The Cumulative Distribution Transform For Data Analysis And Machine Learning

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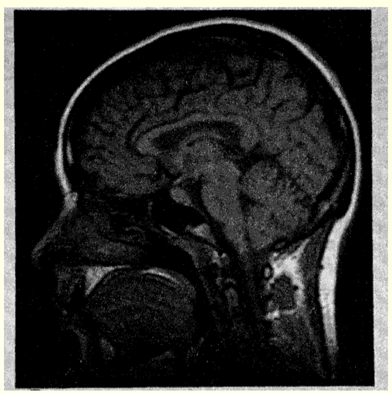
Jubilee of Fourier Analysis and Applications
In Celebration of John Benedetto 80th Birthday

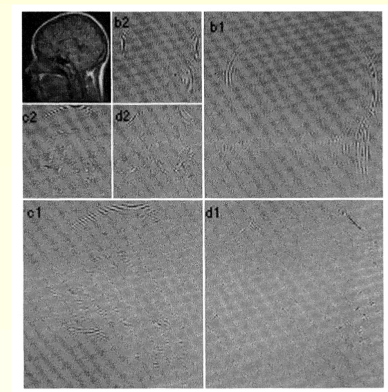
Supported by NIH Grant (Gustavo Rohde PI)

Gustavo Rohde

Transports Transform

Transforms: Fourier Transform, Wavelet transform, Zak Transform, Shearlets, Scattering “transform,” ...





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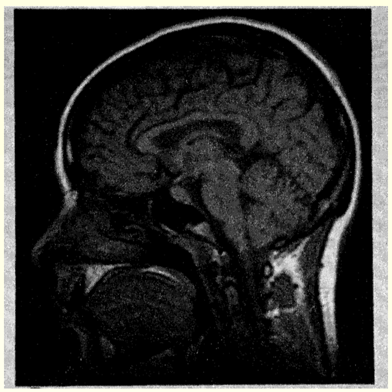

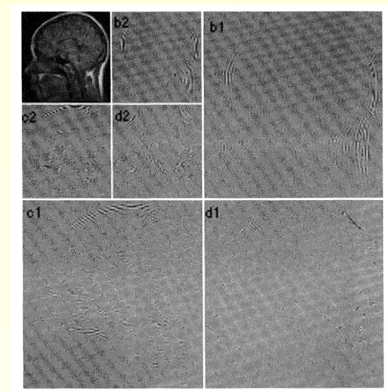

Transport Transforms: Non-linear transforms based on transport theory: Monge and Katorovich Transport theory

The Monge Problem (1781)

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Let μ, ν be probability measures on $\mathbb{R}^n$ find a map $T^\dagger : \mathbb{R}^n \to \mathbb{R}^n$ such

$$T^\dagger = \arg \min_{\nu = T_{\#}\mu} \int_{\mathbb{R}^n} \|x - T(x)\|^2 d\mu(x),$$

where $\nu(B) = \mu(T^{-1}(B))$ for all measurable measurable sets B.

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Weak version of the Monge problem: The Kantorovich problem (1939).

Brenier's Theorem (1991)

(Brenier's Theorem) Let μ, ν be two probability measures on $\mathbb{R}^n$ (finite 2nd moments) that are absolutely continuous w.r.t Lebesgue measure. Then there exists a map $T^\dagger : \mathbb{R}^n \to \mathbb{R}^n$ such that

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The transport transform: Let $d\mu(x) = s(x)dx$ and $d\nu(x) = s_0(x)dx$, where r is a fixed reference signal, then the transform $\tilde{s}$ of s is the unique solution to the Monge problem above, i.e., $\tilde{s} = T^\dagger$.

The CDT transform

Let s a smooth probability density function on $[0,1] \subset \mathbb{R}$, and s_0 a reference probability density function on $[0,1] \subset \mathbb{R}$.

The Cumulative Distribution Transform

$$\int_0^{\tilde{s}(x)} s(\xi) d\xi = \int_0^{x} s_0(\xi) d\xi, \quad x \in [0,1].$$

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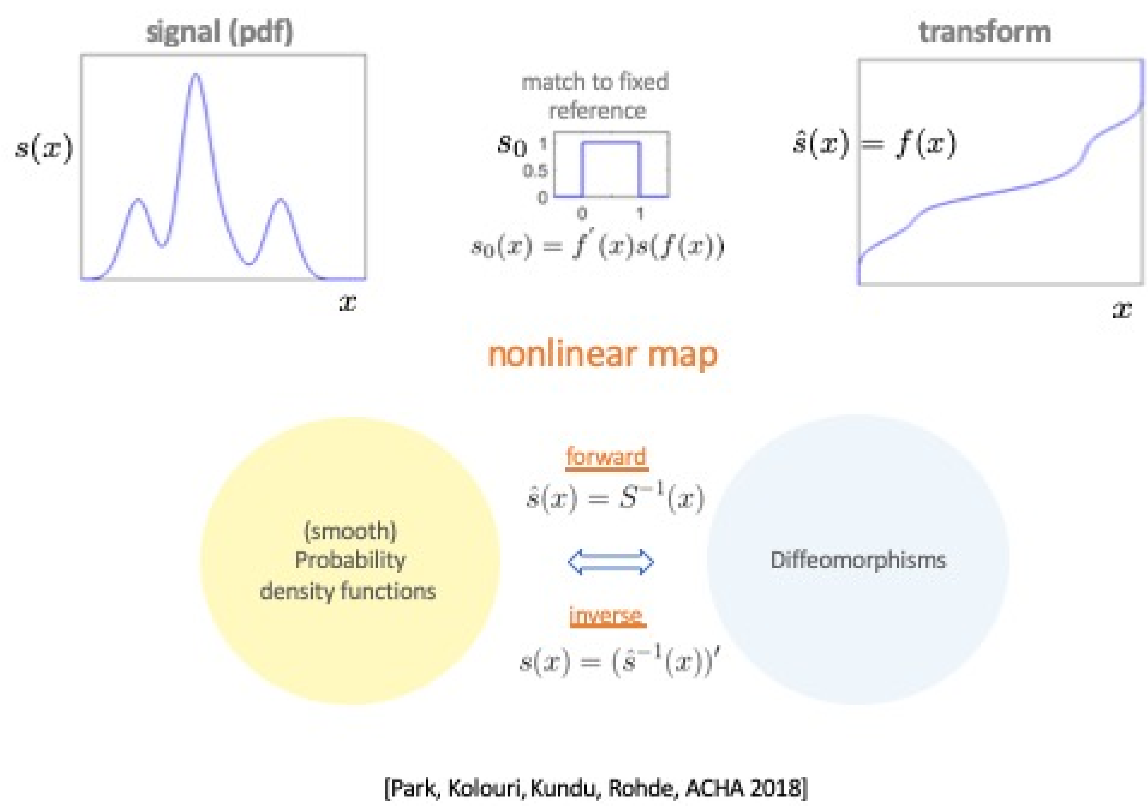
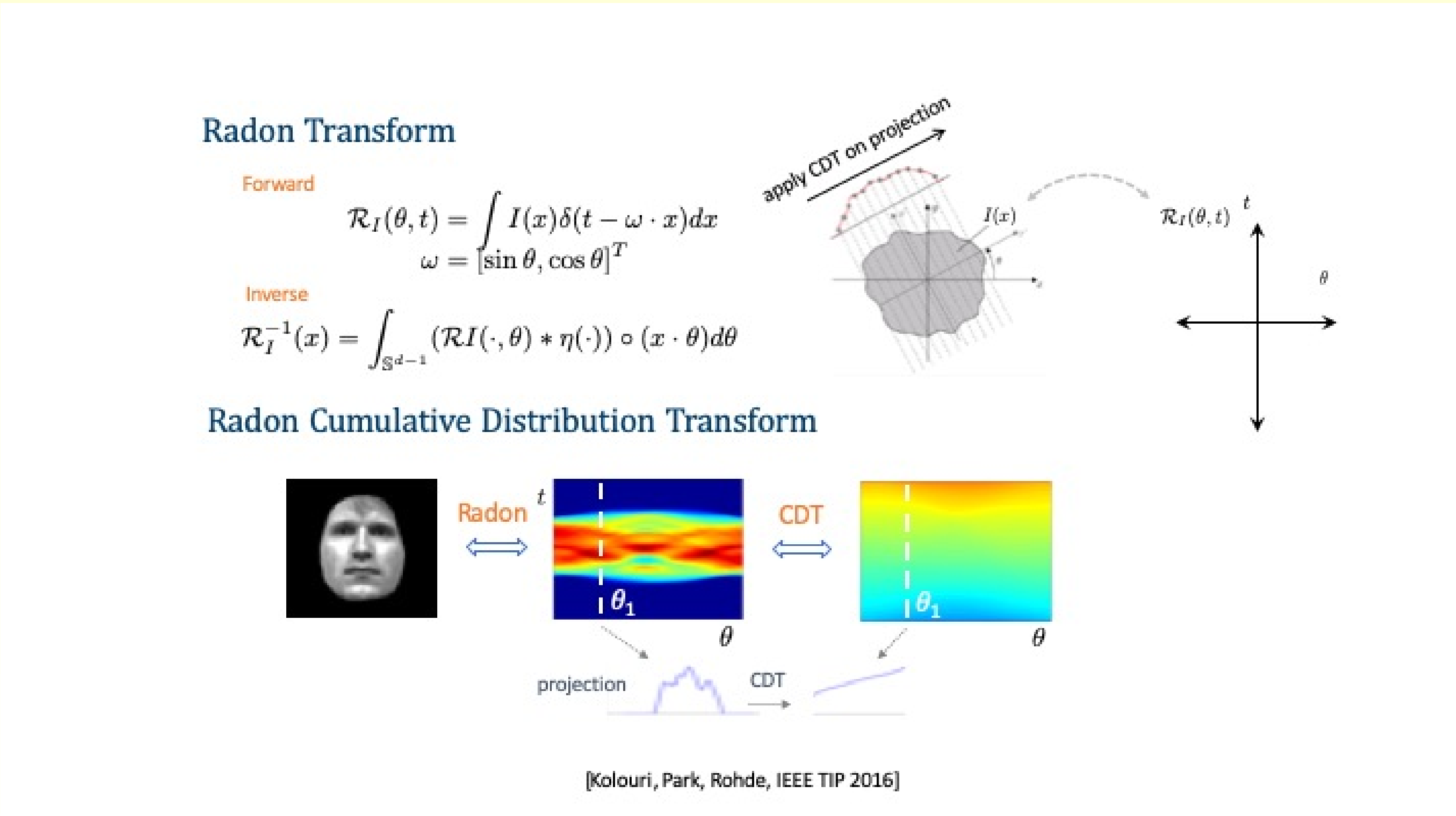
Inverse Transform:

$$\tilde{s}'(x) s(\tilde{s}(x)) = s_0(x).$$

CDT and its inverse

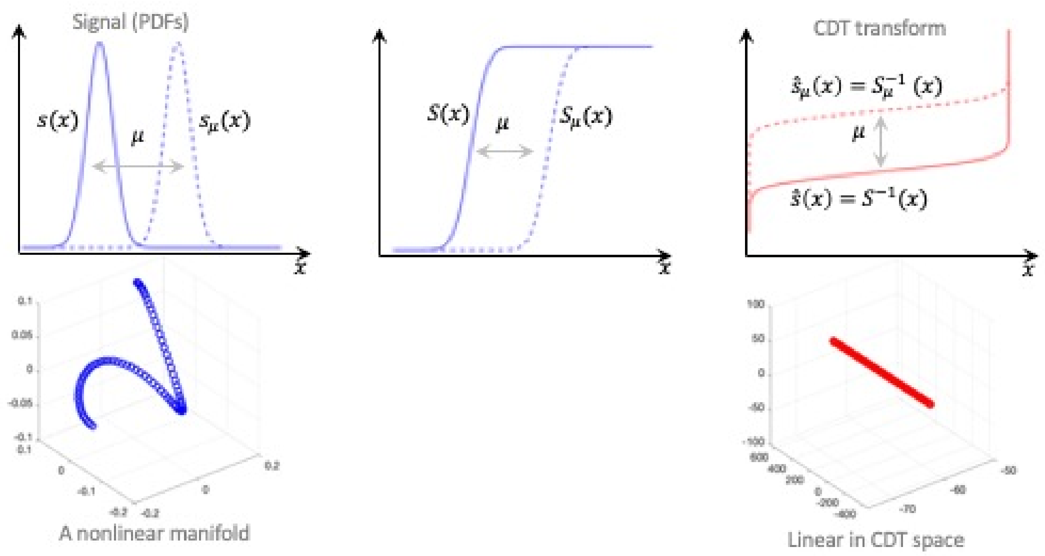


Radon-CDT


Radon Transform
Forward
$\mathcal{R}_I(\theta, t) = \int I(x)\delta(t - \omega \cdot x)dx$
$\omega = [\sin\theta, \cos\theta]^T$
Inverse
$\mathcal{R}_I^{-1}(x) = \int_{\mathbb{S}^{d-1}} (\mathcal{R}I(\cdot,\theta) * \eta(\cdot)) \circ (x \cdot \theta) d\theta$
apply CDT on projection
$I(x)$
$\mathcal{R}_I(\theta, t)$
t
θ
Radon Cumulative Distribution Transform
Radon
CDT
t
θ_1
θ
θ_1
θ
projection
CDT
[Kolouri, Park, Rohde, IEEE TIP 2016]

Convexification properties

$T_s : \mathbb{R} \to L^2(\mathbb{R})$: $T_s(\mu)(x) = s_\mu(x) = s(x-\mu)$.



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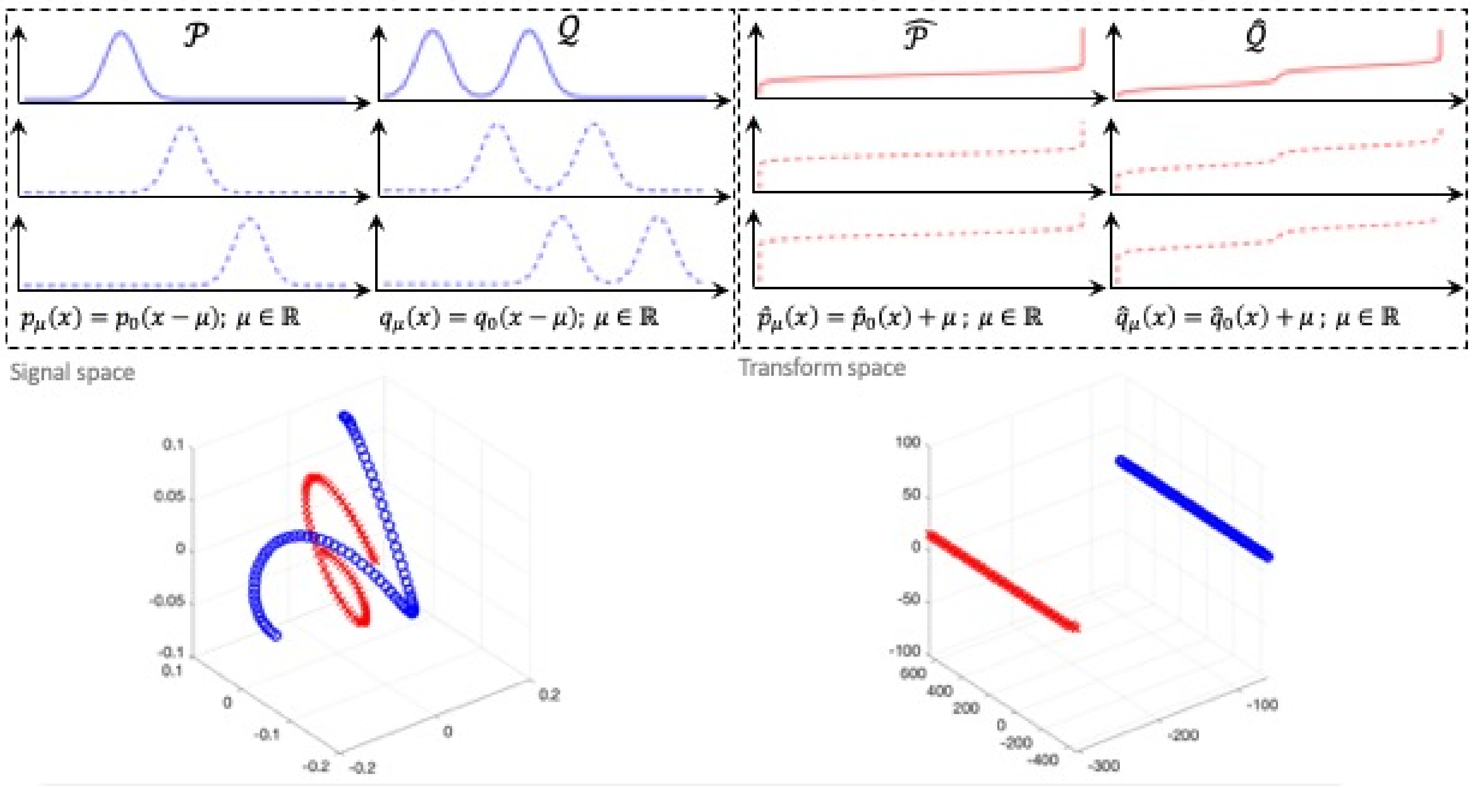

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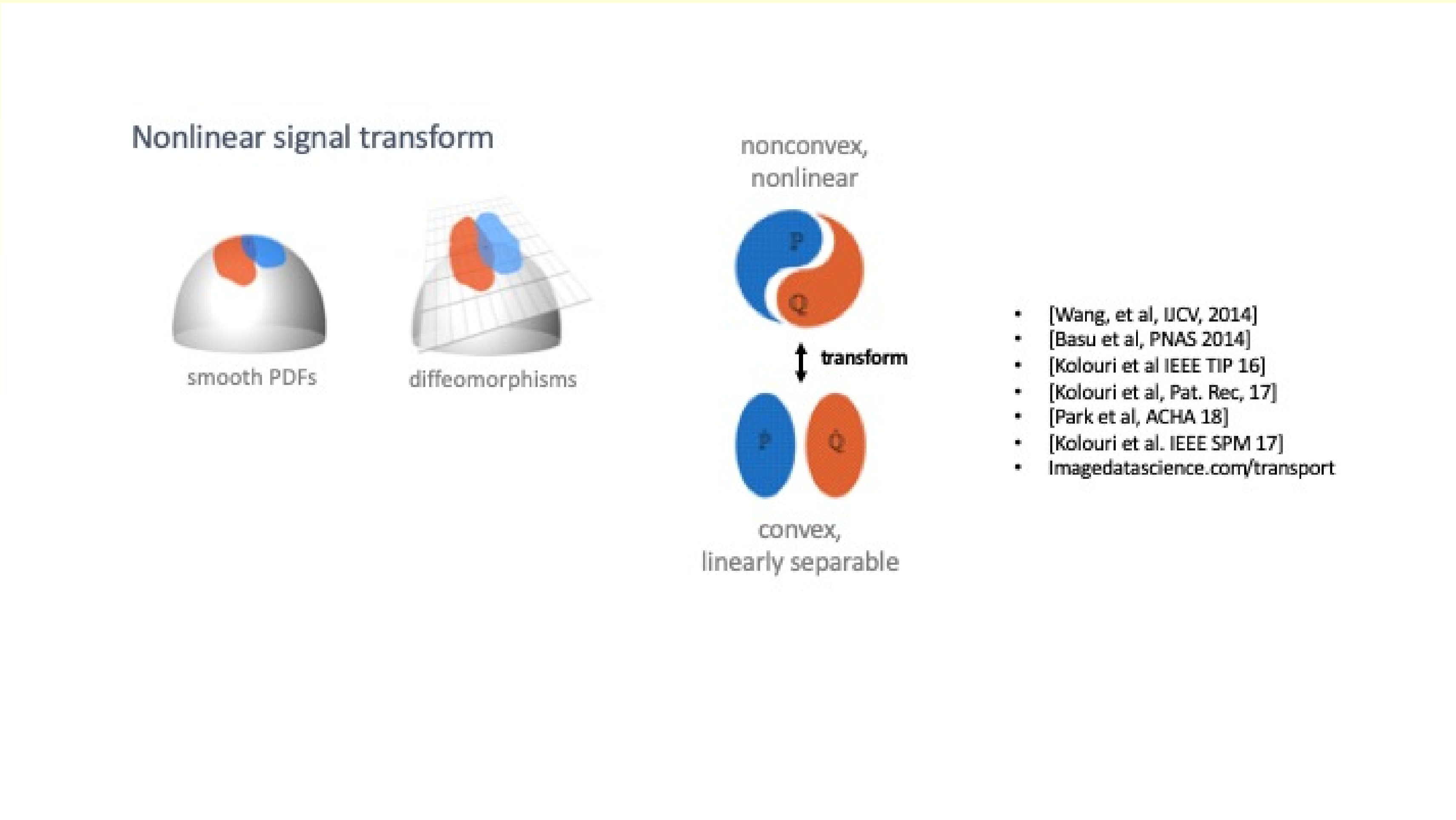

Wasserstein distance between two measures

Let $\Pi(\mu, \nu)$ be the set of all probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals μ and ν.

2-Wasserstein distance between μ and ν:

$$W_2^2(\mu, \nu) = \min_{\pi \in \Pi} \int_{\mathbb{R}^n} \|x - y\|^2 d\pi(x, y).$$

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The solution $T^\dagger$ of the transport map of the Monge problem give rise to the minimizer to the Kantorovich problem: $d(\mu(x)\delta(y=T^\dagger(x)) \in \Pi$ and minimizes $W_2(\mu,\nu)$.

Wasserstein distance between two measures

Theorem (Kolouri, Rohde) Let s_1 and s_2 be two signals (PDFs) and $\tilde{s}_1, \tilde{s}_2$ be their CDT transform with respect to fixed reference s_0. Then

$$\|\tilde{s}_2 - \tilde{s}_1\|^2_{L^2(\mathbb{R}^n)} = W_2(\mu, \nu) = \min_{\pi \in \Pi} \int_{\mathbb{R}^n} \|x - y\|^2 d\pi(x, y)$$

where $d\mu(x) = s_1(x)dx$ and $d\nu = s_2(x)dx$.


signal space,
Wasserstein
geodesics
transform space
Euclidean geodesics
[Kolouri, Rohde, CVPR 16]
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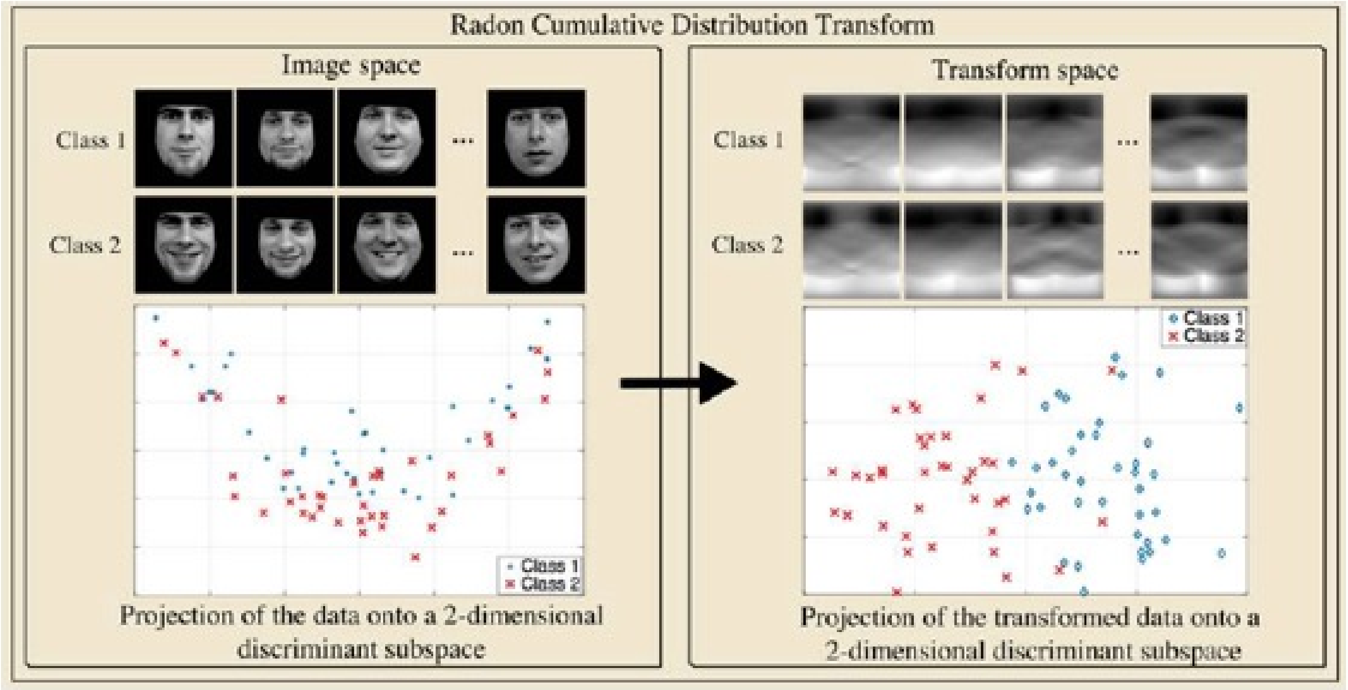
Applications

Example: classifying facial expressions



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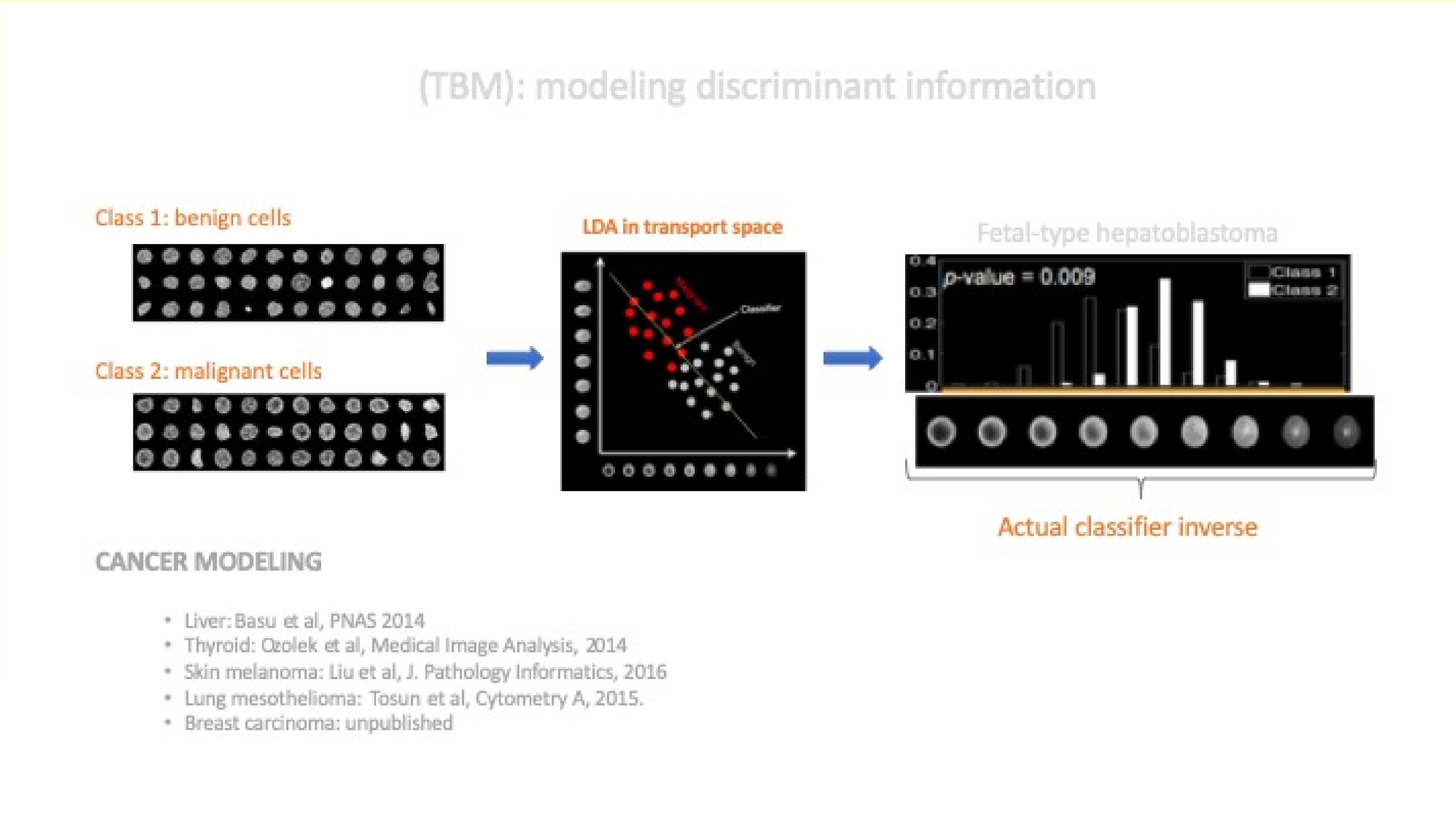

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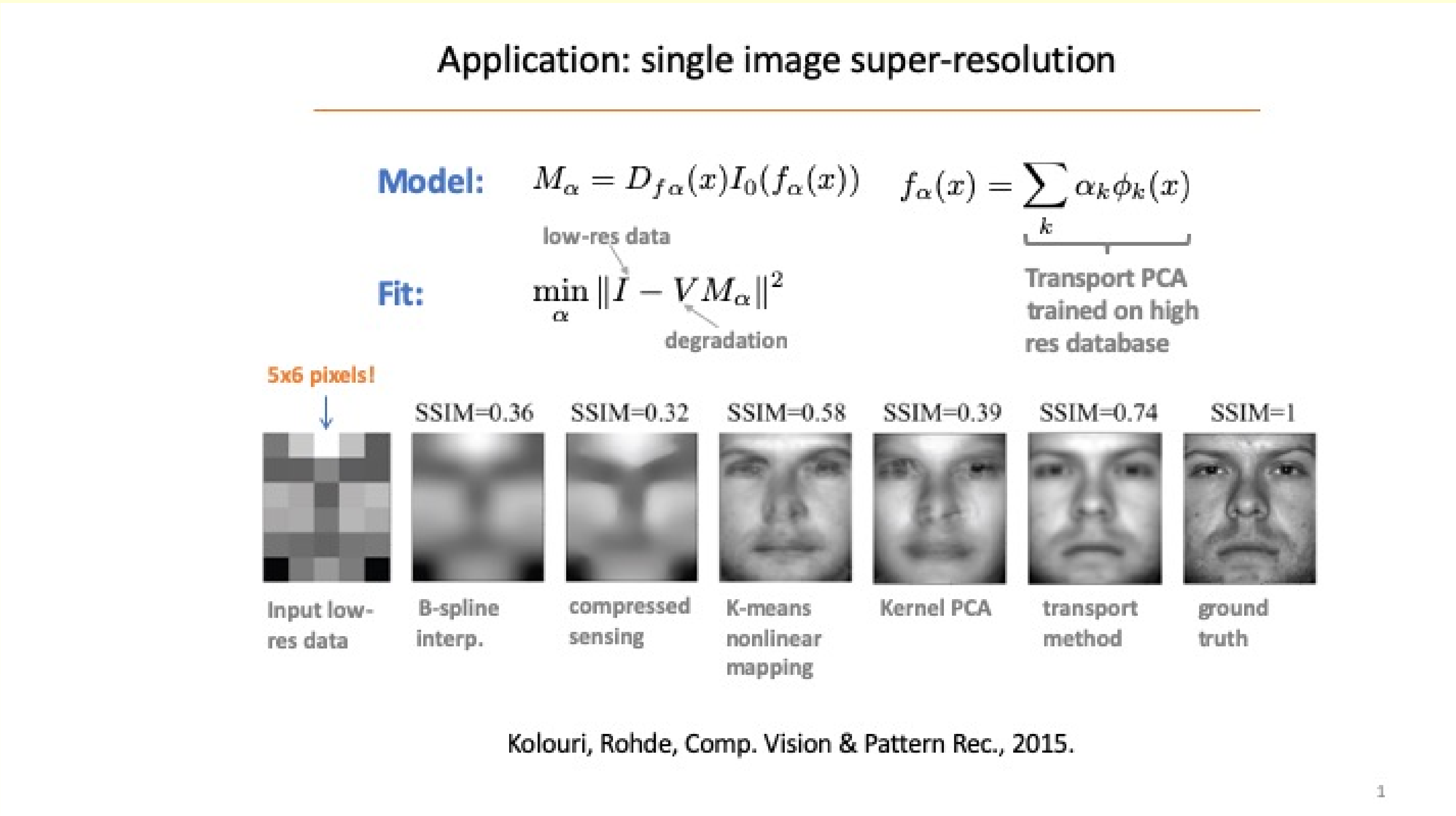

Happy Birthday Caro