Multi-D wavelet construction using Quillen-Suslin theorem for Laurent polynomials

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Outline

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Quillen-Suslin theorem for Laurent polynomials

A column q-vector $\mathtt{D}(z)$ with Laurent polynomial entries is unimodular if it has a left inverse, i.e. if there exists a row q-vector $\mathtt{F}(z)$ s.t. $\mathtt{F}(z)\mathtt{D}(z) = 1$.

We assume $z \in \mathbb{C}^n$, $|z| = 1$.

Example:
$\mathtt{D}(z) = [\frac{1}{2}, \frac{1}{4}z_1^{-1} + \frac{1}{4}, \frac{1}{4}z_2^{-1} + \frac{1}{4}, \frac{1}{4}z_1^{-1}z_2^{-1} + \frac{1}{4}]^T$
is unimodular since $[2, 0, 0, 0]$ is a left inverse of $\mathtt{D}(z)$.
Another left inverse of $\mathtt{D}(z)$ is
$[-\frac{1}{8}z_1^{-1} - \frac{1}{8}z_2^{-1} - \frac{1}{8}z_1^{-1}z_2^{-1} + \frac{5}{4} - \frac{1}{8}z_1 - \frac{1}{8}z_2 - \frac{1}{8}z_1z_2, \frac{1}{4} + \frac{1}{4}z_1, \frac{1}{4} + \frac{1}{4}z_2, \frac{1}{4} + \frac{1}{4}z_1z_2]$
The first one is simpler but the second one has better accuracy.

Theorem (Quillen-Suslin Thm for Laurent poly by Swan, 1978)

Let $\mathtt{D}(z)$ be a unimodular column q-vector. Then there exists an invertible $q \times q$ matrix $\mathtt{T}(z)$ s.t. $\mathtt{T}(z)\mathtt{D}(z) = [1, 0, ..., 0]^T$.

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Designing filter bank (FB) using Laurent polynomials

Via polyphase representation (Vaidyanathan, 1993)

FB design problem: Find

- $\mathtt{H}(z)$, $\mathtt{J}_i(z)$, $i = 1, \ldots, p-1$: row q-vectors
- $\mathtt{D}(z)$, $\mathtt{K}_i(z)$, $i = 1, \ldots, p-1$: column q-vectors s.t.

$$\mathtt{S}(z)\mathtt{A}(z) := \begin{bmatrix} \mathtt{D}(z) & \mathtt{K}_1(z) & \cdots & \mathtt{K}_{p-1}(z) \end{bmatrix} \begin{bmatrix} \mathtt{H}(z) \\ \mathtt{J}_1(z) \\ \vdots \\ \mathtt{J}_{p-1}(z) \end{bmatrix} = \mathtt{I}_q$$

$\mathtt{A}(z)$: analysis bank; $\mathtt{S}(z)$: synthesis bank.
The above identity is called the perfect reconstruction property.
For the perfect reconstruction property to hold, we need $p \geq q$.

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The FB is called

- a non-redundant (or biorthogonal) FB if $p = q$. In this case, $\mathtt{A}(z)$ and $\mathtt{S}(z)$ are square matrices.
- a wavelet FB if
  - $\mathtt{H}(z)$: lowpass row q-vector
  - $\mathtt{J}_i(z)$, $i = 1, \ldots, p-1$: highpass row q-vectors
  - $\mathtt{D}(z)$: lowpass column q-vector
  - $\mathtt{K}_i(z)$, $i = 1, \ldots, p-1$: highpass column q-vectors

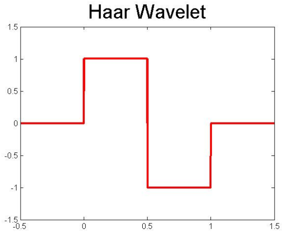
  $\Rightarrow$ wavelet FB design: a key step in wavelet construction
- a wavelet FB with m vanishing moments (VM) (for $m \geq 1$) if
  - $\mathtt{H}(z)$: lowpass row q-vector
  - $\mathtt{J}_i(z)$, $i = 1, \ldots, p-1$: row q-vectors with m VM
  - $\mathtt{D}(z)$: lowpass column q-vector
  - $\mathtt{K}_i(z)$, $i = 1, \ldots, p-1$: column q-vectors with m VM

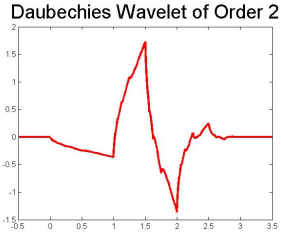
  $\Rightarrow$ leads to wavelets with m VM (high performance)

Very brief introduction to wavelets

- Wavelets are a collection of functions obtained by scaling and translating a fixed set of functions (mother wavelets).
- Wavelet is a subfield of Harmonic analysis and highly interdisciplinary. Wavelets are used in many applications (e.g. image/signal processing, compressive sensing).

Examples (1-D): Haar (1909), VM=1; Daubechies (1987), VM=2





Constructing multi-D wavelets is challenging and important.

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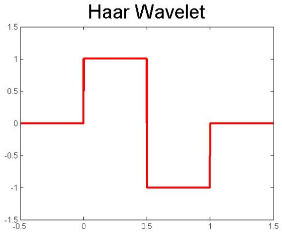

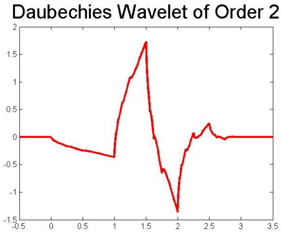

Constructing multi-D wavelets is challenging and important.

Current approach

for designing non-redundant wavelet FBs

Theorem (by Chen-Han-Riemenschneider, 2000)

Suppose $\mathtt{H}(z)$, $\mathtt{D}(z)$ *are lowpass vectors and*

$$\begin{bmatrix} \mathtt{D}(z) & \mathtt{K}_1(z) & \cdots & \mathtt{K}_{q-1}(z) \end{bmatrix} \begin{bmatrix} \mathtt{H}(z) \\ \mathtt{J}_1(z) \\ \vdots \\ \mathtt{J}_{q-1}(z) \end{bmatrix} = \mathtt{I}_q$$

Then the following are equivalent.

1. $\mathtt{H}(z)$, $\mathtt{D}(z)$ *have* m *accuracy (AC, or approximation order).*
2. $\mathtt{J}_i(z)$, $\mathtt{K}_i(z)$, $i = 1, ..., q-1$, *have* m *VM.*

Corollary (obtained by C-H-R and Q-S for Laurent polynomials)

Let $\mathtt{H}(z)$, $\mathtt{D}(z)$ *be lowpass vectors with* m *AC and* $\mathtt{H}(z)\mathtt{D}(z) = 1$. *Then there exist* $\mathtt{J}_i(z)$, $\mathtt{K}_i(z)$, $i = 1, ..., q-1$, *with* m *VM such that the perfect reconstruction property holds.*

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Vanishing moments (VM) and accuracy (AC)

Assume the dilation is dyadic, and $q = 2^n$.
Then, $\Gamma := \{0,1\}^n =: \{\nu_0 = 0, \nu_1, .., \nu_{q-1}\}$ can be chosen.
Notice that $\mathbb{Z}^n = \cup_{\nu \in \Gamma}(2\mathbb{Z}^n + \nu)$.

Definition (for dyadic dilation case)

For $\mathtt{H}(z) = [H_0(z), H_1(z), \ldots, H_{q-1}(z)]$, let $H(z) = \sum_{j=0}^{q-1} z^{\nu_j} H_j(z^2)$.
Let m be a nonnegative integer. Then $\mathtt{H}(z)$ has

- m VM if $\frac{\partial^k}{\partial \omega^k} H(e^{i\omega})|_{\omega=0} = 0$, $\forall |k| \le m-1$, and
- m AC if $\frac{\partial^k}{\partial \omega^k} H(e^{i\omega})|_{\omega=\gamma} = 0$, $\forall |k| \le m-1$, $\forall \gamma \in \{0,\pi\}^n \backslash 0$.

VM and AC for column vector $\mathtt{D}(z) = [D_0(z), D_1(z), \ldots, D_{q-1}(z)]^T$ is defined similarly by forming $D(z) = \sum_{j=0}^{q-1} z^{-\nu_j} D_j(z^2)$.

$\mathtt{H}(z)$ or $\mathtt{D}(z)$ is the highpass vector iff it has VM ≥ 1.

Current approach is not satisfactory

Finding $\mathtt{H}(z)$, $\mathtt{D}(z)$ satisfying assumptions of Corollary is not easy, especially if the AC or the spatial dimension n is large.

Example: Let $n = 2$, and let
$\mathtt{H}(z) = [\frac{1}{2}, \frac{1}{4}z_1^{-1} + \frac{1}{4}, \frac{1}{4}z_2^{-1} + \frac{1}{4}, \frac{1}{4}z_1^{-1}z_2^{-1} + \frac{1}{4}]$: lowpass with 2 AC.
Then $[2, 0, 0, 0]^T$: lowpass, a right inverse of $\mathtt{H}(z)$, but with 0 AC.
Using Maple implementation of Algebraic Geometry theory (Cox-Little-O'Shea, 2006), we see any right inverse of $\mathtt{H}(\mathtt{z})$ is

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}u_1(z) \begin{bmatrix} z_1^{-1} + 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}u_2(z) \begin{bmatrix} z_2^{-1} + 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \frac{1}{2}u_3(z) \begin{bmatrix} z_1^{-1}z_2^{-1} + 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$

for some Laurent polynomials $u_1(z), u_2(z), u_3(z)$. To find a right inverse of $\mathtt{H}(z)$ with 2 AC, one can use this parameterization. Usually done by fixing the total degree of Laurent poly u_1, u_2, u_3, and then increasing the total degree if needed (Riemenschneider-Shen, 1997; Han-Jia, 1999; Park, 2002).

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Outline

Our approach (theory & algorithm)

for designing non-redundant wavelet FBs

Inputs:

- $\mathtt{H}(z)$: row, lowpass, q-vector w/ unimodularity, positive AC.
- $\mathtt{G}(z)$: column, lowpass, q-vector w/ positive AC.
- $\mathtt{F}(z)$: column, lowpass, q-vector, a right inverse of $\mathtt{H}(z)$.

Algorithm: Set

- $\mathtt{D}(z) := \mathtt{G}(z) + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{G}(z))$: column, q-vector
- $\mathtt{T}(z)$: $q \times q$ invertible matrix s.t. $\mathtt{T}(z)\mathtt{H}(z)^T = [1, 0, .., 0]^T$.
- $\mathtt{K}_1(z), .., \mathtt{K}_{q-1}(z)$: 2nd to last columns of $\mathtt{T}(z)^T$.
- $\mathtt{J}_1(z), .., \mathtt{J}_{q-1}(z)$: 2nd to last rows of $\mathtt{T}(z)^{-T}[\mathtt{I}_q - \mathtt{F}(z)\mathtt{H}(z)][\mathtt{I}_q - \mathtt{G}(z)\mathtt{H}(z)]$.

Output:

Wavelet FB: $(\mathtt{D}(z), \mathtt{K}_1(z), .., \mathtt{K}_{q-1}(z))$, $(\mathtt{H}(z), \mathtt{J}_1(z), .., \mathtt{J}_{q-1}(z))$
where $\mathtt{D}(z)$ is a right inverse of $\mathtt{H}(z)$ with positive AC.

Example: Designing a 2-D wavelet FB

Let $n = 2$.
Let $\mathtt{H}(z) = \mathtt{G}(z)^* = [\frac{1}{2}, \frac{1}{4}z_1^{-1} + \frac{1}{4}, \frac{1}{4}z_2^{-1} + \frac{1}{4}, \frac{1}{4}z_1^{-1}z_2^{-1} + \frac{1}{4}]$: lowpass with 2 AC.
Let $\mathtt{F}(z) = [2, 0, 0, 0]^T$: lowpass with $\mathtt{H}(z)\mathtt{F}(z) = 1$, but with 0 AC.
Set $\mathtt{D}(z) = \mathtt{G}(z) + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{G}(z)) = \mathtt{H}(z)^* + (1 - \mathtt{H}(z)\mathtt{H}(z)^*)\mathtt{F}(z)$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} + \frac{1}{4}z_1 \\ \frac{1}{4} + \frac{1}{4}z_2 \\ \frac{1}{4} + \frac{1}{4}z_1z_2 \end{bmatrix} + (-\frac{1}{16}z_1^{-1} - \frac{1}{16}z_2^{-1} - \frac{1}{16}z_1^{-1}z_2^{-1} + \frac{3}{8} - \frac{1}{16}z_1 - \frac{1}{16}z_2 - \frac{1}{16}z_1z_2)\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$=[-\frac{1}{8}z_1^{-1} - \frac{1}{8}z_2^{-1} - \frac{1}{8}z_1^{-1}z_2^{-1} + \frac{5}{4} - \frac{1}{8}z_1 - \frac{1}{8}z_2 - \frac{1}{8}z_1z_2, \frac{1}{4} + \frac{1}{4}z_1, \frac{1}{4} + \frac{1}{4}z_2, \frac{1}{4} + \frac{1}{4}z_1z_2]^T$$

We see that $\mathtt{D}(z)$ has 2 AC.
From the implementation of Quillen-Suslin Theorem by Maple, we see that

$$\mathtt{T}(z) := \begin{bmatrix} 2 & 0 & 0 & 0 \\ -\frac{1}{2}z_1^{-1} - \frac{1}{2} & 1 & 0 & 0 \\ -\frac{1}{2}z_2^{-1} - \frac{1}{2} & 0 & 1 & 0 \\ -\frac{1}{2}z_1^{-1}z_2^{-1} - \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \text{ satisfies } \mathtt{T}(z)\mathtt{H}(z)^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, we can get $\mathtt{K}_1(z), \mathtt{K}_2(z), \mathtt{K}_3(z)$ from 2nd to 4th columns of $\mathtt{T}(z)^T$ and $\mathtt{J}_1(z), \mathtt{J}_2(z), \mathtt{J}_3(z)$ from 2nd to 4th rows of $\mathtt{T}(z)^{-T}[\mathtt{I}_4 - \mathtt{F}(z)\mathtt{H}(z)][\mathtt{I}_4 - \mathtt{G}(z)\mathtt{H}(z)]$.

Example: Designing a 2-D wavelet FB

More precisely, the analysis bank $\mathtt{A}(z)$ and the synthesis bank $\mathtt{S}(z)$ are given as

$$\mathtt{A}(z) = \begin{bmatrix} \mathtt{H}(z) \\ \mathtt{J}_1(z) \\ \mathtt{J}_2(z) \\ \mathtt{J}_3(z) \end{bmatrix}, \quad \mathtt{S}(z) = \begin{bmatrix} \mathtt{D}(z) & \mathtt{K}_1(z) & \mathtt{K}_2(z) & \mathtt{K}_3(z) \end{bmatrix}$$

where

$$\mathtt{J}_1(z) = \begin{bmatrix} -\frac{1}{8} - \frac{1}{8}z_1 & -\frac{1}{16}z_1^{-1} + \frac{7}{8} - \frac{1}{16}z_1 & -\frac{1}{16}z_2^{-1} - \frac{1}{16}z_2^{-1}z_1 - \frac{1}{16} - \frac{1}{16}z_1 & -\frac{1}{16}z_1^{-1}z_2^{-1} - \frac{1}{16}z_2^{-1} - \frac{1}{16} - \frac{1}{16}z_1 \end{bmatrix}$$

$$\mathtt{J}_2(z) = \begin{bmatrix} -\frac{1}{8} - \frac{1}{8}z_2 & -\frac{1}{16}z_1^{-1} - \frac{1}{16}z_1^{-1}z_2 - \frac{1}{16} - \frac{1}{16}z_2 & -\frac{1}{16}z_2^{-1} + \frac{7}{8} - \frac{1}{16}z_2 & -\frac{1}{16}z_1^{-1}z_2^{-1} - \frac{1}{16}z_1^{-1} - \frac{1}{16} - \frac{1}{16}z_2 \end{bmatrix}$$

$$\mathtt{J}_3(z) = \begin{bmatrix} -\frac{1}{8} - \frac{1}{8}z_1z_2 & -\frac{1}{16}z_1^{-1} - \frac{1}{16} - \frac{1}{16}z_2 - \frac{1}{16}z_1z_2 & -\frac{1}{16}z_2^{-1} - \frac{1}{16} - \frac{1}{16}z_1 - \frac{1}{16}z_1z_2 & -\frac{1}{16}z_1^{-1}z_2^{-1} + \frac{7}{8} - \frac{1}{16}z_1z_2 \end{bmatrix}$$

and

$$\mathtt{K}_1(z) = \begin{bmatrix} -\frac{1}{2}z_1^{-1} - \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathtt{K}_2(z) = \begin{bmatrix} -\frac{1}{2}z_2^{-1} - \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathtt{K}_3(z) = \begin{bmatrix} -\frac{1}{2}z_1^{-1}z_2^{-1} - \frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Our approach for high-performance wavelet FB design

Given a positive integer m, let

- $\mathtt{H}(z)$: row, lowpass, q-vector w/ unimodularity, m AC.
- $\mathtt{G}(z)$: column, lowpass, q-vector w/ m AC.
- $\mathtt{F}(z)$: column, lowpass, q-vector, a right inverse of $\mathtt{H}(z)$.

Then one has an algorithm to construct a non-redundant wavelet FB with at least m VM so that

$$\begin{bmatrix} \mathtt{D}(z) & * & \cdots & * \end{bmatrix} \begin{bmatrix} \mathtt{H}(z) \\ * \\ \vdots \\ * \end{bmatrix} = \mathtt{I}_q,$$

where $\mathtt{D}(z)$, still determined by $\mathtt{H}(z), \mathtt{G}(z)$ and $\mathtt{F}(z)$, is a right inverse of $\mathtt{H}(z)$ with at least m AC.

Summary

Summary of the talk

- Our method can be used to design non-redundant wavelet FBs for any dimension (it is especially useful for multi-D).
- It provides an algorithm for constructing a wavelet FB w/ m VM, starting from a unimodular lowpass vector w/ m AC.
- It does not require the initial unimodular lowpass vector to satisfy any additional assumption other than AC condition.

Things that I did not cover in the talk

- Our approaches work for any dilation.
- Connection of our method to Laplacian pyramid algorithms.

Remaining challenges

- Our approaches so far have been mostly algebraic, hence questions that are analytic in nature need to be answered separately.
- Currently we are using only the implementation of the big theorem (Quillen-Suslin Theorem for Laurent polynomials). We'll try to fully exploit the powerfulness of the big theorem.
- Currently we are concerned with only the VM of wavelets. We'll try to incorporate other properties such as symmetry, interpolatory property, and fast algorithms.

References

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Thank you for your attention

Outline

Unimodularity is ring-dependent

$[z, z^2]$ is unimodular in Laurent polynomial ring since $[\frac{1}{2}z^{-1}, \frac{1}{2}z^{-2}]^T$ is a right inverse, but not in polynomial ring since there are no polynomials $f(z)$, $g(z)$ s.t. $f(z)z + g(z)z^2 = 1$.

Filter bank (FB)

$h, d, j_i, k_i : \mathbb{Z}^n \to \mathbb{R}$, $i = 1, \ldots, p-1$, are filters (w/ finite supports)
Downsampling & upsampling, with $n \times n$ sampling matrix Λ (w/ integers and all eigenvalues have magnitude larger than 1):

$$\begin{aligned} y_{\downarrow}(k) &= y(\Lambda k), \quad k \in \mathbb{Z}^n. \\ y_{\uparrow}(k) &= \begin{cases} y(\Lambda^{-1}k), & k \in \Lambda\mathbb{Z}^n, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

i.e. for $n = 1$, $\Lambda = 2$, $y = (\ldots, y(-1), y(0), y(1), \ldots)$, we have

$$\begin{aligned} y_{\downarrow} &= (\ldots, y(-2), y(0), y(2), \ldots) \\ y_{\uparrow} &= (\ldots, y(-1), 0, y(0), 0, y(1), \ldots). \end{aligned}$$

Filter bank (FB) problem is to find $\{h, j_1, ..., j_{p-1}\}$, $\{d, k_1, ..., k_{p-1}\}$ s.t. $d * ((h * x)_{\downarrow})_{\uparrow} + \sum_{i=1}^{p-1} k_i * ((j_i * x)_{\downarrow})_{\uparrow} = x$, for any finitely supported signal x.

Laurent polynomial lowpass/highpass vectors

Λ: $n \times n$ sampling matrix
$q = |\det \Lambda|$
Γ: a set of representatives of distinct cosets of $\mathbb{Z}^n/\Lambda\mathbb{Z}^n$ with 0.
Then $\Gamma =: \{\nu_0 = 0, \nu_1, .., \nu_{q-1}\}$, $\mathbb{Z}^n = \cup_{\nu\in\Gamma}(\Lambda\mathbb{Z}^n + \nu)$.

Definition

Let $\mathtt{H}(z) = [H_0(z), H_1(z), \ldots, H_{q-1}(z)]$ be a row q-vector.

- $\mathtt{H}(z)$ is the (polyphase) lowpass vector if $H(e^{i\omega})|_{\omega=0} = \sqrt{q}$
- $\mathtt{H}(z)$ is the (polyphase) highpass vector if $H(e^{i\omega})|_{\omega=0} = 0$

where

$$H(z) = \sum_{j=0}^{q-1} z^{\nu_j} H_j(z^\Lambda)$$

The type of column q-vector $\mathtt{D}(z) = [D_0(z), D_1(z), \ldots, D_{q-1}(z)]^T$ is defined similarly by forming $D(z) = \sum_{j=0}^{q-1} z^{-\nu_j} D_j(z^\Lambda)$.

Our theory: full version

Theorem (by Hur-Park-Zheng)

Let

- $\alpha_{\mathtt{H}}, \beta_{\mathtt{H}}, \alpha_{\mathtt{G}}, \beta_{\mathtt{G}} > 0$ *and* $\alpha_{\mathtt{F}} \geq 0$*: integers.*
- $\mathtt{H}(z)$ *: unimodular lowpass row* q*-vector w/* $\alpha_{\mathtt{H}}$ *AC,* $\beta_{\mathtt{H}}$ *FL.*
- $\mathtt{G}(z)$ *: lowpass column* q*-vector w/* $\alpha_{\mathtt{G}}$ *AC and* $\beta_{\mathtt{G}}$ *FL.*
- $\mathtt{F}(z)$ *: lowpass column* q*-vector w/* $\alpha_{\mathtt{F}}$ *AC,* $\mathtt{H}(z)\mathtt{F}(z) = 1$*.*

Then one can construct a non-redundant wavelet FB so that

$$\begin{bmatrix} \mathtt{D}(z) & * & \cdots & * \end{bmatrix} \begin{bmatrix} \mathtt{H}(z) \\ * \\ \vdots \\ * \end{bmatrix} = \mathtt{I}_q, \mathtt{D}(z) := \mathtt{G}(z) + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{G}(z))$$

with lowpass $\mathtt{D}(z)$ *w/ at least* $\min\{\alpha_{\mathtt{G}}, \alpha_{\mathtt{F}} + \beta_{\mathtt{G}}, \alpha_{\mathtt{F}} + \beta_{\mathtt{H}}\} > 0$ *AC.*

FL: flatness

Our algorithm to get a wavelet FB: full version

Inputs:

- $\alpha_{\mathtt{H}}, \beta_{\mathtt{H}}, \alpha_{\mathtt{G}}, \beta_{\mathtt{G}} > 0$ and $\alpha_{\mathtt{F}} \geq 0$: integers.
- $\mathtt{H}(z)$: unimodular lowpass row q-vector w/ $\alpha_{\mathtt{H}}$ AC, $\beta_{\mathtt{H}}$ FL.
- $\mathtt{G}(z)$: lowpass column q-vector w/ $\alpha_{\mathtt{G}}$ AC and $\beta_{\mathtt{G}}$ FL.
- $\mathtt{F}(z)$: lowpass column q-vector w/ $\alpha_{\mathtt{F}}$ AC, $\mathtt{H}(z)\mathtt{F}(z) = 1$.

Output:

- wavelet FB whose lowpass row vector is $\mathtt{H}(z)$.

Procedure:

Step 1 Set $\mathtt{D}(z) := \mathtt{G}(z) + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{G}(z))$.

Step 2 Find invertible $\mathtt{K}(z)$ s.t. $\mathtt{K}(z)\mathtt{H}(z)^T = [1, 0, .., 0]^T$.

Step 3 Let $\mathtt{K}_1(z), .., \mathtt{K}_{q-1}(z)$ be the 2nd to last columns of $\mathtt{K}(z)^T$.

Step 4 Let $\mathtt{J}_1(z), .., \mathtt{J}_{q-1}(z)$ be the 2nd to last rows of $\mathtt{K}(z)^{-T}[\mathtt{I}_q - \mathtt{F}(z)\mathtt{H}(z)][\mathtt{I}_q - \mathtt{G}(z)\mathtt{H}(z)]$

Our algorithm to get a wavelet FB w/ *at least* $\alpha_{\mathtt{H}}$ *VM*: full version

Inputs:

- $\alpha_{\mathtt{H}}, \beta_{\mathtt{H}} > 0$ and $\alpha_{\mathtt{F}} \geq 0$: integers.
- $\mathtt{H}(z)$: unimodular lowpass row q-vector w/ $\alpha_{\mathtt{H}}$ AC, $\beta_{\mathtt{H}}$ FL.
- $\mathtt{F}(z)$: lowpass column q-vector w/ $\alpha_{\mathtt{F}}$ AC, $\mathtt{H}(z)\mathtt{F}(z) = 1$.

Output:

- wavelet FB w/ lowpass row vector $\mathtt{H}(z)$ and at least $\alpha_{\mathtt{H}}$ VM.

Procedure:

Step 1 Initialize $\mathit{Iter} := 1$ and $\mathtt{D}(z) := \mathtt{H}(z)^* + \mathtt{F}(z)(1 - \mathtt{H}(z)\mathtt{H}(z)^*)$

Step 2 While $(\alpha_{\mathtt{F}} + (\mathit{Iter})\beta_{\mathtt{H}} < \alpha_{\mathtt{H}})$

$\mathit{Iter} := \mathit{Iter} + 1; \quad \mathtt{D}(z) := \mathtt{H}(z)^* + \mathtt{D}(z)(1 - \mathtt{H}(z)\mathtt{H}(z)^*)$

end

Step 3 Find invertible $\mathtt{K}(z)$ s.t. $\mathtt{K}(z)\mathtt{H}(z)^T = [1, 0, .., 0]^T$.

Step 4 Define $\mathtt{K}_1(z), .., \mathtt{K}_{q-1}(z)$ and $\mathtt{J}_1(z), .., \mathtt{J}_{q-1}(z)$ as previous.

Vanishing moments (VM), flatness (FL), accuracy (AC)
Definition

Definition

For $\mathtt{H}(z) = [H_0(z), H_1(z), \ldots, H_{q-1}(z)]$, let $H(z) = \sum_{j=0}^{q-1} z^{\nu_j} H_j(z^\Lambda)$.
Let m be a nonnegative integer.

- $\mathtt{H}(z)$ has m VM if $\frac{d^k}{d\omega^k} H(e^{i\omega})|_{\omega=0} = 0,\ \forall |k| \le m-1$
- $\mathtt{H}(z)$ has m FL if $\frac{d^k}{d\omega^k} (\sqrt{q} - H(e^{i\omega}))|_{\omega=0} = 0,\ \forall |k| \le m-1$
- $\mathtt{H}(z)$ has m AC if $\frac{d^k}{d\omega^k} H(e^{i\omega})|_{\omega=\gamma} = 0,\ \forall |k| \le m-1,\ \forall \gamma \in \Gamma^* \backslash 0$
  Γ^*: set of rep. of distinct cosets of $2\pi(((\Lambda^T)^{-1}\mathbb{Z}^n)/\mathbb{Z}^n)$ w/ 0.

VM, FL, AC for column vector $\mathtt{D}(z) = [D_0(z), D_1(z), \ldots, D_{q-1}(z)]^T$ is defined similarly by forming $D(z) = \sum_{j=0}^{q-1} z^{-\nu_j} D_j(z^\Lambda)$.

$\mathtt{H}(z)$ or $\mathtt{D}(z)$ is the highpass vector iff it has VM ≥ 1
$\mathtt{H}(z)$ or $\mathtt{D}(z)$ is the lowpass vector iff it has FL ≥ 1

Equivalent conditions for VM, FL, and AC

Theorem

For $\mathtt{H}(z) = [H_0(z), H_1(z), \ldots, H_{q-1}(z)]$*, let* $H(z) = \sum_{j=0}^{q-1} z^{\nu_j} H_j(z^{\Lambda})$*.*
Let m *be a nonnegative integer.*

- $\mathtt{H}(z)$ *has* m *VM iff* $H(e^{i\omega}) \approx O(|\omega|^m)$ *(at* $\omega = 0$*)*
- $\mathtt{H}(z)$ *has* m *FL iff* $\sqrt{q} - H(e^{i\omega}) \approx O(|\omega|^m)$ *(at* $\omega = 0$*)*
- $\mathtt{H}(z)$ *has* m *AC iff* $H(e^{i(\omega+\gamma)}) \approx O(|\omega|^m)$ *(at* $\omega = 0$*),* $\forall \gamma \in \Gamma^* \backslash 0$
  Γ^**: set of rep. of distinct cosets of* $2\pi(((\Lambda^T)^{-1}\mathbb{Z}^n)/\mathbb{Z}^n)$ *w/* 0*.*