The 1960s
Some people and mathematics I met

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1960

John F. Kennedy was elected president!

1960 - Boston College

- Stanley J. Bezuszka S.J. – Ph.D. Brown – Inspiration
- Hans Haefeli – Swiss postdoc of Lars Ahlfors – Analysis and Schwartz' distribution theory
- René J. Marcou – Ph.D. MIT (Struik) – Quantum mechanics and relativity

1960 - Harvard

- Richard Brauer – Complex analysis (Weierstass $\mathcal{P}$-functions),
- Andrew M. Gleason [Japanese code, Hilbert's fifth problem, and QM] – Analysis and inspiration
- George W. Mackey – TVS and distribution theory
- Joseph L. Walsh – Potential theory
- David V. Widder – Laplace transforms, Tauberian theorems, and a dose of Beurling

So it just HAD to be a Ph.D. in TVS, distribution theory, and Laplace transforms. But $\mathcal{P}$-functions, Potential theory, and QM (with frames) all eventually popped-up!

Notes

Captain Jack Dyer and Andy

1962 - 1964 Exciting jobs*

- René Marcou's electron density of the ionosphere research for the AF
- RCA – Bellman dynamic programming, control theory, and BMEWS
- IBM (Itsy Bitsy Machines) – equatorial satellites

*There were many less mathematical and less exciting ones.

$\mathcal{P}$-functions and equatorial satellites

IBM interviewer: Do you know about Weierstrass $\mathcal{P}$-functions? Wow!

- Problem: Integrate the Newtonian equations of motion of a secondary body in the equatorial plane of a rotationally symmetric central body. Solution: In terms of $\mathcal{P}$-functions!
- Application: Determine motion of equatorial artificial satellites.
- Analogy: Apsidal line shift of mercury's orbit about sun.
- Explanation: OK by Newton if sun is flat enough. But Robert Dicke et al. proved sun is too spherical. Einstein's general relativity explains it!

The Benedetto 4 year plan

Plan: Get a Ph.D. 4 years after B.A.

- Brauer called University of Toronto, and I'm on my way there for a Licentiate in Thomistic Philosophy and a Ph.D. in Mathematics.
- **Chandler Davis** becomes my Ph.D. Adviser - unbelievably great good fortune for me! Chandler (and Natalie) are a field all to themselves.
- Chandler and Naimark's theorem and my obstreperous nature. But here are bread and butter frames that I am still trying to figure out with Gleason functions and quantum measurement, and that many of you are working on with ETFs.
- Lunch with Chandler and Laurent Schwartz, who had striking blue eyes like my father.
- "And you're not getting any younger Benedetto"!
- **Hans Heinig** – wise and wild ideas – and 20 years later I had my first collaboration and it was with Hans.

Notes

- Chandler began his prison term on February 3, 1960.
- No Ivory Tower : McCarthyism & the Universities by Ellen W. Schrecker, Oxford, 1986. The present is a grim reminder of that past. The mathematical *Back to the future* theme ahead is happier.
- "An extremal problem for plane convex curves" by Chandler Davis. His acknowledgement: Research supported in part by the Federal Prison System. Opinions expressed in this paper are the author's and are not necessarily those of the Bureau of Prisons.
- Fortunately for me, Chandler was "raring to go" at the University of Toronto in September 1962 – not US university :-)

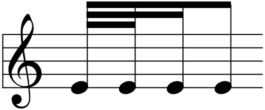
1964 - 1965

- NATO on Distributions in Lisbon - Köthe; Schwartz asking about my research - yikes :-)
- NYU - the opening of the Courant Hilton, what a year! and leaving my tenure track job there on a whim! Those were the days.
- Liège - TVS with Henri Garnir, Marc De Wilde, Jean Schmetts, the Ardenne, and my love affair with Stella.
- UMD - Aziz, Brace, Leon Cohen, Diaz, Goldberg, Gulick, Horvath, Maltese, Kleppner, Warner.
- Functional analysis seminar at UMD - Choquet, Dieudonné, Garnir.
- *LF*-spaces and Dieudonné - Schwartz reaching out to Köthe - beautiful!

Bob Warner

Bob told me and taught me about spectral synthesis. He was one of my closest friends. Unfortunately, GOB could never teach me his mathematical elegance. He died in 2017.

We sang Gregorian chant together:



"Let G be a locally compact Abelian group (with dual group Γ) ... "

1965 – Back to the future – spectral synthesis

- $L^1(G) \longrightarrow A(\Gamma)$ CBA of absolutely convergent Fourier transforms,
- $L^\infty(G) \longleftarrow A'(\Gamma)$ pseudo-measures, module over ring $A(\Gamma)$
- Norms and notation: $\widehat{f} = \phi \in A(\Gamma)$, $\|\phi\| := \|f\|_1$;
  $T \in A'(\Gamma)$, $\widehat{T} = \Phi \in L^\infty(G)$, $\|T\| := \|\Phi\|_\infty$
- $\mathrm{spectrum}(\Phi) := \mathrm{supp}(T)$
- $M_b(\Gamma) \subseteq A'(\Gamma) \subseteq \mathcal{S}'(\Gamma)$ tempered distributions on the LCAG Γ

Fourier decomposition (spectral synthesis problem): When does

$$\Phi(\cdot) \in \overline{\mathrm{span}}\{\gamma(\cdot) : \gamma \in \mathrm{supp}(T)\} \text{ in the } \sigma(L^\infty(G), L^1(G)) - \text{topology?}$$

Suggestively (notationally), this problem for *group characters*, e.g., $\gamma(x) = e^{-2\pi i x\gamma}$, for $x \in \mathbb{R}$ and $\gamma \in \widehat{\mathbb{R}} = \mathbb{R}$, is:

$$\text{“}\Phi(\cdot) = \sum_{\gamma \in \mathrm{supp}(T)} c_\gamma\, \gamma(\cdot)\text{” in the } \sigma(L^\infty(G), L^1(G)) - \text{topology?}$$

This is *the* fundamental problem of harmonic analysis.

Back to the future – spectral synthesis, cont.

As such, we say that closed $\Lambda \subseteq \Gamma$ is an *S-set* if

$$\forall T \in A'(\Gamma) \text{ and } \forall \phi \in A(\Gamma), \quad \phi = 0 \text{ on } \Lambda \text{ and } \mathrm{supp}(T) \subseteq \Lambda \;\Rightarrow\; T(\phi) = 0.$$

Straightforward for $M_b(\Gamma)$ instead of $A'(\Gamma)$:

$$\phi = 0 \text{ on } \mathrm{supp}(\mu) \;\Rightarrow\; \int_\Gamma \phi \, d\mu = 0.$$

Example

a. $S^{d-1} \subseteq \mathbb{R}^d$ is non-S for $d \geq 3$ (L. Schwartz).
b. $S^1 \subseteq \mathbb{R}^2$ is an S-set (Herz).
c. $\Lambda = \{\gamma \in \Gamma : \|\gamma\| \leq 1\}$ is an S-set.
d. $\Lambda = C_{1/3} \subseteq \mathbb{T}$ is an S-set and has non-S subsets (Herz et al.).

Back to the future – spectral synthesis, cont.

- Spectral synthesis analyzes the *ideal structure* of $L^1(G)$; its theorems are *the Nullstellensatz* from algebraic geometry of harmonic analysis.
- Given $\Lambda \subseteq \Gamma$ closed. Define
  $Z(\phi) := \{\gamma \in \Gamma : \phi(\gamma) = 0\}$;
  $k(\Lambda) := \{\phi \in A(\Gamma) : \Lambda \subseteq Z(\phi)\}$ - closed ideal;
  $j(\Lambda) := \{\phi \in A(\Gamma) : \Lambda \cap \mathrm{supp}(\phi) = \emptyset\}$ - ideal.
- Thus, Λ is an S-set $\Leftrightarrow \overline{j(\Lambda)} = k(\Lambda)$.
- *Wiener's inversion of Fourier series theorem*: Λ compact and $Z(\phi) \cap \Lambda = \emptyset \;\Rightarrow$

$$\exists \psi \in A(\Gamma), \text{ such that } \forall \gamma \in \Lambda, \quad \psi(\gamma) = 1/\phi(\gamma).$$

- This gives Wiener's Tauberian theorem! $\emptyset$ is an S-set!

Back to the future – Kronecker sets

Uniform approximation by characters, and Kronecker sets.

$\Lambda \subseteq \Gamma$ is a *Kronecker set* if

$$\forall \epsilon > 0 \text{ and } \forall \phi : \Lambda \to \mathbb{R}/\mathbb{Z}, \quad \phi \text{ continuous on } \Lambda \text{ and } |\phi| = 1 \text{ on } \Lambda,$$

$$\exists y = y_{\phi,\epsilon} \in G \text{ such that } \quad \sup_{\gamma \in \Lambda} |\phi(\gamma) - y(\gamma)| < \epsilon.$$

Kronecker's theorem in Diophantine analysis

$\Lambda \subset \mathbb{R}$ finite and independent $\Rightarrow$ Λ is a Kronecker set.

Theorem

$\Lambda \subseteq \Gamma$ Kronecker and $\mathrm{supp}(T) \subseteq \Lambda$, where $T \in A'(\Gamma) \Rightarrow T \in M_b(\Gamma)$. In particular, $\Lambda \subseteq \Gamma$ is an S-set.

Back to the future – the prescient Yves Meyer

Uniform approximation by characters, and Meyer sets.

- Yves Meyer's theory of harmonious, i.e., Meyer, sets (1972) and Dan Schectman theory of quasi-crystals (1982), one mathematical and one physical, *and* essentially equivalent, as well as being related to Penrose tilings.
- Model sets are Meyer sets in $\mathbb{R}^d$; and Meyer sets were defined to substitute for the fact that $\mathbb{Q}_p$ has no discrete subgroups, see [BB2019].
- By the nature of quasi-crystals and model sets, we deal with [aperiodic tilings without translational symmetries, as well as] icosahedral and dodecahedral (pyritohedral) quasi-crystal geometric objects *appearing physically in nature*!
- Schectman – Nobel Prize 2011. Meyer – Gauss Prize in 2010 and Abel Prize in 2017.
- Bombieri, Lagarias, Meyer, Moody, Senechal, Chenzhi Zhao.

Uniform approximation by characters, and Meyer sets.

$\Lambda \subseteq \Gamma$, $\langle \Lambda \rangle = \Lambda_d$ the group generated by Λ with the discrete topology. A Λ-*discrete character* is the restriction x_Λ to Λ of some algebraic homomorphism $x : \Lambda_d \to \mathbb{R}/\mathbb{Z}$.

$\Lambda \subseteq \Gamma$ is a *Meyer (harmonious) set* if every Λ-discrete character x_Λ can be uniformly approximated on Λ by some $y \in G$, i.e.,

$$\forall \epsilon > 0 \text{ and } \forall x_\Lambda,\ \exists y = y_{\Lambda,\epsilon} \in G \text{ such that } \quad \sup_{\gamma \in \Lambda} |x_\Lambda(\gamma) - y(\gamma)| < \epsilon.$$

Remark. Because of the discrete topology, x is continuous. G gives rise to a natural subset of Λ-discrete characters; in fact, the restriction of $x \in G$ to Λ_d is an algebraic homomorphism $\Lambda_d \to \mathbb{R}/\mathbb{Z}$. The elements of this subset are referred to as Λ-*characters*. This means that a Λ-character $y|_\Lambda$ is the restriction to Λ of some $y \in G$, recalling that $y : G \to \mathbb{R}/\mathbb{Z}$ is a continuous homomorphism and so $y|_\Lambda : \Lambda \to \mathbb{R}/\mathbb{Z}$ is unimodular and continuous.

Now – spectral synthesis to spectral super-resolution

Every $\Lambda \subseteq \mathbb{Z}^d$ is Meyer; consider the sub-category Λ_α of model sets .

Theorem (Matei and Meyer)

Given *infinite* $\Lambda_\alpha \subseteq \mathbb{Z}^d = \Gamma$ and $\nu = \sum_{j=1}^N w_j \delta_{x_j} \in M_b(\mathbb{T}^d)$, each $w_j \geq 0$. Then, $\mu \in M_b(\mathbb{T}^d)$ positive and $\widehat{\mu}(\lambda) = \widehat{\nu}(\lambda)$ on $\Lambda_\alpha \Rightarrow \mu = \nu$.

- Candès and Fernandez-Granda (2013 – 2014). Setting: discrete $\mu \in M_b(\mathbb{T}^d)$, where $\widehat{\mu} = F$ is known on *finite* $\Lambda = \{-M, \ldots, M\}^d$.
- To deal with *continuous singular measures*, with applications such as edge detection, define a *minimal extrapolation* ν by :

$$\epsilon(F, \Lambda) := \inf\{\|\nu\| : \nu \in M_b(\mathbb{T}^d) \text{ and } F = \widehat{\nu} \text{ on } \Lambda\}.$$

- Set

$$\Psi(F, \Lambda) := \{m \in \Lambda : |F(m)| = \epsilon(F, \Lambda)\}.$$

Now – spectral synthesis to spectral super-resolution

The fundamental technology goes back to Beurling.

Theorem (with Weilin Li)

Let F be spectral data on a finite set $\Lambda \subseteq \mathbb{Z}^d$.

(a) Suppose $\Psi = \emptyset$. Each minimal extrapolation of F on Λ is a singular measure.

(b) Suppose $\#\Psi \geq 2$. For each distinct pair $m, n \in \Psi$, define $\alpha_{m,n} \in \mathbb{R}/\mathbb{Z}$ by $e^{2\pi i \alpha_{m,n}} = F(m)/F(n)$ and define the closed set,

$$S = \bigcap_{\substack{m,n\in\Psi \\ m\neq n}} \{x \in \mathbb{T}^d \colon x \cdot (m-n) + \alpha_{m,n} \in \mathbb{Z}\},$$

an intersection of $\binom{\#\Psi}{2}$ periodic hyperplanes. Each minimal extrapolation of F on Λ is a singular measure supported in S.

Corollaries allow distinguishing discrete and continuous measures, as well as obtaining *unique* extrapolations.

Norbert Wiener Center for Harmonic Analysis and Applications

The best results are by Ray Johnson and Bob Warner.

The S-set geometry problem

Determine the S-set, Fourier decomposition properties of geometric objects in terms of classical criteria, e.g., curvature, and in light of current problems in dimension reduction, machine learning, and deep neural networks. Earliest, excellent results are due to Domar.

It WAS the 1960s

The teach-ins, and marches, and protests, a crazy war, my honorable discharges (2 of them!), Bobby Kennedy campaigning at NYU in 1964 moving dazzlingly through the crowds, the soothing jazz of Carmen Mcrae at the Village Gate, novelist Norman Mailer inebriated and with Dwight MacDonald and poet Robert Lowell at the Ambassador in D.C. (and Cathy and I in attendance) before the 1967 March on the Pentagon, the UMD faculty senate taking a stance!, counseling COs. Chandler, cerebral and passionate - his student was only the latter :-)

1968 Harmonic Analysis Warwick Conference

- Beurling and Malliavin –working on their Acta collaborations, Beurling visible like an omniscient spirit.
- Benedicks, Gavin Brown, Burckel, Jim Byrnes, Carleson, Paul Cohen, Coifman, Domar, C. Fefferman, R. Fefferman, Figá-Talamanca, Gärding, Gaudry, Graham, L. Hedberg, Heinig, Herz, Hewitt, Kahane, Katznelson, Koosis, Körner, Henry Landau, Malliavin, Yves Meyer, McGehee, Bill Moran, John Price, Reiter, Ross, Rudin, Bob Ryan, Saeki, I. Segal, Eli Stein, Varopoulos, Vesentini, Guido Weiss, Wik, Zygmund. I know I must be missing some of those who are closest.
- Reiter's book had just appeared with acknowledgement to John Gilbert, born 1939-07-16 – a fantastic day in the history of mother earth!
- Experts were in the hunt for a counterexample to the union problem for Helson sets. Malliavin told me he thought it was true, and he was right – solved by Drury and Varopoulos in 1970.

Gleason’s theorem - another trip back to the future

- Gleason’s theorem provides a fundamental connection in quantum measurement theory between the Garrett Birkhoff - John von Neumann quantum logic lattice interpretation of quantum events and the Born model for probability in quantum mechanics.
- Gleason’s essential device is a concept based on ONBs.
- Positive operator valued measurements or measures (POVMs) lead to extending Gleason’s concept to Parseval frames (the tantalizing terminology due to **Baggett**) and to intriguing counting problems, that in turn allow us to weaken hypotheses of a basic quantum measurement theorem in the POVM setting. - with Paul Koprowski and Jack Nolan.

Cathy and I married !!

The beginning !

THANK YOU !!!!!!!!!!