccc|cccc|cccc} + & - & + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - & + & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - & - & 0 & 0 & 0 & 0 & + & + & - & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & - & + \\ 0 & 0 & 0 & 0 & + & - & - & + & + & - & - & + & 0 & 0 & 0 & 0 \end{array}\right].$$

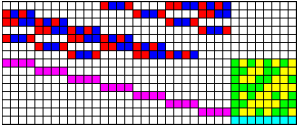
Some Recent Progress on ETFs

[Jasper, Mixon, F 14] Every McFarland harmonic ETF is a rotated Steiner ETF. New infinite family of optimal codes.

[F, Mixon, Jasper 16] New infinite family of complex ETFs arising from finite projective planes containing hyperovals.

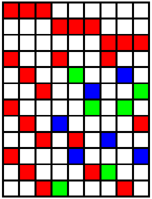


[F, Jasper, Mixon, Peterson 18]: Tremain's construction of an ETF$(15, 36)$ generalizes. New infinite family of *real* ETFs.

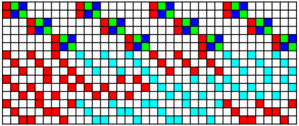


Some More Recent Progress on ETFs

[F, Jasper, King, Mixon 18] Some ETFs can be represented in terms of the regular simplices they contain.



[F, Jasper 19] Generalizing Davis-Jedwab difference sets gives new infinite families of ETFs from group divisible designs.



Some Future Directions

Fundamental mysteries: Lifting, spectral estimation.

Some mature open problems:

- Zauner's conjecture.
- Optimal projective packings when no ETF/OGF exists.
- Integrality conditions on the existence of complex ETFs.
- Breaking the square-root bottleneck for deterministic RIP.

Not-so-high hanging fruit:

- New constructions of ETFs, OGFs, ECTFFs, EITFFs.
- New connections to combinatorial designs.

Literature

R. A. Rankin, The closest packing of spherical caps in n dimensions, Glasg. Math. J. 2 (1955) 139–144.

R. A. Rankin, On the minimal points of positive definite quadratic forms, Mathematika 3 (1956) 15–24.

L. D. Grey, Some bounds for error-correcting codes, IRE Trans. Inform. Theory 8 (1962) 200–202.

R. J. Turyn, Character sums and difference sets, Pacific J. Math. 15 (1965) 319–346.

J. M. Goethals, J. J. Seidel, Strongly regular graphs derived from combinatorial designs, Can. J. Math. 22 (1970) 597–614.

L. R. Welch, Lower bounds on the maximum cross correlation of signals, IEEE Trans. Inform. Theory 20 (1974) 397-–399.

J. H. Conway, R. H. Hardin, N. J. A. Sloane, Packing lines, planes, etc.: packings in Grassmannian spaces, Exp. Math. 5 (1996) 139–159.

G. Zauner, Quantum designs: Foundations of a noncommutative design theory, Ph.D. Thesis, University of Vienna, 1999.

T. Strohmer, R. W. Heath, Grassmannian frames with applications to coding and communication, Appl. Comput. Harmon. Anal. 14 (2003) 257–275.

Thank you, John! Happy Birthday!