Laguerre calculus on nilpotent Lie groups of step two and its applications

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Hans, Ray, Der-Chen and John, 1989

1. Introduction

Assume that $\mathscr{X} = \{X_1, \ldots, X_m\}$, $2 \leq m \leq n$, on a smooth manifold $\mathcal{M}_n$ with smooth measure μ. Denote

$$\mathscr{D} = \operatorname{span}\mathscr{X} \subset T\mathcal{M}_n.$$

Such vector bundles are often called *horizontal*. Define the following real vector bundles

$$\mathscr{D}^1 = \mathscr{D}, \qquad \mathscr{D}^{k+1} = [\mathscr{D}^k, \mathscr{D}] + \mathscr{D}^k \quad \text{for } k \geq 1,$$

which naturally give rise to the flag

$$\mathscr{D} = \mathscr{D}^1 \subseteq \mathscr{D}^2 \subseteq \mathscr{D}^3 \subseteq \ldots.$$

Then we say that a distribution satisfy *bracket generating condition* if $\forall\, x \in \mathcal{M}_n\ \exists\ k(x) \in \mathbb{Z}_+$ such that

$$\mathscr{D}_x^{k(x)} = T_x\mathcal{M}_n. \tag{0.1}$$

If the dimensions $\dim \mathscr{D}_x^k$ do not depend on x for any $k \geq 1$, we say that $\mathscr{D}$ is a *regular distribution*. The least k such that (0.1) is satisfied is called the *step* of $\mathscr{D}$.

A piecewise smooth curve $\gamma : [0,1] \to \mathcal{M}_n$ is called *horizontal* if $\dot{\gamma}(t) = \sum_{k=1}^{m} a_k(t) X_k$, or equivalently $\dot{\gamma}(t) \in \mathscr{D}_{\gamma(t)}$, $\forall\ t \in I$. Chow [1] proved the following theorem.

Theorem 0.1

If a manifold $\mathcal{M}_n$ is topologically connected and the distribution $\mathscr{D} = span\{X_1, \ldots, X_m\}$ is bracket generating, then any two points can be connected by a horizontal curve.

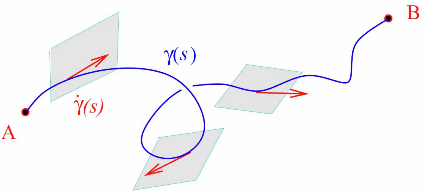

Figure 1. Chow's Theorem.

[1] W.L. Chow : *Über System Von Lineaaren Partiellen Differentialgleichungen erster Orduung*, Math. Ann., **117**, 98-105 (1939)

A subRiemannian structure over a manifold $\mathcal{M}_n$ is a pair $(\mathscr{D}, \langle \cdot, \cdot \rangle)$, where $\mathscr{D}$ is a *bracket generating distribution* and $\langle \cdot, \cdot \rangle$ a fibre inner product defined on $\mathscr{D}$. The length of the horizontal curve γ is

$$\ell(\gamma) := \int_0^\tau \sqrt{\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle} ds = \int_0^\tau \sqrt{a_1^2(s) + \cdots + a_m^2(s)} ds.$$

The shortest length $d_{cc}(A, B)$ is called the Carnot-Carathéodory distance between A, $B \in \mathcal{M}_n$ which is given by

$$d_{cc}(A, B) := \inf \ell(\gamma)$$

where the infimum is taken over all absolutely continuous horizontal curves joining A and B. Hence, we may define a geometry on $\mathcal{M}_n$ which is so-called *sub-Riemannian geometry* [2]

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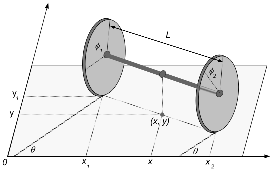
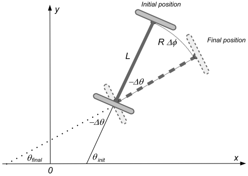
[2] O. Calin and D.C. Chang: *Sub-Riemannian Geometry: General Theory and Examples*, Encyclopedia of Mathematics and Its Applications, **126**, Cambridge University Press, (2009).

Example 0.1

Consider a kinematic cart with two equal wheels of radius R that can roll at different speeds on a plane, so the orientation of the cart might change at any time; see **Figure 2**.





The motion can be described by a curve $(x(t), y(t), \theta(t), \phi_1(t), \phi_2(t))$ on $\mathcal{M} = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. The midpoint (x, y) satisfies the constraints $dx = \frac{1}{2}(dx_1 + dx_2) = \frac{R}{2}\cos\theta(d\phi_1 + d\phi_2)$ and $dy = \frac{1}{2}(dy_1 + dy_2) = \frac{R}{2}\sin\theta(d\phi_1 + d\phi_2)$. The angle constraint $L\,d\theta = -R\,d(\phi_2 - \phi_1)$. Given $A, B \in \mathcal{M}$, there exists at least one piecewise smooth trajectory joining them[3].

[3] D.C. Chang and S.T. Yau: *Schrödinger equation with quartic potential and nonlinear filtering problem*, 48th IEEE Conference on Decision and Control, Shanghai, China, 8089-8094, (2009).

Example 0.2

Let $\mathcal{M} = \mathbf{R}^2 \times \frac{1}{2}\mathbb{S}^1$, $(x, y) \in \mathbf{R}^2$, $\theta \in \mathbb{S}^1$. *The distribution*

$$\mathscr{D} := span\left\{X = \frac{\partial}{\partial p},\ Y = \frac{\partial}{\partial y} + p\frac{\partial}{\partial x}\right\}, \quad p = \tan\theta,\ [X, Y] = \frac{\partial}{\partial x}$$

satisfies Chow's condition which can be applied to our daily life.

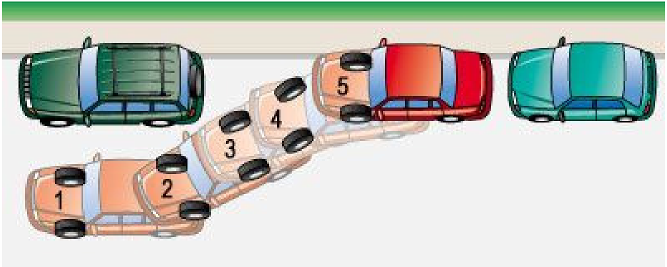


Figure 3. Parallel Parking.

Consider the sum of square vector fields $\mathscr{L} = \sum_{j=1}^{m} X_j^2$. The operator $\mathscr{L}$ is not necessary elliptic.
Let $B_{\mathscr{L}}(x,\rho) = \{y \in \mathcal{M}_n : \; d_{cc}(x,y) < \rho\}$ be a "ball" consists of all $y \in \mathcal{M}_n$ that can be joined to x by a horizontal curve γ with $d_{cc}(x,y) < \rho$. Let $B_E(x,\rho)$ be an ordinary Euclidean ball of radius ρ about x. *C. Fefferman-D.H. Phong*[4] showed that if $\mathscr{X}$ satisfies bracket generating property of step $Q \Leftrightarrow \exists\, c_Q > 0$ s.t.

$$B_E(x,\rho) \subseteq B_{\mathscr{L}}\big(x, c_Q \rho^{\frac{1}{Q}}\big) \quad \forall \; x \in \mathcal{M}_n, \;\; 0 < \rho < 1. \tag{0.2}$$

In fact, using *Fefferman-Phong's method*, we can show that (0.2) $\Leftrightarrow$ $\mathscr{L}$ satisfies the sub-elliptic estimate

$$\big\| |\nabla|^{\frac{2}{Q}} u \big\|_{L^2} \le \widehat{c}_Q \Big\{ \|\mathscr{L} u\|_{L^2} + \widetilde{c}_Q \|u\|_{L^2} \Big\}, \quad \forall \; u \in C^\infty(\mathcal{M}_n) \tag{0.3}$$

where $\widehat{c}_Q > 0$ and $\widetilde{c}_Q \ge 0$. Here $|\nabla|^{\frac{2}{Q}}$ is a ψDO with symbol $|\xi|^{\frac{2}{Q}}$. Hence, (0.3) $\Rightarrow$ a famous result of *Hörmander*[5] .

[4] C. Fefferman and D.H. Phong: The uncertainty principle and sharp Garding inequality, Comm. Pure & Applied Math., **34**, 285-331 (1981)

[5] L. Hörmander: Hypo-elliptic second order differential equations, Acta Math. **119**, 147-171 (1967).

2. Laguerre calculus on nilpotent Lie groups of step 2

In this talk, we concentrate on the case when $\mathcal{M}$ is a nilpotent Lie group of step 2. Let $B : \mathbf{R}^{2n} \times \mathbf{R}^{2n} \to \mathbf{R}^r$ be a non-degenerate skew-symmetric mapping given by

$$B(x, y) = (B_1(x, y), \ldots, B_r(x, y)), \quad B_\beta(x, y) = \sum_{j,k=1}^{2n} B^\beta_{jk} x_j y_k, \quad (0.4)$$

where $x, y \in \mathbf{R}^{2n}$. The multiplication given by the following formula defines a *nilpotent Lie group $\mathcal{N}$ of step two* on $\mathbf{R}^{2n} \times \mathbf{R}^r$:

$$(x, u) \cdot (y, s) = (x + y, u + s + 2B(x, y)). \quad (0.5)$$

The unit element is $(0, 0)$. The skew-symmetry of B implies that the inverse of (y, s) is $(-y, -s)$, and the associativity follows from the bilinearity of B.

Vector fields

$$Y_j := \partial_{y_j} + 2\sum_{\beta=1}^{r}\sum_{k=1}^{2n} B_{kj}^{\beta} y_k \partial_{s_\beta}, \quad j = 1, \dots, 2n \tag{0.6}$$

are left invariant vector fields on $\mathcal{N}$. For any $\lambda \in \mathbf{R}^r \setminus \{0\}$, denote

$$B^\lambda(y, y') := \sum_{j=1}^{2n} \lambda_j B_j(y, y').$$

Let ∂_v for $v \in \mathbf{R}^{2n}$ be the derivative of a function on $\mathbf{R}^{2n}$ along the direction v, *i.e.*, $\partial_v = \sum_{j=1}^{2n} v_j \partial_{y_j}$. Then,

$$Y_v := \sum_{j=1}^{r} v_j Y_j = \partial_v + 2B(y, v) \cdot \partial_s, \tag{0.7}$$

is left invariant vector field on $\mathcal{N}$, where

$$B(y, v) \cdot \partial_s := B_1(y, v)\partial_{s_1} + \cdots + B_r(y, v)\partial_{s_r}.$$

Their brackets are

$$[Y_v, Y_{v'}] = 4B(v, v') \cdot \partial_s. \tag{0.8}$$

Since B^λ is non-degenerate skew-symmetric, it can be written in a normal form with respect to an orthonormal basis $\{v_1^\lambda, \ldots, v_{2n}^\lambda\}$ of $\mathbf{R}^{2n}$ such that

$$B^\lambda\left(v_{2j-1}^\lambda, v_{2j}^\lambda\right) = -B^\lambda\left(v_{2j}^\lambda, v_{2j-1}^\lambda\right) = \mu_j(\lambda), \tag{0.9}$$

$j = 1, 2, \ldots n$ and $B^\lambda(v_j^\lambda, v_k^\lambda) = 0$ for all other choices of subscripts. We can assume $\mu_1(\lambda) \geq \mu_2(\lambda) \geq \cdots \geq \mu_n(\lambda) > 0$. The associated matrix of B^λ with respect to the basis $\{v_j^\lambda\}$ is

$$B^\lambda = \begin{pmatrix} 0 & \mu_1(\lambda) & 0 & 0 & \cdots \\ -\mu_1(\lambda) & 0 & 0 & \mu_2(\lambda) & \cdots \\ 0 & 0 & -\mu_2(\lambda) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{2n\times 2n} . \tag{0.10}$$

This is true locally as *Katsumi*[6] did for symmetric matrices. See *Chang, Markina and Wang*[7].

[6] N. Katsumi: Characteristic roots and vectors of a differentiable family of symmetric matrices, Linear and Multilinear Algebra, **1**, 159-162, (1973).

[7] D.C. Chang, I. Markina and W. Wang: The Laguerre calculus on the nilpotent Lie group of step two, J. Math. Anal. Appl., (2019).

We can write $y \in \mathbf{R}^{2n}$ in terms of the basis $\{v_k^\lambda\}$ as
$y = \sum_{j=1}^{n} \left(y_{2j-1}^\lambda v_{2j-1}^\lambda + y_{2j}^\lambda v_{2j}^\lambda \right) \in \mathbf{R}^{2n}$ for some $y_1^\lambda, \ldots, y_{2n}^\lambda \in \mathbf{R}$.
We call $(y_1^\lambda, \ldots, y_{2n}^\lambda)$ the *λ-coordinates* of $y \in \mathbf{R}^{2n}$.
The most important left-invariant differential operator on nilpotent Lie groups of step two $\mathcal{N}$ is the *Kohn Laplacian*:

$$D_\alpha = \Delta_b + i\alpha \cdot \partial_s, \tag{0.11}$$

where $\partial_s = (\partial_{s_1}, \ldots, \partial_{s_r})$, $s = (s_1, \ldots, s_r)$, $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbf{R}^r$.
In particular, if $\alpha = \mathbf{0}$, one has the *sub-Laplacian* defined on $\mathcal{N}$:

$$\triangle_b := -\frac{1}{4} \sum_{k=1}^{2n} Y_k Y_k. \tag{0.12}$$

Here

$$Y_k := \partial_{y_k} + 2 \sum_{\beta=1}^{r} \sum_{j=1}^{2n} B_{jk}^\beta y_j \partial_{t_\beta}. \tag{0.13}$$

We are interested in finding the *heat kernel* for the operator $\frac{\partial}{\partial t} - D_\alpha$ on $\mathcal{N}$. It is reasonable to expect the kernel has the form

$$h_t(y,s) := exp\{-tD_\alpha\}\delta_0 = \frac{c}{t^{\frac{\nu}{2}}} e^{-g(y,s)}, \quad \text{for suitable } \nu \tag{0.14}$$

in the sense of distribution. Here *modified complex action* function $g(y,s)$ plays the role of $\frac{d^2_{cc}(y,s)}{2t}$ and satisfies the Hamilton-Jacobi equation

$$\frac{\partial g}{\partial s} + H\Big(y, Y_1 g, \dots, Y_{2n} g\Big) = 0.$$

The simplest example of nilpotent Lie group of step two $\mathcal{N}$ is the Heisenberg group $\mathbb{H}_n$. See *Müller and Ricci*[8] for many interesting results. Inspired by the work of *Berenstein, Chang and Tie* [9], we are going to obtain the heat kernel of $h_t(y,s)$ on $\mathcal{N}$ via the Laguerre calculus.

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[8] D. Müller and F. Ricci: Analysis of second order differential operators on Heisenberg groups I,II, Invent. Math., **101**, 545-585, (1990) and JFA, **108**, 296-346, (1992).

[9] Berenstein-Chang-Tie: Laguerre calculus and its applications on the Heisenberg group, AMS/IP Studies in Advanced Mathematics **22**, (2001).

Before we go further, let us recall a beautiful idea of *Mikhlin*[10] contained in his 1936 study of convolution operators on $\mathbf{R}^2$. Let $\mathbf{F}$ denote a principal value convolution operator on $\mathbf{R}^2$:

$$\mathbf{F}(\phi)(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} F(y)\phi(x-y)dy,$$

where $\phi \in C_0^\infty(\mathbf{R}^2)$ and $F \in C^\infty(\mathbf{R}^2 \setminus \{(0,0)\})$ is homogeneous of degree -2 with the vanishing mean value, *i.e.*, $F(\lambda z) = \lambda^{-2} F(z)$ for $\lambda > 0$ and $\int_{|z|=1} F(z)dz = 0$. It follows that

$$F(z) = \frac{f(\theta)}{r^2}, \qquad z = y_1 + iy_2 = re^{i\theta},$$

where $f(\theta) = \sum_{k \in \mathbf{Z}, k \neq 0} f_k e^{ik\theta}$. Suppose that g is another smooth function on $[0, 2\pi]$ with $g(\theta) = \sum_{m \in \mathbf{Z}, m \neq 0} g_m e^{im\theta}$. Then g induces a principal value convolution operator $\mathbf{G}$ on $\mathbf{R}^2$ with kernel $G = \frac{g(\theta)}{r^2}$.

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[10] Mikhlin: Multidimensional singular integrals and integral equation, International Series of Monographs in Pure and Applied Mathematics, **83**, (1936).

Mikhlin found the following identity:

$$\frac{|k|i^{-|k|}}{2\pi}\frac{e^{ik\theta}}{r^2} * \frac{|m|i^{-|m|}}{2\pi}\frac{e^{im\theta}}{r^2} = \frac{|k+m|i^{-|k+m|}}{2\pi}\frac{e^{i(k+m)\theta}}{r^2}. \qquad (0.15)$$

Here $*$ stands for the Euclidean convolution. Denote the "symbol" $\sigma(\mathbf{F})$ of F as

$$\sigma(\mathbf{F}) = \sum_{k\in\mathbb{Z},k\neq 0} \left(\frac{|k|i^{-|k|}}{2\pi}\right)^{-1} f_k e^{ik\theta}.$$

With this notation, one may rewrite (0.15) as follows:

$$\sigma(\mathbf{F} * \mathbf{G}) = \sigma(\mathbf{F}) \cdot \sigma(\mathbf{G}).$$

It is natural to seek a similar calculus in noncommutative setting. The simplest and most natural noncommutative analogue of the algebra of principal value convolution operators in $\mathbf{R}^n$ is the left-invariant principal value convolution operators on the n-dimensional Heisenberg group $\mathbb{H}_n$. Mikhlin's symbol is replaced by a matrix, or tensor, and commutative symbol multiplication becomes noncommutative matrix or tensor multiplication. This is the so-called *Laguerre calculus*. Laguerre calculus is the symbolic tensor calculus originally induced by the Laguerre functions on the Heisenberg group $\mathbb{H}_n$. It was first introduced on $\mathbb{H}_1$ by *Greiner* and later extended to $\mathbb{H}_n$ and $\mathbb{H}_n \times \mathbf{R}^m$ by *Beals, Gaveau, Greiner and Vauthier*[11].

[11] R. Beals, B. Gaveau, P. Greiner and J. Vauthier: The Laguerre calculus on the Heisenberg group II, Bull. Sci. Math., **110 (3)**, 225-288, (1986).

For a fixed point $(y, s) \in \mathcal{N}$, the left multiplication by (x, u) is an affine transformation of $\mathbf{R}^{2n+r}$:

$$y \mapsto y + x, \qquad s \mapsto s + u + 2B(x, y),$$

which preserves the Lebesgue measure $dyds$ of $\mathbf{R}^{2n+r}$. $dyds$ is also right invariant, and so it is an invariant measure on the group $\mathcal{N}$.

$$\varphi * \psi(y, s) = \int_{\mathcal{N}} \psi\left((x, u)^{-1}(y, s)\right)\varphi(x, u)dxdu \tag{0.16}$$

for $f, g \in L^1(\mathcal{N})$.
The *partial Fourier transformation* of a function φ on $\mathcal{N}$ is defined as

$$\widetilde{\varphi}_\lambda(y) = \int_{\mathbf{R}^r} e^{-i\lambda \cdot s}\varphi(y, s)ds, \quad \tau \in \mathbf{R}^r \setminus \{0\}.$$

The *twisted convolution* of two function f and g on $\mathbf{R}^{2n}$ is

$$
\begin{aligned}
f *_\lambda g(y) &= \int_{\mathbf{R}^{2n}} e^{-i2B^\lambda(y,x)} f(y-x)g(x)dx \\
&= \int_{\mathbf{R}^{2n}} e^{-i2|\lambda|B^{\dot\lambda}(y,x)} f(y-x)g(x)dx
\end{aligned}
$$

where

$$
\dot\lambda = \frac{\lambda}{|\lambda|} \in \mathbb{S}^{r-1}.
$$

Straightforward calculation shows that

$$
\begin{aligned}
(\widetilde{\varphi * \psi})_\lambda(y) &= \int_{\mathbf{R}^r} e^{-i\lambda\cdot s} ds \int_{\mathbf{R}^r}\int_{\mathbf{R}^{2n}} \varphi(y-x, s-u-2B(y,x))\psi(x,u)dxdu \\
&= \int_{\mathbf{R}^{2n}} dx \int_{\mathbf{R}^r}\int_{\mathbf{R}^r} e^{-i\lambda\cdot[\widetilde{s}+u+2B(y,x)]}\varphi(y-x,\widetilde{s})\psi(x,u)d\widetilde{s}ds \\
&= \int_{\mathbf{R}^{2n}} e^{-i2|\lambda|B^{\dot\lambda}(y,x)}\widetilde{\varphi}_\lambda(y-x)\widetilde{\psi}_\lambda(x)dx = \widetilde{\varphi}_\lambda *_\lambda \widetilde{\psi}_\lambda.
\end{aligned}
$$

Therefore,

the convolution algebra $L^1(\mathcal{N}) \xrightarrow{\text{homo.}}$ the algebra $L^1(\mathbf{R}^{2n})$

under twisted convolution $*_\lambda$.

The *generalized Laguerre polynomials* $L_k^{(p)}$ are defined by the generating function formula[12]:

$$\sum_{k=1}^{\infty} L_k^{(p)}(\lambda) z^k = \frac{1}{(1-z)^{p+1}} e^{-\frac{\lambda z}{1-z}}, \qquad \lambda \in \mathbf{R}, \tag{0.17}$$

For $\lambda \in [0, \infty)$, k, $p \in \mathbf{Z}_+$,

$$l_k^{(p)}(\lambda) := \left[\frac{\Gamma(k+1)}{\Gamma(k+p+1)}\right]^{\frac{1}{2}} L_k^{(p)}(\lambda) \lambda^{\frac{p}{2}} e^{-\frac{\lambda}{2}}. \tag{0.18}$$

By a result of *Szegö*[13], we know that $\{l_k^{(p)}(\lambda),\ k \in \mathbf{Z}_+\}$ forms an orthonormal basis of $L^2([0, \infty), d\lambda)$ for fixed p. We define the functions $\mathcal{W}_k^{(p)}$ on $\mathbf{R}^2 \times \mathbf{R}^r$ via their partial Fourier transform

$$\widetilde{\mathcal{W}}_k^{(p)}(z, \lambda) = \frac{2|\lambda|}{\pi} (\operatorname{sgn} p)^p l_k^{(|p|)}(2|\lambda||z|^2) e^{ip\theta}, \quad \lambda \in \mathbf{R}^r, \tag{0.19}$$

where $z = y_1 + iy_2 = |z| e^{i\theta} \in \mathbf{C}^1$.

[12] A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi: Higher Transcendental Functions I and II, McGraw-Hill, (1953).

[13] G. Szegö: Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ., **23**, (1939).

One may define the *exponential Laguerre distribution* $\mathcal{W}_{\mathbf{k}}^{(\mathbf{p})}(z,s)$ on $\mathbf{C}^n \times \mathbf{R}^r$ via their partial Fourier transformations

$$\widetilde{\mathcal{W}}_{\mathbf{k}}^{(\mathbf{p})}(z,\lambda) := \prod_{j=1}^{n} \mu_j(\dot{\lambda}) \widetilde{\mathcal{W}}_{k_j}^{(p_j)} \left(\sqrt{\mu_j(\dot{\lambda})} z_j^{\lambda}, \lambda \right), \tag{0.20}$$

where $z \in \mathbf{C}^n$, $\lambda \in \mathbf{R}^r$, $\mathbf{p} = (p_1, \dots p_n) \in \mathbf{Z}^n$, $\mathbf{k} = (k_1, \dots k_n) \in \mathbf{Z}_+^n$, and $\dot{\lambda} = \frac{\lambda}{|\lambda|} \in \mathbb{S}^{r-1}$, $\mu_j(\lambda) = |\lambda| \mu_j(\dot{\lambda})$ and $z_j^{\lambda} = y_{2j-1}^{\lambda} + i y_{2j}^{\lambda} \in \mathbf{C}^1$, $j = 1, \dots, n$. Note that $\widetilde{\mathcal{W}}_{\mathbf{k}}^{(\mathbf{p})}(y,\lambda)$ is only defined for $\lambda \in \mathbf{R}^r$ such that B^{λ} is non-degenerate. It can be calculated that

$$\begin{aligned} \left\| \widetilde{\mathcal{W}}_{\mathbf{k}}^{(\mathbf{p})}(\cdot,\lambda) \right\|_{L^2(\mathbf{R}^{2n})}^2 &= \frac{2^n (\det |B^{\lambda}|)^{\frac{1}{2}}}{\pi^n} = \frac{2^n}{\pi^n} \prod_{j=1}^{n} \mu_j(\lambda), \\ \left\| \widetilde{\mathcal{W}}_{\mathbf{k}}^{(\mathbf{p})}(\cdot,\lambda) \right\|_{L^1(\mathbf{R}^{2n})} &= \prod_{j=1}^{n} \left\| l_{k_j}^{(p_j)} \right\|_{L^1(\mathbf{R}^1)}, \end{aligned} \tag{0.21}$$

where $|B^{\lambda}| := [(B^{\lambda})^T B^{\lambda}]^{\frac{1}{2}}$.
Moreover, for $f \in L^2(\mathcal{N})$, we have

$$\lim_{r \to 1^-} \lim_{m \to \infty} \sum_{|\mathbf{k}| \le m} r^{\mathbf{k}} \mathcal{W}_{\mathbf{k}}^{(\mathbf{0})} * f = f \quad \text{in} \quad L^2. \tag{0.22}$$

Inspired by a method developed by *Ogden and Vági*[14], for any fixed $\lambda \in \mathbf{R}^r \setminus \{0\}$ with B^λ non-degenerate, $\widetilde{\mathcal{W}}_{\mathbf{k}}^{(\mathbf{p})}(\cdot, \lambda)$ for fixed $\mathbf{k}, \mathbf{p}$ is a Schwarz function over $\mathbf{R}^{2n}$, and $\{\widetilde{\mathcal{W}}_{\mathbf{k}}^{(\mathbf{p})}(\cdot, \lambda)\}_{\mathbf{p} \in \mathbf{Z}^n, \mathbf{k} \in \mathbf{Z}_+^n}$ forms an orthogonal basis of $L^2(\mathbf{R}^{2n})$ that satisfies

Proposition 0.1

For $\mathbf{k}, \mathbf{p}, \mathbf{q}, \mathbf{m} \in \mathbf{Z}_+^n$, *we have*

$$\widetilde{\mathcal{W}}_{(\mathbf{k} \wedge \mathbf{p}) - \mathbf{1}}^{(\mathbf{p}-\mathbf{k})} *_\lambda \widetilde{\mathcal{W}}_{(\mathbf{q} \wedge \mathbf{m}) - \mathbf{1}}^{(\mathbf{q}-\mathbf{m})} = \delta_{\mathbf{k}}^{(\mathbf{q})} \widetilde{\mathcal{W}}_{(\mathbf{p} \wedge \mathbf{m}) - \mathbf{1}}^{(\mathbf{p}-\mathbf{m})},$$

where $\mathbf{p} \wedge \mathbf{m} - \mathbf{1} := (\min(k_1, p_1) - 1, \ldots, \min(k_n, p_n) - 1)$ *and* $\delta_{\mathbf{k}}^{(\mathbf{q})}$ *is the Kronecker delta function.*

Assume that $F \in L^1(\mathcal{N}) \cap L^2(\mathcal{N})$. Then for almost all λ, $\widetilde{F}_\lambda(y) \in L^2(\mathbf{R}^{2n})$ has the Laguerre expansion

$$\widetilde{F}_\lambda(y) = \sum_{\mathbf{p}, \mathbf{k} \in \mathbf{Z}_+^n} F_{\mathbf{k}}^{\mathbf{p}}(\lambda) \widetilde{\mathcal{W}}_{\mathbf{p} \wedge \mathbf{k} - 1}^{(\mathbf{p}-\mathbf{k})}(y, \lambda) \text{ with } \sum_{\mathbf{p}, \mathbf{k} \in \mathbf{Z}_+^n}^{\infty} |F_{\mathbf{k}}^{\mathbf{p}}(\lambda)|^2 < \infty.$$

[14] R. Ogden and S. Vági: Harmonic analysis on a nilpotent group and function theory on Siegel domains of type 2, Adv. Math., **33**, 31-92, (1979).

The Laguerre tensor of F is defined as

$$\mathcal{M}_\lambda(F) := \left(F^{\mathbf{p}}_{\mathbf{k}}(\lambda)\right)_{\mathbf{p},\mathbf{k}\in\mathbf{Z}^n_+}.$$

The following theorem is the core of the Laguerre calculus on nilpotent Lie groups $\mathcal{N}$ of step two[15].

Theorem 0.2

Suppose that B^λ is non-degenerate for almost all $\lambda \in \mathbf{R}^r$. For $F, G \in L^1(\mathcal{N}) \cap L^2(\mathcal{N})$, we have

$$\mathcal{M}_\lambda(F * G) = \mathcal{M}_\lambda(F) \cdot \mathcal{M}_\lambda(G)$$

for almost all $\lambda \in \mathbf{R}^r$.

The convolution algebra $L^1(\mathcal{N}) \xrightarrow{\text{homo.}}$ the algebra $L^1(\mathbf{R}^{2n})$ under twisted convolution $*_\lambda \xrightarrow{\text{homo.}}$ the algebra of $\infty \times \infty$-matrices.

[15] Theorem 1.1 in D.C. Chang, I. Markina and W. Wang, JMAA, (2019).

For a differential operator D on the group $\mathcal{N}$, we denote by $\widetilde{D}$ the *partial symbol* of D with respect to $\lambda \in \mathbf{R}^r$, *i.e.*, ∂_{s_β} is replaced by $i\lambda_\beta$. Then we have

$$\widetilde{\partial_s} = (\widetilde{\partial_{s_1}}, \ldots, \widetilde{\partial_{s_r}}) = i(\lambda_1, \ldots, \lambda_r) = i\lambda. \tag{0.23}$$

Let $\{v_1^\lambda, \ldots, v_{2n}^\lambda\}$ be an orthonormal basis of $\mathbf{R}^{2n}$ given by (0.9), which smoothly depends on λ in an open set U. Then

$$\widetilde{Y_{v_j^\lambda}} = \frac{\partial}{\partial v_j^\lambda} + 2iB^\lambda \left(y, v_j^\lambda\right) = \frac{\partial}{\partial y_j^\lambda} + 2iB^\lambda \left(y, v_j^\lambda\right)$$

for $j = 1, \ldots, 2n$. Using *complex λ-coordinates*, one has $z_j^\lambda := y_{2j-1}^\lambda + iy_{2j}^\lambda$ and complex horizontal vector fields

$$Z_j^\lambda := \frac{1}{2}\left(Y_{v_{2j-1}^\lambda} - iY_{v_{2j}^\lambda}\right), \qquad \bar{Z}_j^\lambda := \frac{1}{2}\left(Y_{v_{2j-1}^\lambda} + iY_{v_{2j}^\lambda}\right).$$

As usual, $\frac{\partial}{\partial z_j^\lambda} := \frac{1}{2}\left(\frac{\partial}{\partial y_{2j-1}^\lambda} - i\frac{\partial}{\partial y_{2j}^\lambda}\right)$.

Hence,

$$\widetilde{Z_j^\lambda} = \frac{\partial}{\partial z_j^\lambda} + iB^\lambda\left(y, v_{2j-1}^\lambda\right) + B^\lambda\left(y, v_{2j}^\lambda\right)$$
$$= \frac{\partial}{\partial z_j^\lambda} + i\mu_j(\lambda) y_{2j}^\lambda - \mu_j(\lambda) y_{2j-1}^\lambda = \frac{\partial}{\partial z_j^\lambda} - \mu_j(\lambda)\bar{z}_j^\lambda,$$

and

$$\widetilde{\bar{Z}_j^\lambda} = \frac{\partial}{\partial \bar{z}_j^\lambda} + \mu_j(\lambda) z_j^\lambda, \quad \text{where} \quad \frac{\partial}{\partial \bar{z}_j^\lambda} := \frac{1}{2}\left(\frac{\partial}{\partial y_{2j-1}^\lambda} + i\frac{\partial}{\partial y_{2j}^\lambda}\right). \quad (0.24)$$

Proposition 0.2

For any given $\lambda \in \mathbf{R}^r \setminus \{0\}$ with B^λ non-degenerate, let $\{v_1^\lambda, \ldots, v_{2n}^\lambda\}$ be the local orthonormal basis of $\mathbf{R}^{2n}$ as before. Then, we have

$$\triangle_b = -\frac{1}{2}\sum_{j=1}^{n}(Z_j^\lambda \bar{Z}_j^\lambda + \bar{Z}_j^\lambda Z_j^\lambda) = -\frac{1}{4}\sum_{j=1}^{2n} Y_{v_j^\lambda} Y_{v_j^\lambda} := -\frac{1}{4}\sum_{j=1}^{2n} Y_j Y_j.$$

It follows from Proposition 0.2 that for any fixed $\lambda \in \mathbf{R}^r \setminus \{0\}$, we have its partial symbol is

$$\widetilde{\triangle_b} := -\frac{1}{4}\sum_{j=1}^{2n} \widetilde{Y_j}\widetilde{Y_j} = -\frac{1}{2}\sum_{j=1}^{n} \left(\widetilde{Z_j^\lambda}\widetilde{\bar{Z}_j^\lambda} + \widetilde{\bar{Z}_j^\lambda}\widetilde{Z_j^\lambda} \right).$$

Lemma 0.1

$$\widetilde{\bar{Z}_j^\lambda}\widetilde{\mathcal{W}}_{\mathbf{k}}^{(-\mathbf{p})}(y,\lambda) = \begin{cases} -\sqrt{2\mu_j(\lambda)(k_j+p_j)}\widetilde{\mathcal{W}}_{\mathbf{k}}^{(-\mathbf{p}+\mathbf{e}_j)}(y,\lambda), & p_j \in \mathbf{N} \\ -\sqrt{2\mu_j(\lambda)k_j}\widetilde{\mathcal{W}}_{\mathbf{k}-\mathbf{e}_j}^{(-\mathbf{p}+\mathbf{e}_j)}(y,\lambda), & p_j = 0, \end{cases}$$

$$\widetilde{Z_j^\lambda}\widetilde{\mathcal{W}}_{\mathbf{k}}^{(\mathbf{p})}(y,\lambda) = \begin{cases} \sqrt{2\mu_j(\lambda)(k_j+1)}\widetilde{\mathcal{W}}_{\mathbf{k}+\mathbf{e}_j}^{(\mathbf{p}-\mathbf{e}_j)}(y,\lambda), & p_j \in \mathbf{N} \\ \sqrt{2\mu_j(\lambda)(k_j+1)}\widetilde{\mathcal{W}}_{\mathbf{k}}^{(\mathbf{p}-\mathbf{e}_j)}(y,\lambda), & p_j = 0, \end{cases}$$

where $\mathbf{e}_j = (0,\ldots,1,\ldots,0)$ *with* 1 *appearing in* j*-th entry and* 0 *otherwise.*

The above lemma shows that partial symbols of complex vectors Z_j^λ, $j = 1,\ldots,n$, act on Laguerre basis simply as shift operators.

Example 0.3

Let us consider the Heisenberg group $\mathbb{H}_1$.

In this case, we may assume that $a_1 = 1$. Then we have

$$M_+(Z_1) = \sqrt{2|\lambda|} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{pmatrix}$$

and $M_-(Z_1) = [M_+(Z_1)]^T$. Now we may set

$$M_+(K) = \frac{1}{\sqrt{2|\lambda|}} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{\sqrt{1}} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{pmatrix}$$

and $M_-(K) = [M_+(K)]^T$. Thus

$$\widetilde{K}(z,\lambda) = \frac{1}{\sqrt{2|\lambda|}} \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}} \widetilde{\mathcal{W}}^{(1)}_{\pm,k}(z,\lambda).$$

Using the definition of $\widetilde{\mathcal{W}}^{(1)}_{\pm,k}(z,\lambda)$, we sum the series

$$\widetilde{K}(z,\lambda) = \frac{2|\lambda|ze^{-|\lambda||z|^2}}{\pi}\int_0^1 \sum_{k=0}^{\infty} r^k L^{(1)}_k(2|\lambda||z|^2)dr,$$

Using the generating formula (0.17): $\sum_{k=0}^{\infty} r^k L^{(1)}_k(x) = \frac{e^x}{(1-r)^2}e^{-\frac{x}{1-r}}$, one has

$$\widetilde{K}(z,\lambda) = \frac{1}{\pi}\frac{e^{-|\lambda||z|^2}}{\bar{z}}$$

and

$$K(z,s) = \frac{1}{2\pi^2\bar{z}}\int_{\mathbf{R}} e^{-is\lambda-|\lambda||z|^2}d\lambda = \frac{z}{\pi^2(|z|^4+s^2)}$$

This is exactly a theorem of *Greiner, Kohn and Stein*[16] on $\mathbb{H}_1$:

$$Z_1K = \mathbf{I} - \mathcal{W}^{(0)}_{-,0} = \mathbf{I} - \mathbf{S}_-, \qquad KZ_1 = \mathbf{I} - \mathcal{W}^{(0)}_{+,0} = \mathbf{I} - \mathbf{S}_+,$$

where $\mathbf{S}_\pm$ are the "*Cauchy-Szegö operators*" with kernel

$$S_\pm(z,s) = \frac{2^{n-1}n!}{\pi^{n+1}}\frac{\prod_{j=1}^n a_j}{[\sum_{j=1}^n a_j|z_j|^2 \mp is]^{n+1}}.$$

[16] P. Greiner, J. Kohn and E.M. Stein: Necessary and sufficient conditions for solvability of the Lewy equation, PNAS USA, **72**, 3287-3289, (1975).

3. The heat kernel of the sub-Laplace operator

By Lemma 0.1, we know that the action of partial Fourier transformation of the operator D_α is diagonal. Computation shows that

$$-\frac{1}{2}\left(\widetilde{Z_j^\lambda}\widetilde{\bar{Z}_j^\lambda}+\widetilde{\bar{Z}_j^\lambda}\widetilde{Z_j^\lambda}\right)\widetilde{\mathcal{W}}_{\mathbf{k}}^{(\mathbf{0})}(y,\lambda)=\mu_j(\lambda)(2k_j+1)\widetilde{\mathcal{W}}_{\mathbf{k}}^{(\mathbf{0})}(y,\lambda). \quad (0.25)$$

Hence, by (0.18), (0.19), (0.20) and (0.22), one has

$$\begin{aligned}\widetilde{\mathbf{I}} &= \sum_{|\mathbf{k}|=0}^{\infty}\widetilde{\mathcal{W}}_{\mathbf{k}}^{(\mathbf{0})}(y,\lambda)=\sum_{|\mathbf{k}|=0}^{\infty}\prod_{j=1}^{n}\mu_j(\dot{\lambda})\widetilde{\mathcal{W}}_{k_j}^{(0)}(\sqrt{\mu_j(\dot{\lambda})}y_j^\lambda,\lambda)\\ &=\frac{1}{\pi^n}\sum_{|\mathbf{k}|=0}^{\infty}\prod_{j=1}^{n}2|\lambda|\mu_j(\dot{\lambda})L_{k_j}^{(0)}(\sigma_j)e^{-\frac{\sigma_j}{2}},\end{aligned} \quad (0.26)$$

where

$$\sigma_j:=2\mu_j(\dot{\lambda})|\lambda||y_j^\lambda|^2=2\mu_j(\lambda)|y_j^\lambda|^2.$$

Then we know that

$$\widetilde{h}_t(y,\lambda) = e^{-t\widetilde{D}_\alpha}\widetilde{\mathbf{I}} = \sum_{|\mathbf{k}|=0}^{\infty} e^{-t\widetilde{D}_\alpha}\widetilde{\mathcal{W}}_{\mathbf{k}}^{(\mathbf{0})}(y,\lambda)$$

$$= \sum_{|\mathbf{k}|=0}^{\infty} e^{-t\left(\sum_{j=1}^{n}(2k_j+1)\mu_j(\lambda)-\alpha\cdot\lambda\right)}\widetilde{\mathcal{W}}_{\mathbf{k}}^{(\mathbf{0})}(y,\lambda).$$

Therefore,

$$\begin{aligned}\widetilde{h}_t(y,\lambda) &= \frac{e^{\alpha\cdot\lambda t}}{\pi^n}\sum_{|\mathbf{k}|=0}^{\infty}\prod_{j=1}^{n} e^{-2k_j\mu_j(\lambda)t}e^{-\mu_j(\lambda)t}2|\lambda|\mu_j(\dot{\lambda})L_{k_j}^{(0)}(\sigma_j)e^{-\frac{\sigma_j}{2}} \\ &= \frac{e^{\alpha\cdot\lambda t}}{\pi^n}\prod_{j=1}^{n} 2\mu_j(\lambda)e^{-\mu_j(\lambda)t}e^{-\frac{\sigma_j}{2}}\sum_{k_j=0}^{\infty}\left(e^{-2\mu_j(\lambda)t}\right)^{k_j}L_{k_j}^{(0)}(\sigma_j) \\ &= \frac{e^{\alpha\cdot\lambda t}}{\pi^n}\prod_{j=1}^{n}\frac{2\mu_j(\lambda)e^{-\mu_j(\lambda)t}}{1-e^{-2\mu_j(\lambda)t}}\cdot e^{-\frac{\sigma_j}{2}\left(1+\frac{2e^{-\mu_j(\lambda)t}}{1-e^{-2\mu_j(\lambda)t}}\right)} \\ &= \frac{e^{\alpha\cdot\lambda t}}{\pi^n}\prod_{j=1}^{n}\frac{\mu_j(\lambda)}{\sinh(\mu_j(\lambda)t)}\cdot e^{-\frac{\sigma_j}{2}\coth(\mu_j(\lambda)t)}\end{aligned} \tag{0.27}$$

Taking inverse Fourier transform with respect to the λ-variable and we get

$$h_t(y,s) = \frac{1}{(2\pi)^r \pi^n t^{n+r}} \int_{\mathbf{R}^r} \left[\prod_{j=1}^{n} \frac{\mu_j(\lambda)}{\sinh \mu_j(\lambda)} \right] \cdot e^{\alpha\cdot\lambda - \frac{f(y,s,\lambda)}{t}} d\lambda, \tag{0.28}$$

Here

$$\begin{aligned} f(y,s,\lambda) :=& - is\cdot\lambda + |\lambda| \sum_{j=1}^{n} \mu_j(\dot{\lambda}) |y_j^\lambda|^2 \coth(\mu_j(\dot{\lambda})|\lambda|) \\ =& - is\cdot\lambda + \sum_{j=1}^{n} \mu_j(\lambda) |y_j^\lambda|^2 \coth \mu_j(\lambda) \end{aligned} \tag{0.29}$$

is the *action function*. Let $|B^\lambda| := [(B^\lambda)^T B^\lambda]^{\frac{1}{2}}$. Then

$$\det\left(\frac{|B^\lambda|}{\sinh |B^\lambda|}\right)^{\frac{1}{2}} = \prod_{j=1}^{n} \frac{\mu_j(\lambda)}{\sinh \mu_j(\lambda)} \tag{0.30}$$

is the *volume element*.

Theorem 0.3

Suppose that B^λ is non-degenerate for any $0 \neq \tau \in \mathbf{R}^r$. For the sub-Laplace operator D_α defined by (0.11) on nilpotent Lie groups $\mathcal{N}$ of step two, the heat kernel of D_α has the following expression:

$$h_t(y,s) = \frac{1}{2^r(\pi t)^{n+r}} \int_{\mathbf{R}^r} \det\left[\frac{|B^\lambda|}{\sinh|B^\lambda|}\right]^{\frac{1}{2}} \cdot e^{\alpha\cdot\lambda - \frac{f(y,s,\lambda)}{t}} d\lambda, \qquad (0.31)$$

where

$$f(y,s,\lambda) = -i\sum_{\beta=1}^{r}\lambda_\beta s_\beta + \langle |B^\lambda| \coth(|B^\lambda|)y, y\rangle. \qquad (0.32)$$

Here $|B^\lambda| := [(B^\lambda)^T B^\lambda]^{\frac{1}{2}}$ is a $2n \times 2n$ symmetric matrix and $\langle x, y\rangle = \sum_{j=1}^{2n} x_j y_j$ for any vectors x, $y \in \mathbf{R}^{2n}$ and $(B^\lambda)^T$ is the transpose of B^λ.

Example 0.4

Let $\mathbb{H}_n$ *be a Heisenberg group which is a vector space* $\mathbf{R}^{2n+1}$ *with a group multiplication*

$$(x,u)(y,s) = (x+y, u+s-2\sum_{j=1}^{n} a_j(x_j y_{j+n} - y_j x_{n+j})),$$

where $a_1, \ldots, a_n$ *are positive real numbers,* $x, y \in \mathbf{R}^{2n}$, $u, s \in \mathbf{R}$. *The Kohn Laplacian is*

$$D_\alpha = -\frac{1}{2}\sum_{j=1}^{n}(Z_j \bar{Z}_j + \bar{Z}_j Z_j) + i\alpha\frac{\partial}{\partial s},$$

where $\alpha \in \mathbf{R}$ *and*

$$Z_j := \frac{\partial}{\partial z_j} + i a_j \bar{z}_j \frac{\partial}{\partial s} \quad \text{and} \quad \bar{Z}_j := \frac{\partial}{\partial \bar{z}_j} - i a_j z_j \frac{\partial}{\partial s}.$$

To simplify notations, let us assume that $n = 2$. Then skew-symmetric matrix B^λ is

$$B^\lambda = \begin{pmatrix} 0 & 0 & -a_1\lambda & 0 \\ 0 & 0 & 0 & -a_2\lambda \\ a_1\lambda & 0 & 0 & 0 \\ 0 & a_2\lambda & 0 & 0 \end{pmatrix} \in M_{4\times 4}.$$

and so

$$|B^\lambda| = [(B^\lambda)^T B^\lambda]^{\frac{1}{2}} = \begin{pmatrix} a_1\lambda & 0 & 0 & 0 \\ 0 & a_2\lambda & 0 & 0 \\ 0 & 0 & a_1\lambda & 0 \\ 0 & 0 & 0 & a_2\lambda \end{pmatrix}.$$

In this case, we get $\det \sinh |B^\lambda| = \prod_{j=1}^n \sinh^2(a_j\lambda)$. Then we have

$$\det\left(\frac{|B^\lambda|}{\sinh|B^\lambda|}\right)^{\frac{1}{2}} = \prod_{j=1}^{n} \frac{a_j\lambda}{\sinh(a_j\lambda)}.$$

Similarly, we get

$$
\coth|B^\lambda| = \frac{\cosh|B^\lambda|}{\sinh|B^\lambda|}
$$

$$
= \begin{pmatrix} \coth(a_1\lambda) & 0 & 0 & 0 \\ 0 & \coth(a_2\lambda) & 0 & 0 \\ 0 & 0 & \coth(a_1\lambda) & 0 \\ 0 & 0 & 0 & \coth(a_2\lambda) \end{pmatrix}.
$$

Then $|B^\lambda|\coth|B^\lambda|$ equals the following matrix

$$
\begin{pmatrix} a_1\lambda\coth(a_1\lambda) & 0 & 0 & 0 \\ 0 & a_2\lambda\coth(a_2\lambda) & 0 & 0 \\ 0 & 0 & a_1\lambda\coth(a_1\lambda) & 0 \\ 0 & 0 & 0 & a_2\lambda\coth(a_2\lambda) \end{pmatrix},
$$

and

$$
\langle |B^\lambda|\coth(|B^\lambda|)y, y\rangle = \lambda\sum_{k=1}^{2} a_k\coth(a_k\lambda)(y_k^2 + y_{2+k}^2).
$$

Hence, the heat kernel of the sub-Laplacian D_α on the Heisenberg group $\mathbb{H}_n$ is

$$h_t(y,s) = \frac{1}{2\pi^{n+1} t^{\frac{2n}{2}+1}} \int_{\mathbf{R}} \prod_{j=1}^{n} \frac{a_j \lambda}{\sinh(a_j \lambda)} \cdot e^{\alpha\lambda - \frac{f(y,s,\lambda)}{t}} d\lambda, \qquad (0.33)$$

where

$$f(y,s,\lambda) = -i\lambda s + \lambda \sum_{k=1}^{n} a_k \coth(a_k \lambda)(y_k^2 + y_{n+k}^2).$$

This recovers the results obtained by *Calin, Chang and Greiner*[17] and *Calin, Chang, Furutani and Iwasaki*[18]

[17] O. Calin, D.C. Chang and P. Greiner: Geometric Analysis on the Heisenberg Group and Its Generalizations, AMS/IP series in Advanced Mathematics, **40**, (2007).

[18] O. Calin, D.C. Chang, K. Furutani and C. Iwasaki: Heat Kernels for Elliptic and Sub-elliptic Operators: Methods and Techniques, Birkhäuser-Verlag, (2010).

The 1-dim *quaternionic Heisenberg group* $\mathcal{Q}_1$ is a vector space

$$\mathbb{Q} \times \mathbf{R}^3 = \{[w, t] : \; w \in \mathbb{Q}, \; t = (t_1, t_2, t_3) \in \mathbf{R}^3\}$$

with the multiplication law

$$\begin{aligned} q_1 \circ q_2 &= [w, t_1, t_2, t_3] \cdot [\omega, s_1, s_2, s_3] \\ &= [w + \omega, t_1 + s_1 - 2\mathrm{Im}_1(\bar{\omega} w), t_2 + s_2 - 2\mathrm{Im}_2(\bar{\omega} w), \\ &\qquad t_3 + s_3 - 2\mathrm{Im}_3(\bar{\omega} w)]. \end{aligned} \tag{0.34}$$

The law (0.34) makes $\mathbb{Q} \times \mathbf{R}^3$ into Lie group with the identity $[0, 0]$ and the inverse $[w, t]^{-1}$ given by

$$q^{-1} = [w, t_1, t_2, t_3]^{-1} = [-w, -t_1, -t_2, -t_3].$$

This group acts on the boundary $\partial\mathcal{U}$ of the "upper half space" $\mathcal{U} = \{(q_1, q_2) \in \mathbb{Q}^2 : \; \mathrm{Re}(q_2) > |q_1|^2\}$ in $\mathbb{Q}^2$ transitively.

Example 0.5

In this case, we know $r = 3$ and the skew-symmetric matrix B^λ has the following form:

$$B^\lambda = \lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3 = \begin{pmatrix} 0 & \lambda_1 & -\lambda_3 & -\lambda_2 \\ -\lambda_1 & 0 & -\lambda_2 & \lambda_3 \\ \lambda_3 & \lambda_2 & 0 & \lambda_1 \\ \lambda_2 & -\lambda_3 & -\lambda_1 & 0 \end{pmatrix} \in M_{4\times 4}.$$

Then we have

$$|B^\lambda| = [(B^\lambda)^T B^\lambda]^{\frac{1}{2}} = |\lambda|\mathbf{I}_4, \quad \sinh|B^\lambda| = \sinh(|\lambda|)\mathbf{I}_4,$$
$$\coth|B^\lambda| = \coth(|\lambda|)\mathbf{I}_4, \quad |B^\lambda|\coth|B^\lambda| = |\lambda|\coth(|\lambda|)\mathbf{I}_4.$$

Here $\mathbf{I}_4$ is the 4×4 identity matrix. Hence,

$$\det\left(\frac{|B^\lambda|}{\sinh|B^\lambda|}\right)^{\frac{1}{2}} = \frac{|\lambda|^2}{\sinh^2|\lambda|},$$

and $\langle |B^\lambda|\coth(|B^\lambda|)y, y\rangle = \langle |\lambda|\coth(|\lambda|)\mathbf{I}_4\, y, y\rangle = |\lambda|\coth(|\lambda|)|y|^2.$

Hence, the heat kernel of the sub-Laplacian D_α on quaternionic Heisenberg group $\mathcal{Q}_1$ is

$$h_t(\omega, s_1, s_2, s_3) = \frac{1}{8\pi^5 t^{\frac{4}{2}+3}} \int_{\mathbf{R}^3} \frac{|\lambda|^2}{\sinh^2|\lambda|} \cdot e^{\alpha\cdot\lambda - \frac{f(\omega,s_1,s_2,s_3,\lambda)}{t}} d\lambda,$$

where

$$f(\omega, s_1, s_2, s_3, \lambda) = -i\sum_{\beta=1}^{3} \lambda_\beta s_\beta + |\lambda|\coth(|\lambda|)|\omega|^2.$$

This recovers the results obtained by *Calin, Chang and Markina*[19].

[19] O. Calin, D.C. Chang and I. Markina: Generalized Hamilton-Jacobi equation and heat kernel on step two nilpotent Lie groups, Analysis and Mathematical Physics, Trends in Mathematics. Birkhüser Basel, (2009).

4. Heat kernel asymptotic expansions

It is well known that much geometric information about Riemannian manifold can be decoded from the small-time asymptotic expansions of the heat kernel of the Laplace-Beltrami operator. See *e.g.*, *Varadhan*[20].

Here we just consider 1-dimensional Heisenberg group $\mathbb{H}_1$. In this case, we just have two horizontal vector fields.

Fix $q_0 = (0,0,0)$ and let the other point $q(x_1, x_2, y)$ vary. The heat kernel $h_t(x_1, x_2, y)$ is given as a Laplace integral

$$h_t(x_1, x_2, y) = \frac{1}{2\pi^2 t^2} \int_{-\infty}^{\infty} e^{-\frac{f(x_1, x_2, y, \lambda)}{t}} V(\lambda)\, d\lambda, \tag{0.35}$$

where the phase function is

$$f(x_1, x_2, y, \lambda) = -i\lambda y + \lambda(x_1^2 + x_2^2) \coth \lambda$$

and $V(\lambda) = \frac{\lambda}{\sinh \lambda}$ is the "*volume element*".

[20] SRS Varadhan: *On the behavior of the fundamental soltion of the heat equation with variable coefficients*, Pure Appl. Math. **20**, 431-455 (1967).

We have the following theorem.

Theorem 0.4

The heat kernel $h_t(x_1, x_2, y)$ of the Heisenberg group in (0.33) has the following asymptotic expansion as $t \to 0^+$:

(1). *when* $(x_1, x_2, y) = (0,0,0)$, $h_t(0,0,0) = \frac{1}{4t^2}$;

(2). *when* $(x_1, x_2, y) = (0,0,y)$ *with* $y \neq 0$,

$$h_t(0,0,y) \sim \frac{1}{2t^2} \sum_{k=1}^{\infty} e^{-\frac{k\pi|y|}{t}} (-1)^{k+1} k;$$

(3). *when* $(x_1, x_2) \neq (0,0)$ *with* $y = 0$,

$$h_t(x_1, x_2, 0) \sim \frac{1}{\pi^2 t^{\frac{3}{2}}} e^{-\frac{(x_1^2 + x_2^2)}{t}} \sum_{k=0}^{\infty} \Gamma\Big(\frac{k+1}{2}\Big) C_k t^{\frac{k}{2}},$$

(4). *when* $(x_1, x_2) \neq (0,0)$,

$$h_t(x_1, x_2, y) \sim \frac{1}{\pi^2 t^{\frac{3}{2}}} e^{-\frac{d_{cc}^2(x_1, x_2, y)}{t}} \sum_{k=0}^{\infty} \Gamma\Big(k + \frac{1}{2}\Big) D_k t^k,$$

where $d_{cc}(x_1, x_2, y)$ *is the sub-Riemannian distance between the origin and the point* (x_1, x_2, y), *and the coefficients* D_k *can be calculated explicitly by Debye's method of steepest decent.*

We have the following form of asymptotics for the heat kernel *Chang-Li*[21]:

Remark 0.1

$$h_t(x_1, x_2, y) \sim \frac{C}{t^{\frac{\nu}{2}}} e^{-\frac{d_{cc}^2}{2t}},$$

where C and ν are constants and d_{cc} is the Carnot-Carathéodory distance between (x_1, x_2, y) and the origin. We note that the power ν of t varies. Namely,

$$\nu = \begin{cases} 4 > n, & \text{when } x = 0,\ y = 0,\ \text{diagonal;} \\ 4 = n+1, & \text{when } x = 0,\ y \neq 0,\ \text{off-diagonal, cut-conjugate;} \\ 3 = n, & \text{when } x \neq 0,\ \text{off-diagonal, not cut-conjugate.} \end{cases}$$

Here, $n = 3$ is the topological dimension and ν is the Hausdorff dimension.

[21] D.C. Chang and Y. Li: *Heat kernel asymptotic expansions for the Heisenberg sub-Laplacian and the Grushin operator*, Proceedings of the Royal Society A, **471**, 20140943 (2016).

Happy Birthday, Professor Benedetto!

Thank you for being a great scholar, a kind person and a good friend.