Characterizations of some types of linear independence of integer translates

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February Fourier Talks 2012
February 16 - 17, 2012

Setting

- Systems of integer translates of a $\psi \in L^2(\mathbb{R})$

$$\mathscr{B} = \{\psi_k, k \in \mathbb{Z}\}, \quad \psi_k(x) = \psi(x - k),$$

occur in approximation theory, frame theory, and wavelet analysis.

- Specific subset of the affine system generate by ψ

$$\{2^{j/2}\psi(2^j x - k), j, k \in \mathbb{Z}\}.$$

Periodization function

Definition

Let $\psi \in L^2(\mathbb{R})$, the periodization function is

$$p_\psi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + k)|^2$$

- $p_\psi \in L^1(\mathbb{T})$
- Many properties of $\mathscr{B} = \{\psi_k, k \in \mathbb{Z}\}$ can be completely described in terms of the periodization function p_ψ.

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Definition

A sequence $(e_n)_{n\in\mathbb{N}}$ in a Hilbert space H is a Riesz basis if

it is complete in H and there exist constants $A, B > 0$ such that,

for any $(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$

$$A\sum_{n=0}^{+\infty}|c_n|^2 \le \|\sum_{n=0}^{+\infty} c_n e_n\|^2 \le B\sum_{n=0}^{+\infty}|c_n|^2.$$

Definition

A sequence $(e_n)_{n\in\mathbb{N}}$ in a Hilbert space H is a Frame if

- There exist two real constants $A, B > 0$ such that, for any $x \in H$

$$A\|x\|^2 \le \sum_{n=0}^{+\infty} |< x, e_n >|^2 \le B\|x\|^2.$$

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Let $\mathscr{B} = \{\psi_k, k \in \mathbb{Z}\}$

$\langle\psi\rangle = \overline{\text{span}(\mathscr{B})} \subset L^2(\mathbb{R})$ denotes the shift invariant space generated by ψ.

$$p_\psi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + k)|^2$$

Theorem

1) $\mathscr{B}$ *is Bessel for* $\langle\psi\rangle$ *with bound* $B \Leftrightarrow p_\psi(\xi) \leq B$ *a.e.*

2) $\mathscr{B}$ *is an Orthonormal basis for* $\langle\psi\rangle \Leftrightarrow p_\psi(\xi) = 1$ *a.e.*

3) $\mathscr{B}$ *is a Riesz basis for* $\langle\psi\rangle$ *with bounds* A, B $\Leftrightarrow A \leq p_\psi(\xi) \leq B$ *a.e.*

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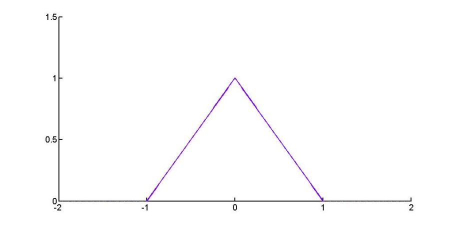
Example

Let $\psi(x) = (1 - |x|)\chi_{[-1,1]}(x)$ be a linear Spline.



Then

$$p_\psi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + k)|^2 = \frac{1}{3}(1 + 2\cos^2(\pi\xi)).$$

$\mathscr{B}$ is a Riesz basis for the space

$$\{f \in L^2(\mathbb{R}) \cap \mathscr{C}(\mathbb{R}), f \text{ linear in intervals } [k, k+1),\ k \in \mathbb{Z}\}.$$

Different concepts of linear independence

Definition

We say that a sequence $(e_n)_{n\in\mathbb{N}}$ in a Hilbert space H is

(i) Linearly independent if each finite subsequence is linearly independent.

(ii) ℓ^2- Linearly independent if whenever the series $\sum_{n=0}^{+\infty} c_n e_n$ is convergent and equal to zero for some coefficients $(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$, then necessarily $c_n = 0$ for all $n \in \mathbb{N}$.

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Periodization function and linear independence

$$\mathscr{B} = \{\psi_k, k \in \mathbb{Z}\}, \qquad p_\psi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + k)|^2.$$

Fact

$\mathscr{B}$ is *minimal*	$\Longleftrightarrow$	$\dfrac{1}{p_\psi} \in L^1(\mathbb{T})$
$\Downarrow$		
$\mathscr{B}$ is ω*-independent*		
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$\mathscr{B}$ is ℓ^2*- linearly independent*	$\Longleftarrow$	$p_\psi(\xi) > 0$ *a.e.*
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$\mathscr{B}$ is *linearly independent*		*Always true*

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More on ℓ^2-linear independence of $\mathscr{B}$

- $\mathscr{B}$ is ℓ^2- linearly independent if whenever the series $\sum_{k\in\mathbb{Z}} c_k\psi_k$ is convergent in $L^2(\mathbb{R})$ and equal to zero for some coefficients $(c_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, then necessarily $c_k = 0$ for all $k \in \mathbb{Z}$.
- We will not always be dealing with unconditionally convergent series so we order

$$\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$$

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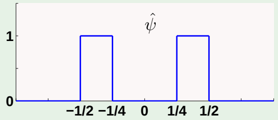
Example

Let $K = [-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]$ $\hat{\psi} = \chi_K$



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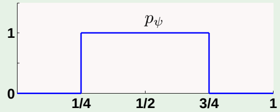

$p_\psi(\xi) = 0$ for all $\xi \in (0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$ so

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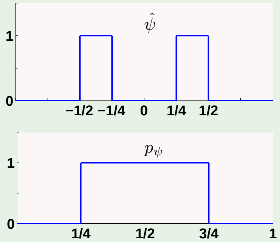

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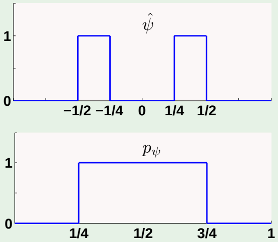

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Is $\mathscr{B}$ ℓ^2-linearly independent? The answer is NO

Theorem (Šikić, Speegle 2007)

If $\|p_\psi\|_\infty < \infty$, *then*

$$\mathscr{B} \text{ is } \ell^2\text{- linearly independent} \Longrightarrow p_\psi(\xi) > 0 \;\; a.e.\; \xi \in \mathbb{T}.$$

Proof.

p_ψ bounded leads to a bounded linear operator

$$\mathcal{I}_\psi : L^2(\mathbb{T}) \to L^2(\mathbb{T}, p_\psi), \quad \mathcal{I}_\psi(f) = f,$$

where $L^2(\mathbb{T}, p_\psi)$ consists of all 1-periodic functions f satisfying $\int_0^1 |f(\xi)|^2\, p_\psi(\xi)\, d\xi < \infty$.

$\Longrightarrow$) Assume $|\{p_\psi(\xi) = 0\}| > 0$. Take $f = \chi_{\{p_\psi = 0\}}$. Let $S_n(f) = \sum_{|k|\le n} c_k e^{2\pi i k\xi}$ be the symmetric partial sums of its Fourier series. Then

$$\|\sum_{|k|\le n} c_{-k}\psi_k\|_{L^2(\mathbb{R})} = \|\mathcal{I}_\psi(S_n(f))\|_{L^2(\mathbb{T},p_\psi)} \underset{n\to+\infty}{\to} \|\mathcal{I}_\psi(f)\|_{L^2(\mathbb{T},p_\psi)} = 0.$$

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Main result

Notation: for $f \in L^2(\mathbb{T})$ the symmetric partial sums of the Fourier series are $S_n(f)(\xi) = \sum_{|k| \leq n} \hat{f}(k) e^{2\pi i k \xi}$.

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Need:

Nice functions

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By uniform boundedness of $S_n(f)$, $p_\psi \in L^1(\mathbb{T})$, and by Lebesgue dominated convergence theorem get non-zero coefficients

$$\Big\|\sum_{|k|\leq n} \hat{f}(-k)\,\psi_k\Big\|_2^2 = \int_0^1 \Big|\sum_{|k|\leq n} \hat{f}(k) e^{2\pi i k\xi}\Big|^2 p_\psi(\xi)\, d\xi \qquad 0$$

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$$\lim_n \left| \sum_{|k|\leq n} \hat{f}(k) e^{2\pi i k\xi} \right|^2 p_\psi(\xi) = 0.$$

By uniform boundedness of $S_n(f)$, $p_\psi \in L^1(\mathbb{T})$, and by Lebesgue dominated convergence theorem get non-zero coefficients

$$\left\| \sum_{|k|\leq n} \hat{f}(-k)\, \psi_k \right\|_2^2 = \int_0^1 \left| \sum_{|k|\leq n} \hat{f}(k) e^{2\pi i k\xi} \right|^2 p_\psi(\xi)\, d\xi \qquad 0$$

and a contradiction. □

Proof.

Let $0 \neq \psi \in L^2(\mathbb{R})$. Assume $|Z_\psi| = |\{p_\psi(\xi) = 0\}| > 0$.
Take a nice function $0 \neq f \in L^2(\mathbb{T})$

1. $\operatorname{supp} f \subset Z_\psi$
2. $\|S_n(f)\|_\infty$ uniformly bounded

By a.e. convergence of the partial sums to f, and $\operatorname{supp} f \subset Z_\psi$, get a.e.

$$\lim_n |\sum_{|k| \leq n} \hat{f}(k) e^{2\pi i k \xi}|^2 \, p_\psi(\xi) = 0.$$

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and a contradiction. □

Looking for nice functions

Need:

Nice functions

For every measurable $A \subset [0,1]$, $|A| > 0$, there exists $0 \neq f \in L^2(\mathbb{T})$, such that

1. $\operatorname{supp} f \subset A$;
2. $\|S_n(f)\|_\infty$ *are uniformly bounded.*

A step in the right direction:

Theorem (Correction theorem of Men'shov 1940)

Every measurable function becomes a function with uniformly convergent Fourier series after a modification on a set of arbitrarily small measure.

For sets A which support continuous functions

Let us introduce the space of uniformly convergent Fourier series

$$U := \{f \in \mathscr{C}(\mathbb{T}) : \sum_{m \le k \le n} \hat{f}(k) e^{2\pi i k \xi} \text{ uniformly convergent}\}.$$

with the natural norm

$$\|f\|_U = \sup\{|\sum_{m \le k \le n} \hat{f}(k) e^{2\pi i k \xi}|,\ \xi \in \mathbb{T},\ m, n \in \mathbb{Z}, m \le n\}.$$

Theorem (Kislyakov 1994)

Let $0 < \varepsilon \le 1$, $\delta > 0$, $f \in \mathscr{C}(\mathbb{T})$ *and* $A = \{\xi \in \mathbb{T} : f(\xi) \neq 0\}$.

Then there exists a function $g \in U$ *such that*

1. $|g(\xi)| + |f(\xi) - g(\xi)| \le (1 + \delta)|f(\xi)|$
2. $|f \neq g| \le \varepsilon |A|$
3. $\|g\|_U \le const(1 + \log \varepsilon^{-1})\|f\|_\infty$

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1. $|g(\xi)| + |f(\xi) - g(\xi)| \le (1 + \delta)|f(\xi)| \Rightarrow \text{supp}\, g \subset \text{supp}\, f$
2. $|f \neq g| \le \varepsilon |A| \Rightarrow g \neq 0$
3. $\|g\|_U \le const\,(1 + \log \varepsilon^{-1})\|f\|_\infty \Rightarrow$ *uniform boundedness of* $S_n(g)$

Sentence

Kislyakov, *A sharp correction theorem, Studia Math., 1995:*

> *"... We note that the existence of functions supported on a given set of positive measure and having uniformly bounded Fourier sums is a nontrivial but well-known fact."*

For general measurable sets *A*: preliminaries

- Let (Ω, μ) be a σ-finite measure space.

  We shall take $\Omega = \mathbb{T}$ and the Lebesgue measure

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$$X = U^\infty := \{f \in L^\infty(\mathbb{T}) : \sum_{m \le k \le n} \hat{f}(k) e^{2\pi i k \xi} \text{ are uniformly bounded}\}.$$

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  $L^\infty(\mathbb{T})$ of course...
- Every $g \in L_0^\infty(\mu)$ generates a linear functional

$$\Phi_g(x) = \int_\Omega x\, g\, d\mu, \quad x \in X.$$

Theorem (Kislyakov 1995)

Assume

A1) *The natural embedding $X \hookrightarrow L^1_{loc}(\mu)$ is continuous, and the unit ball of X is weakly compact in $L^1_{loc}(\mu)$.*

A2) *For every $g \in L^\infty_0(\mu)$*

$$\mu\{|g| > t\} \leq ct^{-1}\|\Phi_g\|_{X^*}, \quad t > 0.$$

(c constant depending only on X)

Then, for every $f \in L^\infty(\mu) \cap L^1(\mu)$, with $\|f\|_\infty \leq 1$ and every $0 < \varepsilon \leq 1$ there exists a function $g \in X$ such that

1. $|g| + |f - g| = |f|$
2. $|f \neq g| \leq \varepsilon\|f\|_1$
3. $\|g\|_X \leq C\,(1 + \log \varepsilon^{-1})$

   (C depends only on c in A2).

Application to general measurable sets A

Apply Kislyakov theorem to

$$X = U^\infty = \{f \in L^\infty(\mathbb{T}) : \sum_{m \le k \le n} \hat{f}(k) e^{2\pi i k \xi} \text{ are uniformly bounded}\}$$

A1) The natural embedding $U^\infty \hookrightarrow L^1(\mathbb{T})$ is continuous, and the unit ball of U^∞ is weakly compact in $L^1(\mathbb{T})$. $\surd$

A2) [Vinogradov, 1981] For every $g \in L^\infty(\mathbb{T})$

$$|\{|g| > t\}| \le c t^{-1} \|\Phi_g\|_{(U^\infty)^*}, \quad t > 0. \quad \surd$$

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So, for measurable $A \subset [0,1)$, $|A| > 0$, and $f = \chi_A \in L^{\infty}(\mathbb{T})$, and every $0 < \varepsilon \le 1$ we get:

there exists a function $g \in U^{\infty}$ such that

1. $|g| + |\chi_A - g| = |\chi_A|$
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So:

> $\mathscr{B}$ *is* ℓ^2*- linearly independent* $\Longleftrightarrow p_\psi(\xi) > 0$ *a.e.*

and the Weiss conjecture is true.

Variation of a theme: other types of linear independence

Definition

We say that a sequence $(e_n)_{n\in\mathbb{N}}$ in a Hilbert space H is

ℓ^p- Linearly independent, $1\leq p<2$, if whenever the series $\sum_{n=0}^{+\infty} c_n e_n$ is convergent and equal to zero for some coefficients $(c_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})$, then necessarily $c_n = 0$ for all $n \in \mathbb{N}$.

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Fact

$$\mathscr{B} \text{ is } \ell^2\text{- linearly independent} \iff p_\psi(\xi) > 0 \text{ a.e.}$$

$$\Downarrow$$

$$\mathscr{B} \text{ is } \ell^p\text{- linearly independent} \quad (1\leq p<2)$$

ℓ^p linear independence and ℓ^p-sets of uniqueness

Definition

We call a Lebesgue measurable set $A \subset [0, 1]$ an ℓ^p-set of uniqueness if no nonzero function $f \in L^2(\mathbb{T})$, vanishing almost everywhere in the complement of A, satisfies the condition $(\hat{f}(n))_{n\in\mathbb{Z}} \in \ell^p$.

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Theorem (Šikić, Slamić 2011)

If $\|p_\psi\|_\infty < \infty$, and $1 < p < 2$, then

$\mathscr{B}$ *is ℓ^p-linearly independent* $\Longleftrightarrow \{p_\psi(t) = 0\}$ *is an ℓ^p-set of uniqueness.*

(Actually $\Longleftarrow$ does not require $\|p_\psi\|_\infty < \infty$.)

$\mathscr{B}$ *is ℓ^1-linearly independent* $\Longleftrightarrow \{p_\psi(t) = 0\}$ *is an ℓ^1-set of uniqueness.*

Looking again for nice functions

Assume for $1 < p < 2$,

For every $A \subset [0,1]$, not an ℓ^p-set of uniqueness there exists $0 \neq f \in L^2(\mathbb{T})$, such that

1. *$\operatorname{supp} f \subset A$;*
2. *$(\hat{f}(n))_{n\in\mathbb{Z}} \in \ell^p$;*
3. *$\|S_n(f)\|_\infty$ are uniformly bounded.*

Then

For every $\psi \in L^2(\mathbb{R})$

$\mathscr{B}$ is ℓ^p-linearly independent $\Longleftrightarrow$ $\{p_\psi(t) = 0\}$ is an ℓ^p-set of uniqueness.

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