

# **The Cumulative Distribution Transform For Data Analysis And Machine Learning**

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Vanderbilt University

**Jubilee of Fourier Analysis and Applications  
In Celebration of John Benedetto 80th Birthday**

Jubilee- 2019

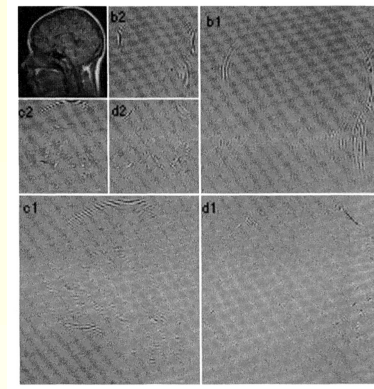
Supported by NIH Grant (Gustavo Rohde PI)



Gustavo Rohde

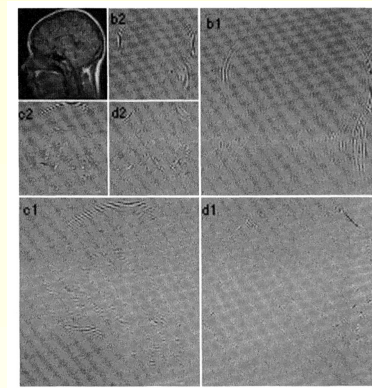
# Transports Transform

Transforms: Fourier Transform, Wavelet transform, Zak Transform, Shearlets, Scattering “transform,” ...



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Transport Transforms: Non-linear transforms based on transport theory: Monge and Katorovich Transport theory

## The Monge Problem (1781)

The Monge Problem Let  $\mu$  be a pile of sand on  $X \subset \mathbb{R}^n$ , find the "most efficient way" to transport it to the hole in the ground  $\nu$  on  $X \subset \mathbb{R}^n$ .

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Let  $\mu, \nu$  be probability measures on  $\mathbb{R}^n$  find a map  $T^\dagger : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such

$$T^\dagger = \arg \min_{\nu=T\#\mu} \int_{\mathbb{R}^n} \|x - T(x)\|^2 d\mu(x),$$

where  $\nu(B) = \mu(T^{-1}(B))$  for all measurable measurable sets  $B$ .

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Weak version of the Monge problem: The Kantorovich problem (1939).

## Brenier's Theorem (1991)

(Brenier's Theorem) Let  $\mu, \nu$  be two probability measures on  $\mathbb{R}^n$  (finite 2nd moments) that are absolutely continuous w.r.t Lebesgue measure. Then there exists a map  $T^\dagger : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

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**The transport transform:** Let  $d\mu(x) = s(x)dx$  and  $d\nu(x) = s_0(x)dx$ , where  $r$  is a fixed reference signal, then the transform  $\tilde{s}$  of  $s$  is the unique solution to the Monge problem above, i.e.,  $\tilde{s} = T^\dagger$ .

## The CDT transform

Let  $s$  a smooth probability density function on  $[0, 1] \subset \mathbb{R}$ , and  $s_0$  a reference probability density function on  $[0, 1] \subset \mathbb{R}$ .

### The Cumulative Distribution Transform

$$\int_0^{\tilde{s}(x)} s(\xi) d\xi = \int_0^x s_0(\xi) d\xi, \quad x \in [0, 1].$$

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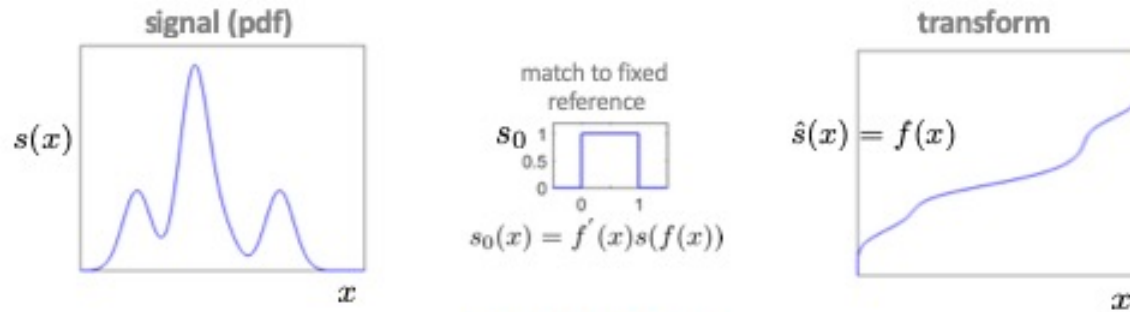
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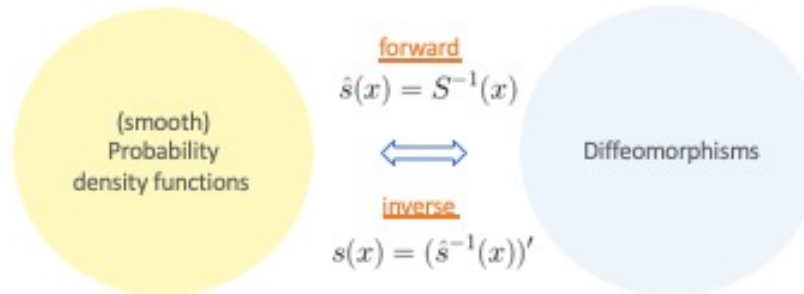
### Inverse Transform:

$$\tilde{s}'(x) s(\tilde{s}(x)) = s_0(x).$$

# CDT and its inverse



nonlinear map



[Park, Kolouri, Kundu, Rohde, ACHA 2018]

# Radon-CDT

## Radon Transform

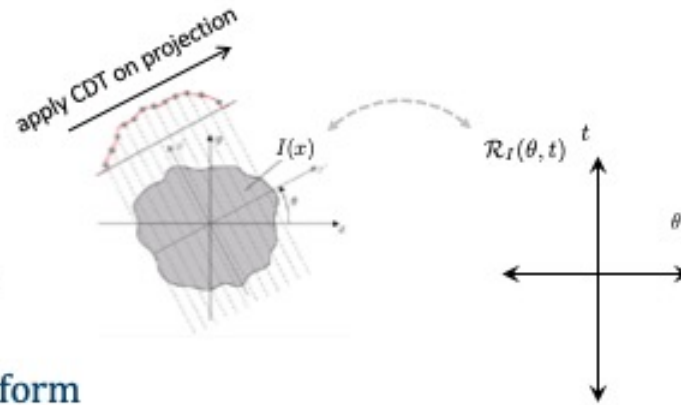
Forward

$$\mathcal{R}_I(\theta, t) = \int I(x) \delta(t - \omega \cdot x) dx$$

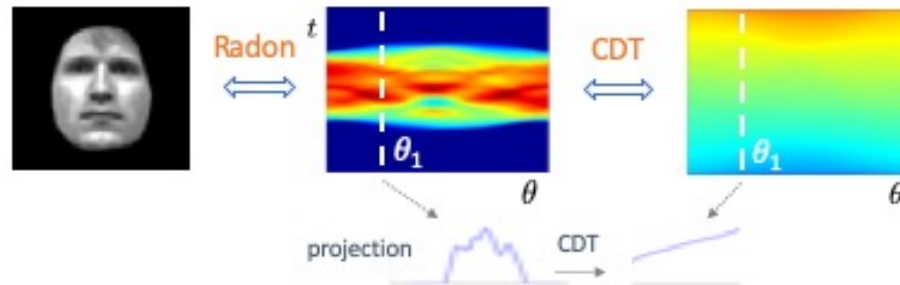
$$\omega = [\sin \theta, \cos \theta]^T$$

Inverse

$$\mathcal{R}_I^{-1}(x) = \int_{\mathbb{S}^{d-1}} (\mathcal{R}I(\cdot, \theta) * \eta(\cdot)) \circ (x \cdot \theta) d\theta$$



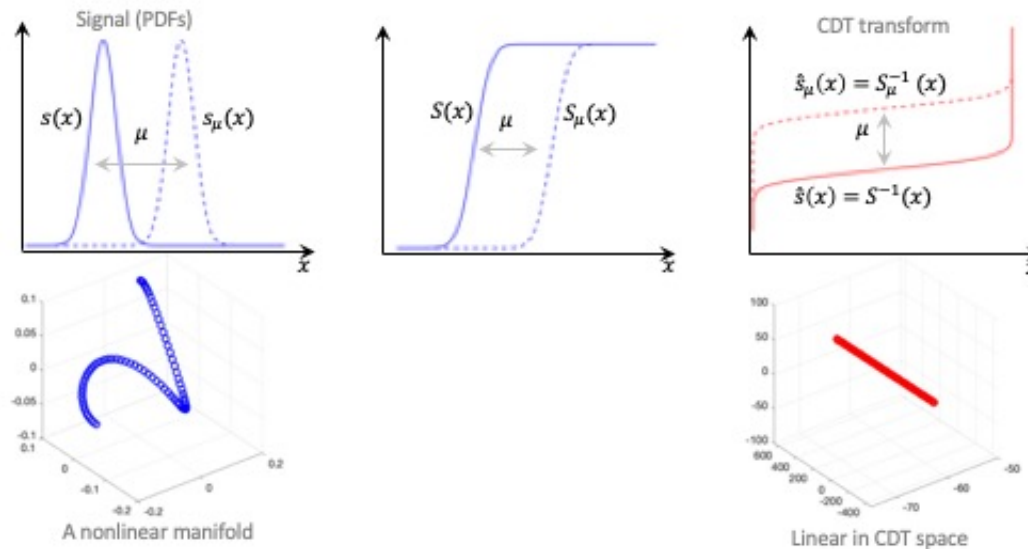
## Radon Cumulative Distribution Transform



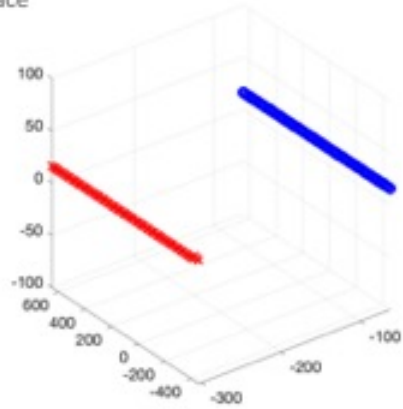
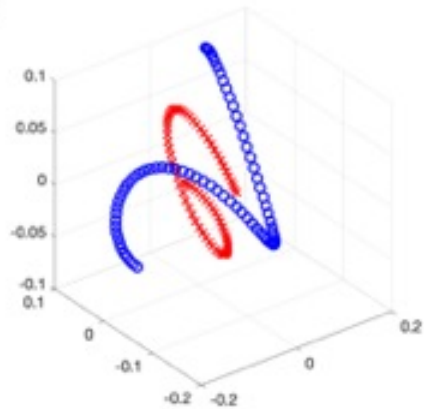
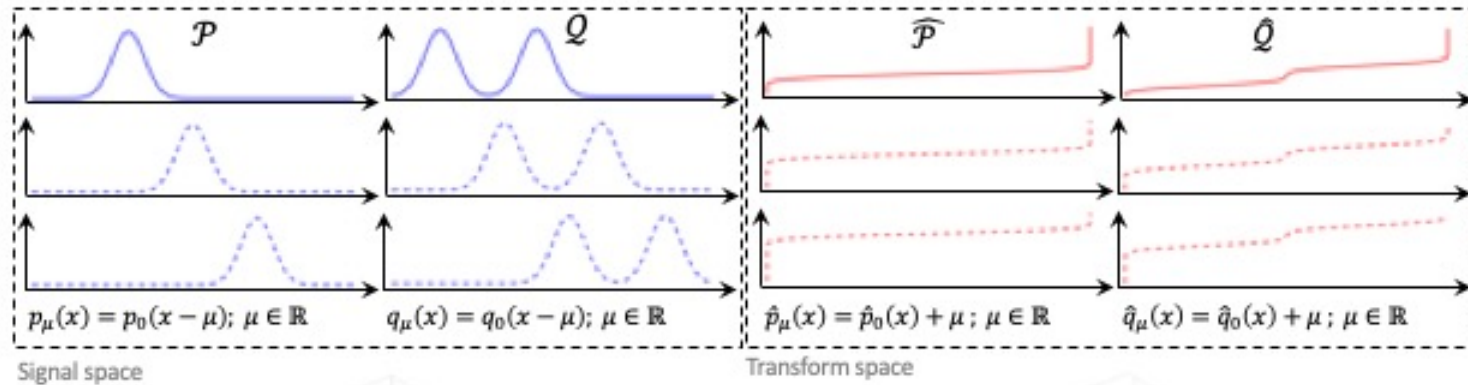
[Kolouri, Park, Rohde, IEEE TIP 2016]

# Convexification properties

$$T_s : \mathbb{R} \rightarrow L^2(\mathbb{R}): T_s(\mu)(x) = s_\mu(x) = s(x - \mu).$$



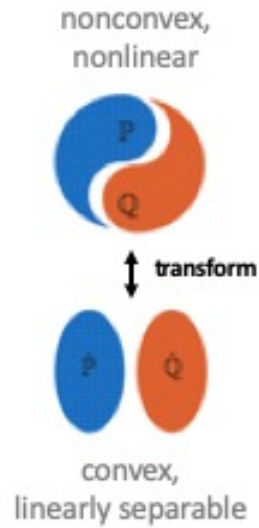
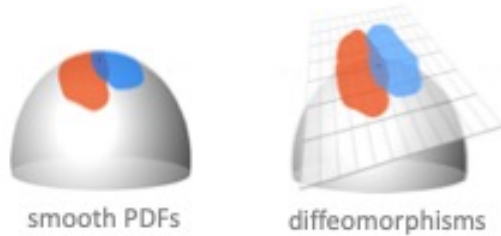
# Convexification properties





# Convexification properties

## Nonlinear signal transform



- [Wang, et al, IJCV, 2014]
- [Basu et al, PNAS 2014]
- [Kolouri et al IEEE TIP 16]
- [Kolouri et al, Pat. Rec, 17]
- [Park et al, ACHA 18]
- [Kolouri et al. IEEE SPM 17]
- [Imagedatascience.com/transport](http://Imagedatascience.com/transport)

## Wasserstein distance between two measures

Let  $\Pi(\mu, \nu)$  be the set of all probability measures on  $\mathbb{R}^n \times \mathbb{R}^n$  with marginals  $\mu$  and  $\nu$ .

**2-Wasserstein distance between  $\mu$  and  $\nu$ :**

$$W_2^2(\mu, \nu) = \min_{\pi \in \Pi} \int_{\mathbb{R}^n} \|x - y\|^2 d\pi(x, y).$$

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The solution  $T^\dagger$  of the transport map of the Monge problem give rise to the minimizer to the Kantorovich problem:  $d(\mu(x)\delta(y = T^\dagger(x))) \in \Pi$  and minimizes  $W_2(\mu, \nu)$ .

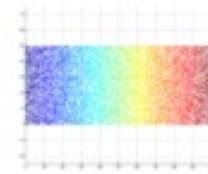
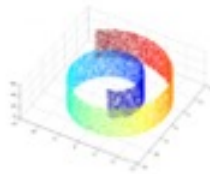
## Wasserstein distance between two measures

Theorem (Kolouri, Rohde ) Let  $s_1$  and  $s_2$  be two signals (PDFs) and  $\tilde{s}_1, \tilde{s}_2$  be their CDT transform with respect to fixed reference  $s_0$ . Then

$$\|\tilde{s}_2 - \tilde{s}_1\|_{L^2(\mathbb{R}^n)}^2 = W_2(\mu, \nu) = \min_{\pi \in \Pi} \int_{\mathbb{R}^n} \|x - y\|^2 d\pi(x, y)$$

where  $d\mu(x) = s_1(x)dx$  and  $d\nu = s_2(x)dx$ .

signal space,  
Wasserstein  
geodesics



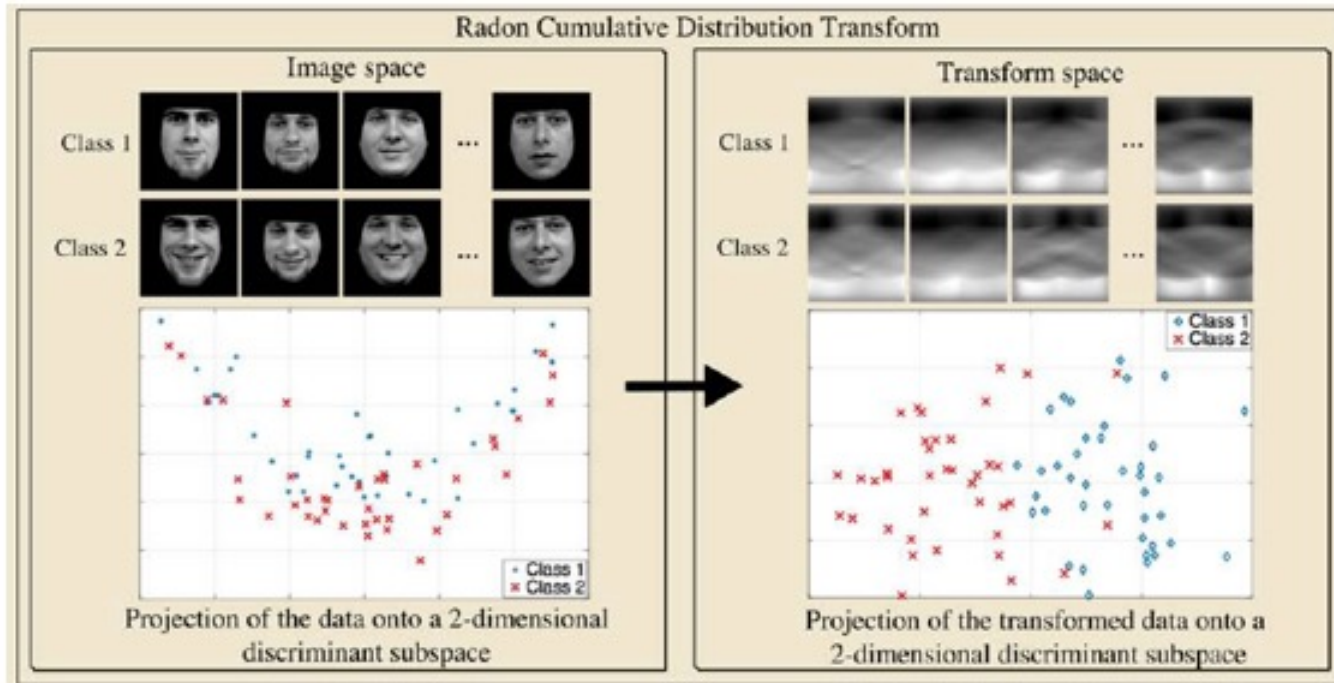
transform space  
Euclidean geodesics

[Kolouri, Rohde, CVPR 16]



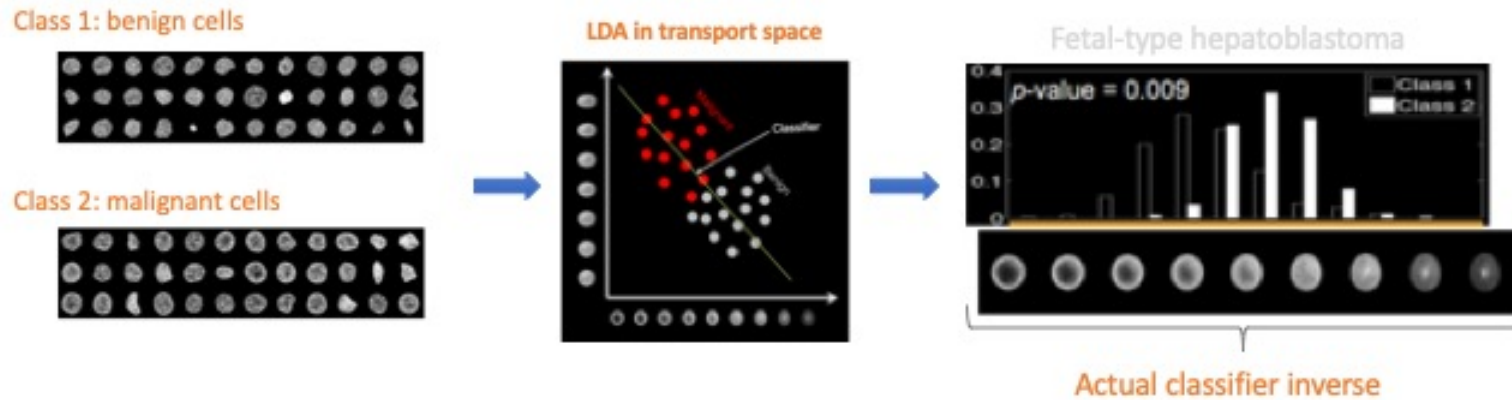
# Applications

Example: classifying facial expressions



# Applications

(TBM): modeling discriminant information



## CANCER MODELING

- Liver: Basu et al, PNAS 2014
- Thyroid: Ozolek et al, Medical Image Analysis, 2014
- Skin melanoma: Liu et al, J. Pathology Informatics, 2016
- Lung mesothelioma: Tosun et al, Cytometry A, 2015.
- Breast carcinoma: unpublished



# Applications

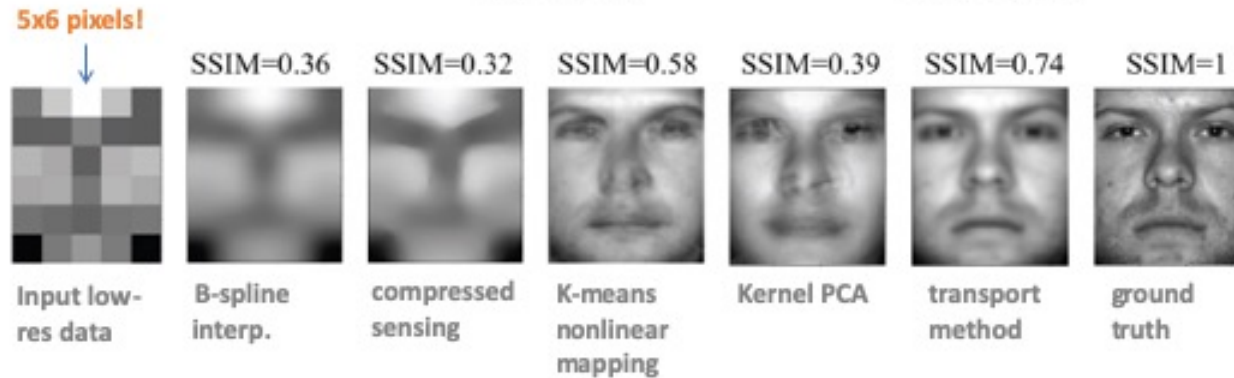
## Application: single image super-resolution

**Model:**  $M_\alpha = D_{f_\alpha}(x)I_0(f_\alpha(x)) \quad f_\alpha(x) = \sum_k \alpha_k \phi_k(x)$

**Fit:**  $\min_\alpha \|I - VM_\alpha\|^2$

low-res data (pointing to  $I$ )  
 degradation (pointing to  $M_\alpha$ )

Transport PCA trained on high res database (under the sum in  $f_\alpha(x)$ )



Kolouri, Rohde, Comp. Vision & Pattern Rec., 2015.

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# Happy Birthday Caro