Existence of frames with prescribed norms and frame operator

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Statement of problem

Definition

A sequence $\{f_i\}_{i\in I}$ *in a Hilbert space* $\mathcal{H}$ *is called a frame if there exist constants* $0 < A \le B < \infty$ *such that*

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2 \qquad \forall f \in \mathcal{H}.$$

A frame operator $Sf = \sum_{i\in I}\langle f, f_i\rangle f_i$.

Problem. Characterize all possible sequences of norms $\{||f_i||\}_{i\in I}$ of frames with prescribed frame operator S.

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Trivial necessary condition:

$$0 \le ||f_i||^2 \le B.$$

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- Jasper (2011) -frames with 2 point spectrum frame operator

Orthonormal dilation of frames

Proposition

Let $\mathcal{K}$ be a Hilbert space with orthonormal basis $\{e_i\}_{i\in I}$ and $0 < A \le B < \infty$. If E is a positive operator on $\mathcal{K}$ with $\sigma(E) \subseteq \{0\} \cup [A, B]$, then $\{Ee_i\}$ is a frame for $\mathcal{H} = E(\mathcal{K})$ with bounds A^2 and B^2.

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The converse is also true.

Proposition

Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with optimal bounds A^2 and B^2. Then, there exists a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ with basis $\{e_i\}_{i\in I}$ and positive operator E on $\mathcal{K}$ such that $E(e_i) = f_i$ and

$$\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B].$$

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$\mathcal{K}$ can be identified with $\ell^2(I)$.
E is unitarily equivalent with $S^{1/2} \oplus \mathbf{0}$, S frame operator, $\mathbf{0}$ on $\mathcal{H}^\perp$.

Reformulation of problem

Theorem (Antezana, Massey, Ruiz, Stojanoff (2007))

Let $0 < A \le B < \infty$ and S be a positive operator on a Hilbert space $\mathcal{H}$ with $\sigma(S) \subset [A, B]$. The following sets are equal:

$$\left\{ \{\|f_i\|^2\}_{i\in I} \;\middle|\; \begin{array}{l} \{f_i\}_{i\in I} \text{ is a frame for } \mathcal{H} \text{ with} \\ \text{frame operator } S \end{array} \right\}$$

$$\left\{ \{\langle Ee_i, e_i\rangle\}_{i\in I} \;\middle|\; \begin{array}{c} E \text{ is self-adjoint on } \ell^2(I) \text{ and} \\ \text{unitarily equivalent with } S \oplus \mathbf{0} \end{array} \right\}$$

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Reformulated Problem. Characterize diagonals of positive operators E with $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$.

Parseval Frames

Definition

A sequence $\{f_i\}_{i\in I}$ *in a Hilbert space* $\mathcal{H}$ *is a tight frame (Parseval frame if* $B = 1$*) if*

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 = B\|f\|^2 \qquad \forall f \in \mathcal{H}.$$

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Reformulated Problem. Characterize diagonals of orthogonal projections.
This problem was solved by Kadison (2002) and independently in the finite case by Casazza, Fickus, Kovačevíc, Leon, and Tremain (2006) using frame potentials.

$$\max_{i=1,\dots,M} ||f_i||^2 \le \frac{1}{N}\sum_{i=1}^{M} ||f_i||^2 = B.$$

Pythagorean Theorem and Carpenter's Theorem

Theorem (Kadison (2002))

Let $\{d_i\}_{i\in I}$ *be a sequence in* $[0,1]$ *and* $\alpha \in (0,1)$. *Define*

$$C(\alpha) = \sum_{d_i<\alpha} d_i, \qquad D(\alpha) = \sum_{d_i\geq\alpha} (1-d_i).$$

There exists an orthogonal projection on $\ell^2(I)$ *with diagonal* $\{d_i\}_{i\in I} \iff$ *either:*

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The same condition characterizes sequences of norms of Parseval frames. The finite case is a consequence of the Schur-Horn theorem—the necessary and sufficient condition is $\sum d_i \in \mathbb{N}$.

Schur-Horn Theorem

Theorem (Schur (1923), Horn (1954))

Suppose S is an $N \times N$ Hermitian matrix with eigenvalues $\{\lambda_i\}_{i=1}^N$ and diagonal $\{d_i\}_{i=1}^N$ listed in nonincreasing order. Then,

$$\begin{aligned} \sum_{i=1}^n d_i &\le \sum_{i=1}^n \lambda_i \quad \forall\, n = 1, \ldots, N \\ \sum_{i=1}^N \lambda_i &= \sum_{i=1}^N d_i \end{aligned} \tag{1}$$

Conversely, if (1) holds, then there is a **real** *$N \times N$ Hermitian matrix with eigenvalues $\{\lambda_i\}$ and diagonal $\{d_i\}$.*

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(1) is equivalent to the convexity condition

$$(d_1, \ldots, d_N) \in \operatorname{conv}\{(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(N)}) : \sigma \in S_N\} \subset \mathbb{R}^N.$$

Locally invertible positive operators

Theorem (Bownik, Jasper (2011))

Let $0 < A < B < \infty$ *and* $\{d_i\}$ *be a nonsummable sequence in* $[0, B]$. *Define the numbers*

$$C = \sum_{d_i < A} d_i \quad \text{and} \quad D = \sum_{d_i \geq A} (B - d_i).$$

There is a positive operator E *with* $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$ *and diagonal* $\{d_i\}$ $\iff$ *either:*

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2. *$C, D < \infty$ and $\exists\ N \in \mathbb{N} \cup \{0\}$,*

$$NA \leq C \leq A + B(N-1) + D.$$

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The nonsummability $\sum d_i = \infty$ is not a true limitation.

Frames with prescribed lower and upper bounds

Corollary (Bownik, Jasper (2011))

Let $0 < A < B < \infty$ *and* $\{d_i\}$ *be a nonsummable sequence in* $[0, B]$*. Define the numbers*

$$C = \sum_{d_i < A} d_i \quad \text{and} \quad D = \sum_{d_i \geq A} (B - d_i).$$

There is a frame with optimal bounds A *and* B *and* $d_i = \|f_i\|^2$ $\iff$ *either:*

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Moving diagonal in desirable configuration

Lemma (Moving toward 0-1)

Let $\{d_i\}_{i\in I}$ *be a sequence in* $[0, B]$*. Let* $I_0, I_1 \subset I$ *be two disjoint finite subsets such that* $\max\{d_i : i \in I_0\} \le \min\{d_i : i \in I_1\}$*. Let*

$$0 \le \eta_0 \le \min\left\{ \sum_{i\in I_0} d_i, \sum_{i\in I_1} (B - d_i) \right\}.$$

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$$0 \le \eta_0 \le \min\left\{ \sum_{i\in I_0} d_i, \sum_{i\in I_1}(B-d_i) \right\}.$$

(i) There exists a sequence $\{\tilde{d}_i\}_{i\in I}$ in $[0,B]$ satisfying:

1. $\tilde{d}_i = d_i$ *for* $i \in I \setminus (I_0 \cup I_1)$,
2. $\tilde{d}_i \le d_i \quad i \in I_0$ *and* $\tilde{d}_i \ge d_i, \quad i \in I_1$,
3. $\eta_0 + \sum_{i\in I_0} \tilde{d}_i = \sum_{i\in I_0} d_i$, *and*
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   $\eta_0 + \sum_{i\in I_1}(B-\tilde{d}_i) = \sum_{i\in I_1}(B-d_i)$.

(ii) $\tilde{E}$ self-adjoint operator with diagonal $\{\tilde{d}_i\}_{i\in I}$ $\implies$ there exists an operator E unitarily equivalent to $\tilde{E}$ with diagonal $\{d_i\}_{i\in I}$.

Theorem (Jasper (2011))

Let $0 < A < B < \infty$ and $\{d_i\}_{i \in I}$ be a sequence in $[0, B]$ with $\sum d_i = \sum (B - d_i) = \infty$. Define

$$C = \sum_{d_i < A} d_i \quad \text{and} \quad D = \sum_{d_i \geq A} (B - d_i).$$

There is a self-adjoint operator E with diagonal $\{d_i\}_{i \in I}$ and $\sigma(E) = \{0, A, B\} \iff$ either:

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There is a self-adjoint operator E *with diagonal* $\{d_i\}_{i \in I}$ *and* $\sigma(E) = \{0, A, B\} \iff$ *either:*

1. $C = \infty$ *or* $D = \infty$, *or*
2. $C, D < \infty$ *and* $\exists\, N \in \mathbb{N}$, $k \in \mathbb{Z}$

$$C - D = NA + kB \quad \text{and} \quad C \geq (N + k)A.$$

Schur-Horn for operators with 3 point spectrum

Theorem (Jasper (2011))

Let $0 < A < B < \infty$ and $\{d_i\}_{i \in I}$ be a sequence in $[0, B]$ with $\sum d_i = \sum (B - d_i) = \infty$. Define

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Corollary (Jasper (2011))

Let $0 < A < B < \infty$ *and* $\{d_i\}_{i \in I}$ *be a sequence in* $[0, B]$ *with* $\sum d_i = \sum (B - d_i) = \infty$. *Define*

$$C = \sum_{d_i < A} d_i \quad \text{and} \quad D = \sum_{d_i \geq A} (B - d_i).$$

There is a frame such that $\sigma(S) = \{A, B\}$ *and* $d_i = \|f_i\|^2 \iff$ *either:*

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2. $C, D < \infty$ *and* $\exists\ N \in \mathbb{N}$, $k \in \mathbb{Z}$

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Definition

Let $0 < A < B < \infty$ *and* $\{d_i\}_{i\in I}$ *in* $[0,B]$. *Define the sets*

$$I_1 = \{i \in I : d_i < A\},\ I_2 = \{i \in I : d_i \geq A\},$$

$$J_2 = \{i \in I_2 : d_i < (A+B)/2\},\ J_3 = I_2 \setminus J_2$$

and the constants (each possibly infinite)

$$C = \sum_{i\in I_1} d_i \qquad D = \sum_{i\in I_2}(B - d_i)$$

$$C_1 = \sum_{i\in I_1}(A - d_i),\ C_2 = \sum_{i\in J_2}(d_i - A),\ C_3 = \sum_{i\in J_3}(B - d_i).$$

Let E *be a bounded operator on a Hilbert space.*
For $\lambda \in \mathbb{C}$ *define* $m_E(\lambda) = \dim\ker(E - \lambda)$.

3 point spectrum and prescribed multiplicites

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E has diagonal $\{d_i\}$ *and* $\sigma(E) = \{0, A, B\} \iff$

	$m_E(0)$	$m_E(A)$	$m_E(B)$	Condition
(a)	Z	N	K	$|I| = Z + N + K$ $\sum_{i \in I} d_i = NA + KB,\ C \geq (N + K - |I_2|)A$
(b)	∞	N	K	$|I_1| = \infty,$ $\sum_{i \in I} d_i = NA + KB,\ C \geq (N + K - |I_2|)A$
(c)	∞	N	∞	$C + D = \infty$ *or* $C, D < \infty,\ |I_1| = |I_2| = \infty,$ $\exists\, k \in \mathbb{Z}\ C - D = NA + kB,\ C \geq A(N + k)$
(d)	Z	∞	K	$|I| = \infty,\ C_1 \leq AZ$ $\sum_{i \in I}(d_i - A) = K(B - A) - ZA$
(e)	Z	∞	∞	$C_1 \leq AZ,\ C_2 + C_3 = \infty$ *or* $|I_1 \cup J_2| = |J_3| = \infty,\ C_1 \leq AZ,\ C_2, C_3 < \infty$ $\exists\, k \in \mathbb{Z},\ \ C_1 - C_2 + C_3 = (Z - k)A + kB$
(f)	∞	∞	∞	$C + D = \infty$

Schur-Horn for operators with finite point spectrum

Theorem (Bownik, Jasper (2012))

Let $0 = A_0 < A_1 < \ldots < A_{n+1} = B$, $n \in \mathbb{N}$. *Let* $\{d_i\}_{i \in I} \subset [0, B]$. *Assume* $\sum d_i = \sum (B - d_i) = \infty$. *For* $\alpha \in (0, B)$ *define*
$C(\alpha) = \sum_{d_i < \alpha} d_i$ *and* $D(\alpha) = \sum_{d_i \geq \alpha} (B - d_i)$.
There exists a self-adjoint operator E *with diagonal* $\{d_i\}_{i \in I}$ *and* $\sigma(E) = \{A_0, A_1, \ldots, A_{n+1}\}$ $\iff$ *either:*

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$$C(B/2) - D(B/2) = \sum_{j=1}^{n} A_j N_j + kB,$$

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for all $r = 1, \ldots, n$.

Question: Given a fixed sequences $\{d_i\} \subset [0,1]$, for what numbers $0 < A < 1$ does there exist a frame $\{f_i\}$ such that $d_i = ||f_i||^2$ and the spectrum of frame operator $\sigma(S) = \{A, 1\}$?

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Theorem (Jasper (2011))

Let $\{d_i\}_{i\in\mathbb{N}}$ be a sequence in $[0,1]$ and set

$$\mathcal{A} = \{A \in (0,1) : \exists\, E \text{ with } \sigma(E) = \{0, A, 1\} \text{ and diagonal } \{d_i\}\}.$$

Either $\mathcal{A} = (0,1)$ or $\mathcal{A}$ is a finite (possibly empty) set.

Geometric series example

Example

Let $\beta \in (0,1)$ and define the sequence $\{d_i\}_{i\in\mathbb{Z}\setminus\{0\}}$ by

$$d_i = \begin{cases} 1-\beta^i, & i>0 \\ \beta^{|i|} & i<0. \end{cases}$$

Then

$$\mathcal{A} = \left\{ \vphantom{\begin{matrix}a\\a\\a\\a\\a\end{matrix}} \right.$$

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Geometric series example

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Let $\beta \in (0,1)$ and define the sequence $\{d_i\}_{i \in \mathbb{Z}\setminus\{0\}}$ by

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Then

$$\mathcal{A} = \begin{cases} \varnothing & 0<\beta<1/3, \\ \{\frac12\} & 1/3 \le \beta < \frac{-1+\sqrt{13}}{6} \approx 0.434, \\ \{\frac13,\frac12,\frac23\} & \frac{-1+\sqrt{13}}{6} \le \beta < 1/2, \\ \{\frac14,\frac13,\frac12,\frac23,\frac34\} & 1/2 \le \beta < x \approx 0.56, \\ \{\frac15,\frac14,\frac13,\frac12,\frac23,\frac34,\frac45\} & x \le \beta < \ldots \end{cases}$$

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Extended example

Example

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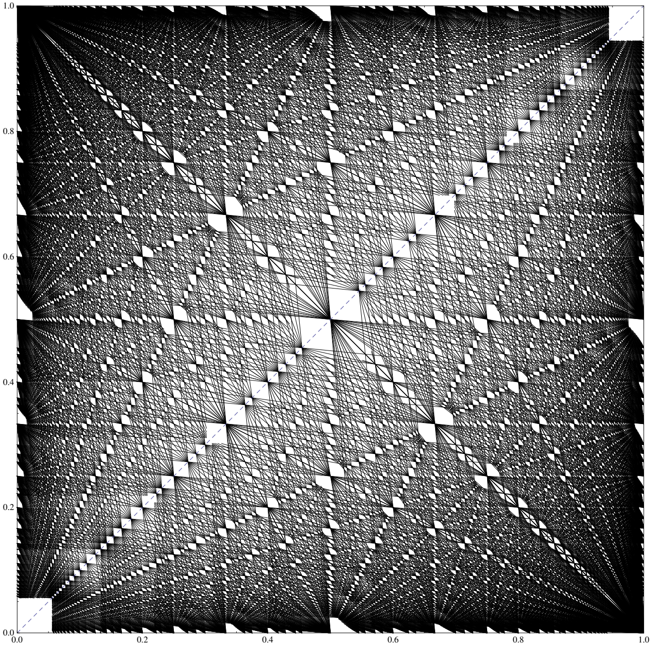
The following picture corresponds to $\beta = 0.8$.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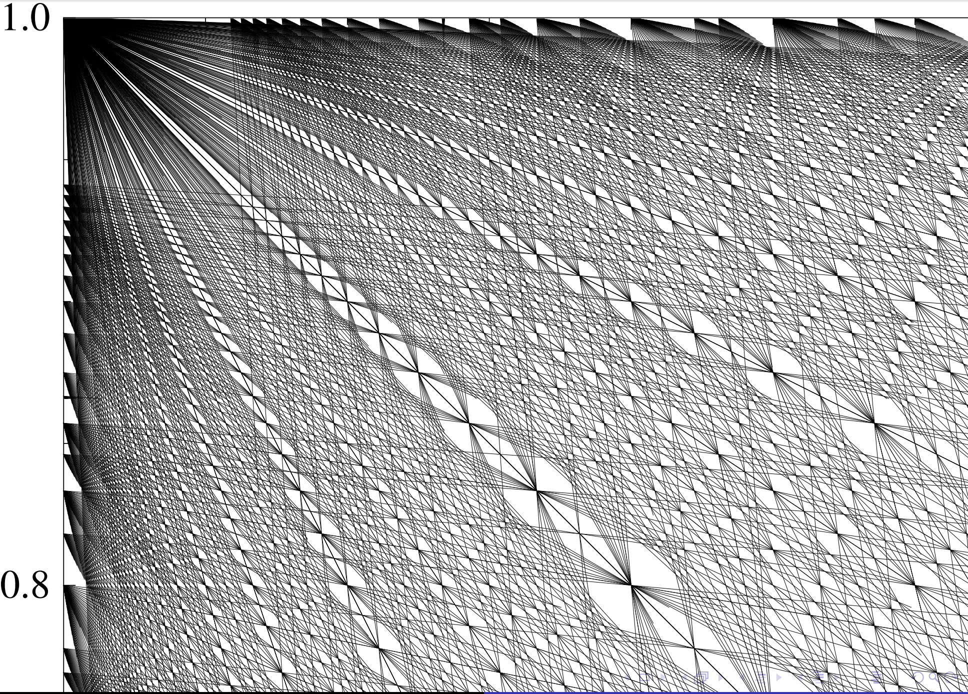
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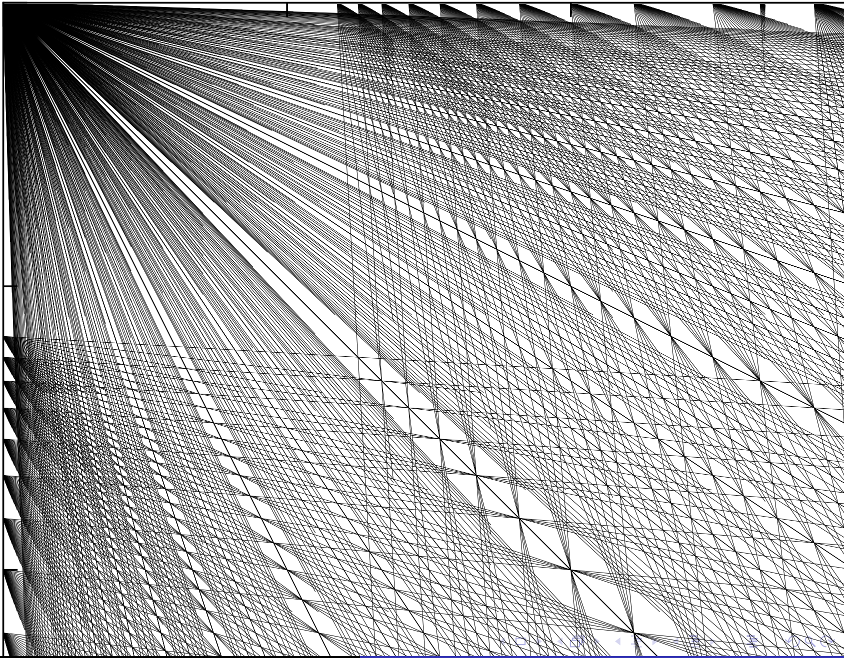

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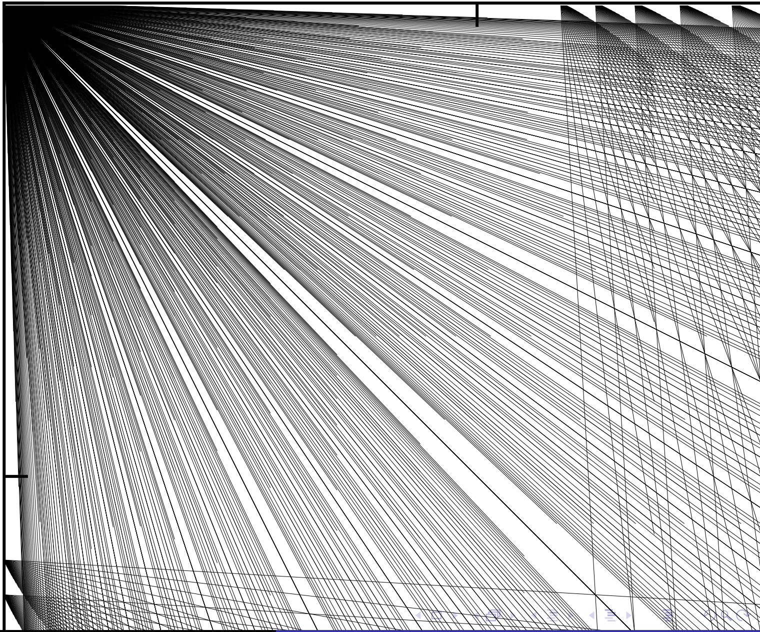

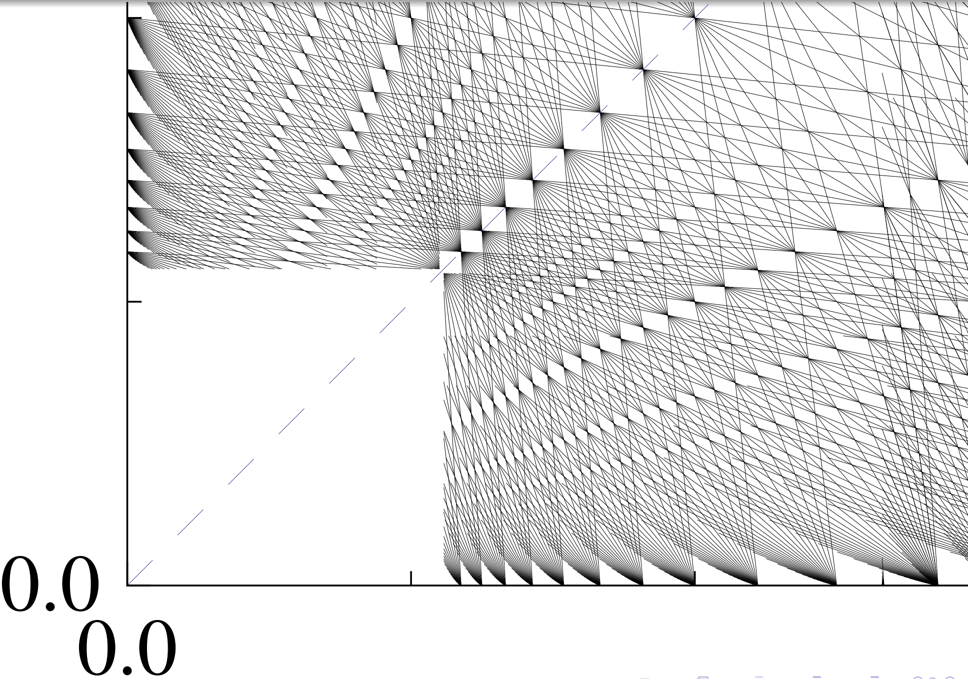
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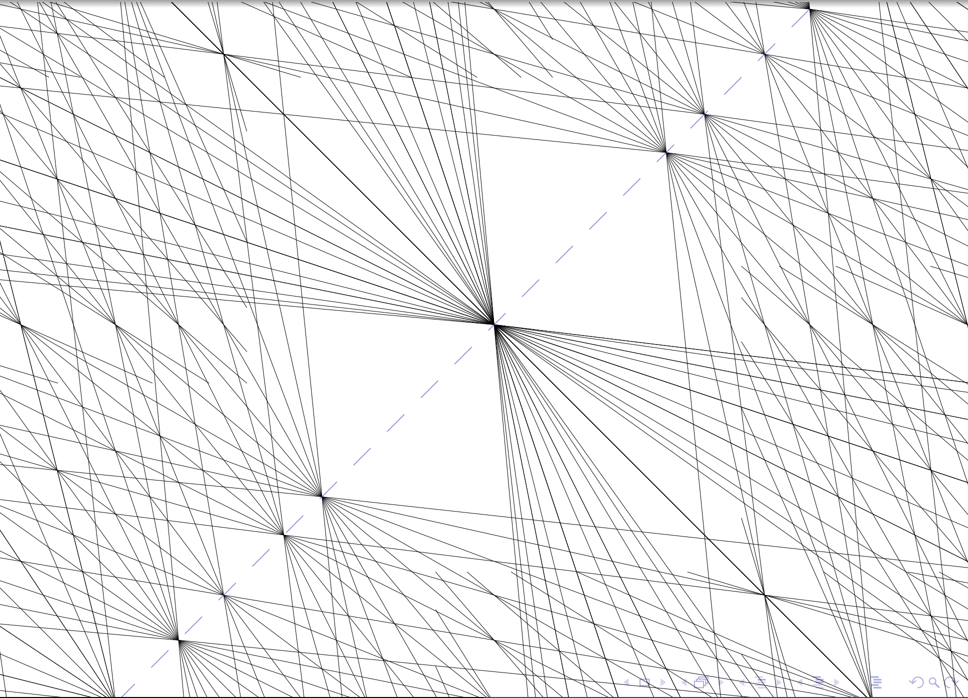

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- Characterize diagonals of operators with finite spectrum and **with prescribed multiplicities**.
- Ultimately extend the Schur-Horn theorem to operators with infinite spectrum beyond the results of Arveson-Kadison (2006) and Kaftal-Weiss (2010).