NuHAG Numerical Harmonic Analysis Group

Visions for Fourier Analysis in its Third Century

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John J. Benedetto Birthday Conference

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Submitted Abstract I

We are all aware of the fact that soon Fourier Analysis will celebrate its 200th birthday (the fundamental paper by J.B. Fourier was published in 1822). Hence this talk will give a short panoramic view on the developments of the field, pointing out its importance for many branches of Mathematical Analysis. The main part of the talk will be concerned with speculations and suggestions for future tasks in the field, for the coming years. The main goals concern three different directions:

1. Conceptual Harmonic Analysis, meaning an integration of ideas from Abstract and Computational Harmonic Analysis; making use of suitable function spaces in order to approximate and execute numerically efficient various tasks arising in the continuous domain;

Submitted Abstract II

2. Reinforce the connections to applied fields, such as physics, chemistry, communication theory and other natural sciences;
3. Make the available results *more user-friendly*, i.e. ensure that existing algorithms or theoretical results are not only accessible to the expert who can tune the parameters her/himself, e.g. by providing examples of best practice, verifications of optimality or self-tuning of parameters.

Overall the spirit should be more that to combine scientific knowledge already accumulated and coming up due to the efforts of a large community of mathematicians in the coming decades in a way that changes from the view-point of producers to that of customers, thus providing "consumer reports", customer satisfaction, rating by costumers and quality asessment.

Personal Background

- Starting as a teacher student math/physics, Univ. Vienna
- PhD and habilitation (1974/1979)
  in **Abstract Harmonic Analysis**
- Establishing **NuHAG** (Numerical Harmonic Analysis Group)
- Reach out for applications (communication theory, image processing, astronomy, medicine, musicology,...)
- European projects (Marie Curie and EUCETIFA)
- Main interest: Function spaces, Fourier Transform
- Nowadays: formally retired, but teaching at ETH, DTU, TUM, with the goal of supporting the applied sciences
- As editor to JFAA also the perspective is sharpend.

Key aspects of my talk I

1. Browse the (long-standing) **history of Fourier Analysis**
2. Show large number of **applications** influencing our life
3. Discussing some of the mathematics behind it
(take away the touch of mystery?)
4. Describing **time-frequency and Gabor analysis**
5. Suggesting ways to teach Fourier Analysis

Key aspects of my talk II

Some of the topics I want to present for further discussion, also by sharing with you (and anyone who is reading the slides of this talk) some of my own (sometimes very subjective) ideas on these issues.
I am NOT going to predict *hot topics* or *recommended areas of reasearch*. I do have (at the moment for myself) a list of interesting open problems which I plan to make public within a couple of months, but this is a different issues.
By no means is Fourier Analysis a finished subject area with only marginal or too difficult problems being left over.
COMPARISON with BUSINESS IDEAS! and business plan!??

Key aspects of my talk III

1. Topics and Goals
2. Methods and Results
3. Numerical Methods
4. Applications
5. Impact on and from Math. Analysis

Fourier history of in a nut-shell

1. 1822: J.B.Fourier proposes: Every periodic function can be expanded into a Fourier series using only pure frequencies;
2. up to 1922: concept of functions developed, set theory, Lebesgue integration, $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$;
3. first half of 20th century: Fourier transform for $\mathbb{R}^d$;
4. A. Weil: Fourier Analysis on Locally Compact Abelian Groups;
5. L. Schwartz: Theory of Tempered Distributions
6. Cooley-Tukey (1965): FFT, the Fast Fourier Transform
7. L. Hörmander: Fourier Analytic methods for PDE (Partial Differential Equations);

Citations

- In the early 1980s Jean Dieudonne called Abstract Harmonic Analysis “offstream” (in a talk in Vienna);
- In his review of the book of Colin Graham and Carruth McGehee Carl Herz called it a “tombstone for Commutative Harmonic Analysis”.

Hawkin’s citation

- The Greatest Obstacle to Discovery Is Not IgnoranceIt Is the Illusion of Knowledge (cite due to Stephen Hawkins)

C. C. Graham and O. C. McGehee. *Essays in Commutative Harmonic Analysis*. Grundl. Math. Wiss. 238, Springer Verlag, New York, 1979.

Technologies and their consequences

As in real life technological advances are changing our way of carrying out our tasks and influence strongly what we can do and how we can do it (steam engine, electricity, computers...). The same is true in mathematics, and thus in Fourier Analysis:

1. Lebesgue integration >> Banach algebra $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$;
2. Banach and Hilbert spaces, Riesz bases;
3. Haar measures >> foundations of AHA;
4. Invention of tempered distibutions >> PDE (Hörmander);
5. Interpolation theory: families of function spaces
6. Wavelets and Gabor expansions;
7. (Banach) Frames and Riesz sequences.

Signals to be analyzed

Which LCA group should be choose in order to analyze

- images (taken at various resolutions)
- movies
- hearbeat
- music
- machine-noise
- bird-songs

Or should we use wavelets? And which ones?

Different Aspect of Fourier Analysis

- Theory and Applications
- Levels of Generality
- Tools and Justifications
- Computations and Simulations

The irregular Sampling Problem

It all began in College Park (in THIS building, in 1989). The same year the Iron Curtain and the Berlin Wall came down while I was visiting John.

The irregular sampling problem, later irregular sampling in spline-type space (shift-invariant spaces) was one of the early strong points of NuHAG (`www.nuhag.eu`).

We learnt there many things:

1. how to work with families of function spaces;
2. how to do the discrete and the continuous case;
3. how to prove robustness and locality results;
4. develop ideas about Banach frames;
5. connect the continuous with the discrete case;

Discrete Approximation of Continuous Problems

One of the main general goals and problem areas is the approximation of continuous problems by finite/discrete problems. Such problems arise in the context of PDE when people use finite elements, but the situation is much less investigated in the context of Harmonic Analysis.

A short list of problems:

1. Compute the FT of a function;
2. Compute the dual Gabor atom for (g, Λ);
3. Compute the action of an operator (described in some form) T on a given function/distribution;
4. Solve a pseudo-differential operator equation $D(f) = h$;
5. Estimate eigenvalues and eigenvectors of localization (or Anti-Wick) operators.

Constructive Realizability

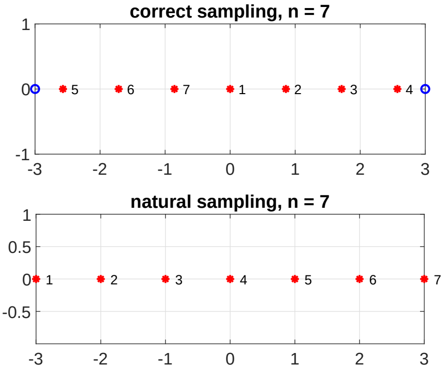
In practice one can only work with finite dimensional approximations to the continuous operators, or finitely sampling values. Unless FEM the finite model used for approximation can be taken in such a way that it has the same structure, so we work with approximation living on different groups

So the typical problem is the following (formulated for the forward problem):

Given some operator T and input f, mapped into some Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, and $\varepsilon > 0$. How can you obtain a approxamtion $y \in \boldsymbol{B}$ with $\|T(f) - y\|_{\boldsymbol{B}} < \varepsilon$, where y can be computed (finite time, finite precision, etc.).

Compare this with the question of numerical integration. Good functions can be obviously integrated in a more efficient way, but fast algorithms work only good for nice functions.

How should we sample a continuous function?



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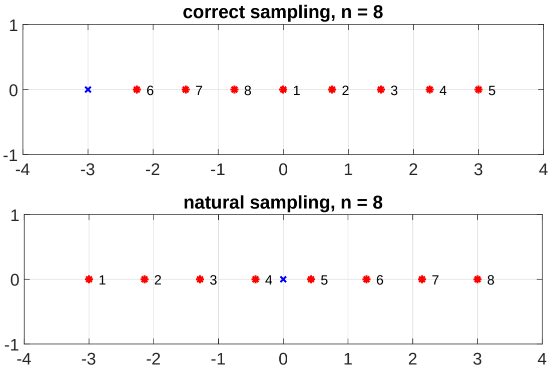

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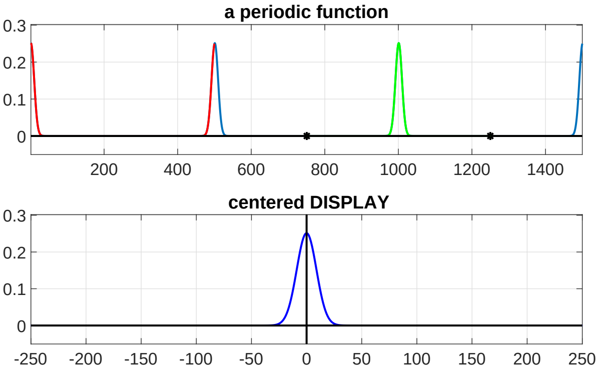

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Comparing signals of length n to $4n$

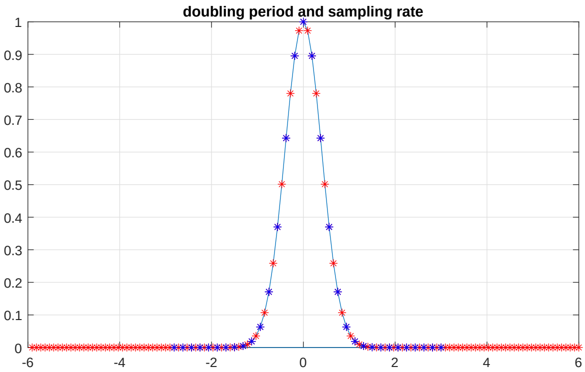

Figure: smpvaldem3A1.eps

Another form of Cauchy condition

While samples taken over differen periods and with different sampling rates are not immediately comparable to each other (except via interpolation/quasi-interpolation and resampling) the choice of particular sequences allow immediate comparison.
For example one may start to plot a Gauss function over $[-3, 3]$ with 144 samples taken at a rate of $1/6$.
Then this discrete (periodized) Gauss function will be invariant under the (unitary normalized) FFT.
taking now signal lengths of the form $4^k * 144$, say $n = 576$ and $n = 2304$ and then $n = 9216$ or $n = 36864$ one can then check whether discretely computed functions (e.g. Hermite functions up to some order) are *compatible*, which is/was in fact the case.

Function Spaces

1. $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty), (\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty), (\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty)$;
2. $(\ell^1, \|\cdot\|_1), \ell^2(\mathbb{N}), (\ell^\infty, \|\cdot\|_\infty), (\boldsymbol{c}_0, \cdot, \|\cdot\|_\infty)$;
3. $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1), (\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2), (\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$;
4. $(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}), (\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}})$;
5. $(\boldsymbol{M}_b(G), *, \|\cdot\|_{\boldsymbol{M}_b}), (\boldsymbol{L}^1(\mathbb{R}^d), *, \|\cdot\|_1), (\boldsymbol{C}_0(\mathbb{R}^d), \cdot, \|\cdot\|_\infty)$;
6. $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}), (\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'}), (\boldsymbol{W}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}})$;
7. $(\boldsymbol{B}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{B}^s_{p,q}}), (\boldsymbol{M}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^s_{p,q}})$; $\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$;
8. $(\boldsymbol{L}^1_w(\mathbb{R}^d), \|\cdot\|_{1,w}), (\boldsymbol{L}^p_w(\mathbb{R}^d), \|\cdot\|_{p,w}), (\boldsymbol{L}^\infty_w(\mathbb{R}^d), \|\cdot\|_{\infty,w})$;
9. $(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}), \boldsymbol{W}(\boldsymbol{C}_0, \ell^2)(\mathbb{R}^d), \boldsymbol{W}(\boldsymbol{M}, \ell^\infty)(\mathbb{R}^d)$
10. $(\boldsymbol{M}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^s_{p,q}}), (\boldsymbol{M}^p(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^p})$ $(\boldsymbol{F}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{F}^s_{p,q}})$;
11. $(\ell^1, \ell^2, \ell^\infty), (\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0'), (\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$

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Compare this with the question of numerical integration. Good functions can be obviously integrated in a more efficient way, but fast algorithms work only good for nice functions.

Teaching Fourier Analysis I

For the last 100 years the way how Fourier Analysis is introduced has not much changed. It is considered obvious that one has to make sure that the reader is familiar with Lebesgue integration, after all one has to study an *integral transform*. Browsing the literature I found it interesting that the authors of popular books in the field (Rudin, Edwards, Hewitt, Folland, Stein, Benedetto (!)) have not just written ONE book on Fourier Analysis, but have also authored several books covering mathematical analysis in a wider sense.

Most of the time one also follows the *historical path*, starting with Fourier series and the interpretation of the Fourier transform as a prototypical example of an orthonormal expansion in the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2\right)$ (again defined via Lebesgue integration).

Teaching Fourier Analysis II

Then one goes on to extend it to the Euclidean case (by letting periods go to infinity), coming up with the Riemann-Lebesgue Lemma, Plancherel, and the convolution theorem, perhaps with Shannon's Sampling Theorem.
For the setting of Abstract Harmonic Analysis (AHA) one needs the Haar measure and Pontrjagin's Duality Theory. Following the spirit of A. Weil one finds LCA groups as natural setting for Fourier Analysis. Alternatively, one looks at the Gelfand Theory of the Banach convolution algebra $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$.
This is augmented by the theory of tempered distributions of L. Schwartz $((\mathcal{S}, \boldsymbol{L}^2, \mathcal{S}')(\mathbb{R}^d))$ with comments on PDE, but also (quite independently) by the FFT algorithm due to Cooley/Tukey from 1965, which allows efficient realization of the DFT, respectively the Fourier Transform over the finite group $\mathbb{Z}_N$.

Diagnosis

Such an approach makes it rather difficult for engineers or physicists, who need the Fourier transform as a tool, to go deep enough to really understand it. They *need* objects like Dirac measures (having the *sifting property*), or Dirac combs (describing the sampling operator).

But the mathematical literature is presenting the matter full of warnings about potentially dangerous situations, so that they simply decide to ignore of them and declaring them as *not practically relevant*. Even mathematicians teaching engineers admit, that they communicate just formulas and rules, because there is no time to explain fine mathematical details.

What is also left behind in the mathematical approach is the question, why one should study the $\boldsymbol{L}^1(\mathbb{R}^d)$-convolution algebra. Just because it is possible and a mathematically interesting concept? But what it the actual benefit of the FT?

New Approaches

I am convinced that it is possible to motivate the teaching of Fourier Analysis from the applied side. Instead of discussing the DFT (realized as FFT) at the end of the course, it should be taken as a *starting point!*.

After all, what we can do with MATLAB^{TM} or other mathematical software (like OCTAVE), is "Linear Algebra in the box", so something like an enhanced pocket calculator. Observing that the DFT-matrix is just a Vandermonde matrix (corresponding to the unit roots of order N), which in addition is *unitary* helps a lot to understand basic principles of Fourier Analysis and appreciate the group theoretical background, hence sampling and periodization, Plancherel's theorem or the convolution theorem.

H. G. Feichtinger. Ingredients for Applied Fourier Analysis.
In *Sharda Conference Feb. 2018*, volume Sharda 2018, pages 1–22.
Taylor and Francis, 2019.

Linear Systems and Convolution

For engineers the main reason to study convolution operators is the theory of translation invariant linear systems, i.e. operators which commute with translations.
Typically they are described as convolution operators by some measure, pseudo-meausure, quasi-measure or (mild) distribution, called *impulse response*, or correspondingly as a Fourier multiplier by some *transfer function*.
There is a new (and rather simple) approach to this setting (up to the convolution theorem), working in the context of LCA groups, by starting with BIBOS systems, i.e. bounded operators on $(\boldsymbol{C}_0(G), \|\cdot\|_\infty)$, G a LCA group, free of measure theory!

H. G. Feichtinger. A novel mathematical approach to the theory of translation invariant linear systems. In Peter J. Bentley and I. Pesenson, editors, *Novel Methods in Harmonic Analysis with Applications to Numerical Analysis and Data Processing*, pages 1–32. 2016.

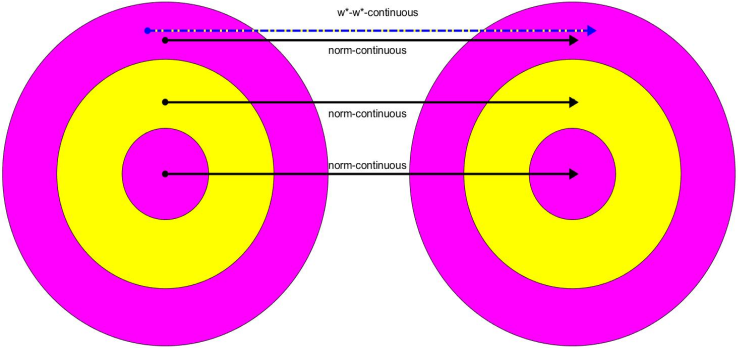
Mild distributions

Clearly the mathematical treatment of these questions requires SOME form of distribution theory. But this does not have to be based on Lebesgue integration or the theory of nuclear Frechet spaces!

In fact, the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$, which has shown to be quite useful in Time-Frequency Analysis and Gabor Analysis, but also for a reconsideration of classical questions of Fourier Analysis (including spectral analysis) turns out to provide (at least from my personal viewpoint) an appropriate setting for the correct mathematical description of such phenomena.

H. G. Feichtinger and M. S. Jakobsen. Distribution theory by Riemann integrals. In Pammy Machanda et al., editor, *ISIAM Proceedings*, Amritsar/India, pages 1–42, 2019.

Banach-Gelfand-Tripel-Homomorphisms



RELATED MATERIAL can be found at

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and checkout other talks at
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Access via user "visitor" and PWD "nuhagtalks".
and related material, see
www.nuhag.eu/bibtex

Well-being of a Community and its Individuals

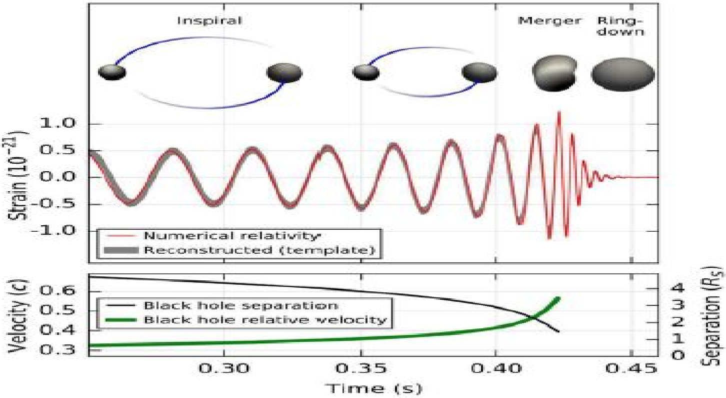
Aside from technical questions the spirit in a community is of great importance. Individuals feeling recognized, finding a way to establish their academic career, work on interesting problems and to get a chance to make progress by working hard is important for a scientific community to flourish.

To provide a good infra-structure (and not just funding for jobs) can be one of the most important goals and achievements of a scientist. It is clear to all of us that John has made huge contributions in this sense, by supporting his students, by creating the Journal (JFAA) and the book series, by establishing the Norbert Wiener Center, this is why we all are here.

And I see that several members of the younger generation have been able to continue carrying that spirit of encouragement, to build their own groups and stimulate their team members to also extend their attention to applied problems.

The shape of gravitational waves

Einstein had predicted, that the shape of the gravitaional wave of two collapsing black holes would be a chirp-like function, depending on the masses of the two objects.



Gravitational waves and Wilson bases

There is not enough time to explain the details of the huge signal processing task behind these findings, the literal "needle in the haystack".

There had been two strategies:

- Searching for 2500 explicitely determined wave-forms;
- Using a family of 14 orthonormal Wilson bases in order to detect the gravitational waves.

The very **first** was detected by the second strategy, because the masses had been out of the expected range of the predetermined wave-forms.

NOTE: Wilson bases are cooked up from tight Gabor frames of redundancy 2 by pairing them, like $\cos(x)$ and $\sin(x)$ using Euler's formula (in a smart, woven way).

frametitle: bibliography I

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Academic Press, Francisco, 1975.

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frametitle: bibliography II

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E. Hewitt and K. A. Ross.
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H.G. Feichtinger and T. Strohmer: Gabor Analysis, 1998.
H.G. Feichtinger and T. Strohmer: Advances in Gabor Analysis, 2003. both with Birkhäuser.
G. Folland: Harmonic Analysis in Phase Space, 1989.
I. Daubechies: Ten Lectures on Wavelets, SIAM, 1992.
G. Plonka, D. Potts, G. Steidl, and M. Tasche.
Numerical Fourier Analysis. Springer, 2018.

Some further books in the field are *in preparation*, e.g. on modulation spaces and pseudo-differential operators (Benyi/Okoudjou, Cordero/Rodino).

See also www.nuhag.eu/talks.

Don't worry, take action!

In contrast to Jean Dieudonne (1906-1992), who was calling "Abstract Harmonic Analysis" an off-stream topic in the early 80th, I am convinced that Fourier Analysis has a bright future. We, as a community engaged in this field have to work hard to enhance the value of our results also outside of mathematical analysis. If my comments contribute a little bit to such a movement I am more than happy.
Last, but not least, let me conclude, that we are all greatful to John for his vision as a founder of the

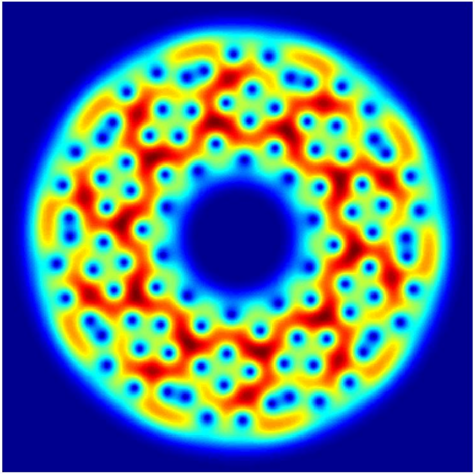
JOURNAL of FOURIER ANALYSIS and APPLICATIONS.

It is doing well and I expect to see more strong papers published with JFAA in the coming years.

HAPPY BIRTHDAY JOHN!

kaleid111

Centered Plot



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