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A Review of Wideband Ambiguity Functions

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A REVIEW OF WIDEBAND AMBIGUITY FUNCTIONS

INTRODUCTION

The narrowband ambiguity function requires the approximation of the Doppler effect by a frequency shift, constant across the bandwidth. (This function and some of its generalizations are discussed in a companion report entitled "A Review of Narrowband Ambiguity Functions," hereafter referred to as "Narrowband Review.") With very-low-frequency wideband signals and with high-velocity or accelerating targets, this approximation is no longer valid. An ambiguity function which takes into account the actual time compression or expansion must be used.

The narrowband ambiguity function has been extensively explored (see "Narrowband Review"). While not as fully developed or used, the wideband function is of increasing interest (1-14). This report reviews the derivations and properties of the wideband ambiguity function and compares the two functions. The notation is designed to be consistent with that for narrowband functions to facilitate comparisons.

DEFINITION

Let $s_1(t) \in L_2(-\infty,\infty)$ represent a signal transmitted with a propagation velocity c, assumed constant. Let $s_2(t)$ represent the echo of $s_1(t)$ from a target at range $r(t)$ moving with a constant radial velocity v. Then

$$s_2(t) = as_1[t - T(t)],$$

where a is assumed constant and $T(t)$ is the time required for the signal to reach the target and return.

It is shown in Ref. 5 that (with no approximations beyond the model itself)

$$s_2(t) = as_1(\alpha t - \tau), \tag{1}$$

where $\alpha = (c - v)/(c + v)$ is the "Doppler stretch factor" and $\tau = 2r_0/(c + v)$ is the signal delay at $t = 0$.

If both $s_1(t)$ and $s_2(t)$ are normalized†,

$$1 = \int |s_2(t)|^2\, dt = a^2 \int |s_1(\alpha t - \tau)|^2\, dt = \frac{a^2}{\alpha} \int |s_1(t)|^2\, dt = \frac{a^2}{\alpha}.$$

Thus, conservation of energy requires that $a = \sqrt{\alpha}$, if attenuation is ignored.

To maximize the resolution between any two signal functions, $s_1(t)$ and $s_2(t)$, a "most general ambiguity function" is defined and discussed in "Narrowband Review" as $|\chi_{s_1 s_2}|^2$, where

$$\chi_{s_1 s_2} = \int s_1(t)\, s_2^*(t)\, dt. \tag{2}$$

†Here and in the sequel the limits of integration are $\pm\infty$ unless otherwise stated.

Substituting Eq. (1) for the present case of true doppler effect and dropping the subscript we define

$$\theta(\tau, a) \triangleq \chi_{ss} \triangleq \sqrt{a} \int s(t) s^*(at - \tau)\, dt. \tag{3}$$

The reason for retaining the factor $\sqrt{a}$ in the definition will be apparent later (Eq. (11)).

The wideband ambiguity function is

$$|\theta(\tau, a)|^2 = a \left| \int s(t) s^*(at - \tau)\, dt \right|^2 . \tag{4}$$

RELATIONSHIP TO THE NARROW-BAND AMBIGUITY FUNCTION

We may write $s(t) = u(t) \exp(2\pi i f_0 t)$, where f_0 is an arbitrary parameter which may be interpreted as a carrier frequency but need not be so restricted. If f_0 represents any single frequency in the spectrum of $s(t)$, the doppler effect at that frequency is a shift

$$\Phi \triangleq a f_0 - f_0 = (a - 1) f_0 .$$

Equation (3) if considered as a function of τ and Φ becomes

$$\begin{aligned} \tilde{\theta}(\tau, \Phi) \triangleq \theta(\tau, a) &= \sqrt{a} \int u(t) \exp(2\pi i f_0 t)\, u^*(at - \tau) \exp[-2\pi i f_0 (at - \tau)]\, dt \\ &= \sqrt{a} \exp(2\pi i f_0 \tau) \int u(t) u^*(at - \tau) \exp(-2\pi i \Phi t)\, dt . \end{aligned} \tag{5}$$

Let

$$\delta \triangleq \frac{2v}{c + v} .$$

Then $a = (c - v)/(c + v) = 1 - 2v/(c + v) = 1 - \delta$, where $\delta \ll 1$ if $v \ll c$, as is assumed in the narrowband case. By a Taylor expansion,

$$u(at - \tau) = u(t - \tau - \delta t) = u(t - \tau) - (\delta t) u'(t - \tau) + (\delta t)^2 u''(t - \tau)/2! + \dots .$$

If we restrict $u(t)$ so that the series converged rapidly, we have, to $o(\delta t)^2$,

$$\begin{aligned} \exp(-2\pi i f_0 \tau) \tilde{\theta}(\tau, \Phi) &= \sqrt{a} \int u(t) [u^*(t - \tau) - \delta t u'^*(t - \tau)] \exp(-2\pi i \Phi t)\, dt, \\ &= \sqrt{a}\, \chi(\tau, \Phi) - \delta \sqrt{a} \int t u(t) u'^*(t - \tau) \exp(-2\pi i \Phi t)\, dt, \end{aligned}$$

where $\chi(\tau, \Phi)$ is the narrowband generalized autocorrelation function introduced by Woodward (15). Hence,

$$|\tilde{\theta}(\tau, \Phi)| \le \sqrt{a} |\chi(\tau, \Phi)| + |\delta| \sqrt{a} \left| \int t u(t) u'^*(t - \tau) \exp(-2\pi i \Phi t)\, dt \right|$$

by the triangle inequality.

By the Cauchy-Schwarz inequality,

$$|\tilde{\theta}(\tau, \Phi)| - \sqrt{a} |\chi(\tau, \Phi)| \le |\delta| \sqrt{a} \left[\int t^2 |u(t)|^2\, dt \int |u'(t)|^2\, dt \right]^{1/2} . \tag{6}$$

Since $u(t) = \int U(f) \exp(2\pi i f t)\, df$, $u'(t) = 2\pi i \int f U(f) \exp(2\pi i f t)\, df$; hence, by Parseval's theorem

$$\int |u'(t)|^2\, dt = 4\pi^2 \int f^2 |U(f)|^2\, df \stackrel{\Delta}{=} \beta^2, \tag{7}$$

where β is the "Woodward bandwidth."

Similarly,

$$U'(f) = -2\pi i \int t u(t) \exp(-2\pi i f t)\, dt$$

and

$$\int |U'(f)|^2\, df = 4\pi^2 \int t^2 |u(t)|^2\, dt \stackrel{\Delta}{=} \mathfrak{I}^2, \tag{8}$$

where $\mathfrak{I}$ may be called the "Woodward duration," by analogy.

Thus, from Eqs. (6), (7), and (8) we have the upper bounds

$$|\tilde{\theta}(\tau, \Phi)| - \sqrt{a}\,|\chi(\tau, \Phi)| \le \frac{\sqrt{a}}{2\pi} |\delta|\, \mathfrak{I}\beta \tag{9}$$

and (by straightforward algebraic manipulation) to $o(\delta^2)$,

$$|\tilde{\theta}(\tau, \Phi)|^2 - a|\chi(\tau, \Phi)|^2 \le \frac{a}{\pi} |\delta|\, \mathfrak{I}\beta\, |\chi(\tau, \Phi)|\,. \tag{10}$$

Consideration of the Doppler effect as a simple shift ignores the change in bandwidth $\Delta\beta$. The narrowband generalized correlation function is a valid approximation of the response to the echo from a moving target of a bank of matched filters, if $|\Delta\beta|$ is much less than Δf, where Δf is the difference in frequency between the centers of adjacent filters in the bank. Taking $\Delta f = 1/T$, where T is the duration of the signal, validity of the narrowband approximation requires

$$|\Delta B| = |B - aB| = B|1 - a| = B|\delta| \ll 1/T$$

or

$$|\delta|\, TB \ll 1.$$

This relation may be compared with Eqs. (9) and (10). The product $|\delta|\, TB$ is known as the dispersion product.

The significance of Eqs. (9) and (10) may be seen by substituting numbers relating to sonar applications. If $v = 10$ knots $= 11.5$ mph and $c = 1500$ m/sec $= 3360$ mph, then $a = 0.993$ and $\delta = 0.007$. Thus, if $\mathfrak{I} = 1.2$ sec and $\beta = 25$ Hz, $a|\delta|\,\mathfrak{I}\beta/\pi \approx 0.07$; the difference between the wideband and narrowband ambiguity functions is at most 7% of the magnitude of the generalized autocorrelation, or 10% of the ambiguity magnitude at $|\chi(\tau, \Phi)|^2 = 1/2$. In general, the difference is less than 10% of the ambiguity at $|\chi(\tau, \Phi)|^2 = 1/2$,

$$\text{if } (c/2)|\delta| \approx |v| < 0.07\, c\pi/2a\,\mathfrak{I}\beta \approx 320/(\mathfrak{I}\beta) \text{ knots}.$$

If $\mathfrak{I} > 3$ sec, $\beta = 100$ Hz, this would require target velocities of less than 1 knot. To express this relationship another way, a sufficient condition for the difference between the wideband and narrowband ambiguity functions to be less than 10% (at the point at which the narrowband ambiguity is 50%) is that the product $\mathfrak{I}\beta|v|$ be less than about 300, where $\mathfrak{I}$ is in seconds, β in Hertz, and v in knots.

Equations (7) and (8) may be used (15,16) to derive an interesting uncertainty relationship: $\mho\beta \geq \pi$, which is done in the Appendix.

PROPERTIES OF THE WIDEBAND AMBIGUITY FUNCTION

1. $\theta(\tau,\ 1) = \int s(t)\, s^*(t - \tau)\ dt = R(\tau)$,

where $R(\tau)$ is the complex autocorrelation function of $s(t)$.

2. $\theta(0,\ 1) = \int |\, s(t)\,|^2\ dt = 1$,

since $s(t)$ is assumed to be normalized.

3. Theorem 1

$$|\,\theta(\tau,\ a)\,| \ \leq\ \theta(0,\ 1) = 1 \tag{11}$$

Proof

$$|\,\theta(\tau,\ a)\,| \ = \sqrt{a}\,|\int s(t)\, s^*(at - \tau)\ dt\,|$$

$$\leq \sqrt{a}\,[\int |\, s(t)\,|^2\ dt \int |\, s(at - \tau)\,|^2\ dt]^{1/2} = \sqrt{a}\,(1/a)^{1/2} = 1.$$

The $\sqrt{a}$ factor in the definition, Eq. (3), was retained to obtain this result (Eq. (11)). The maximum of the wideband ambiguity function (13) occurs at the point (0,1). (Ref. 13).

4. Frequency Representation

$$\begin{aligned}\theta(\tau,\ a) &= \sqrt{a}\int s(t)\, s^*(at - \tau)\ dt\\ &= \sqrt{a}\iiint S(\nu)\exp(2\pi i\nu t)\, S^*(f)\exp[(-2\pi i f(at - \tau))]\ d\nu\, df\, dt\\ &= \sqrt{a}\int S(af)\, S^*(f)\exp(2\pi i f\tau)\ df.\end{aligned} \tag{12}$$

5. Symmetry

$$\theta\left(-\frac{\tau}{a},\ \frac{1}{a}\right) = \frac{1}{\sqrt{a}}\int s(t)\, s^*\left(\frac{t}{a} + \frac{\tau}{a}\right)dt$$

$$= \sqrt{a}\int s(at - \tau)\, s^*(t)\ dt = \theta^*(\tau,\ a)$$

$$|\,\theta\left(-\frac{\tau}{a},\ \frac{1}{a}\right)|^2 = |\theta(\tau,\ a)\,|^2. \tag{13}$$

6. "Conservation of Ambiguity"

Integration of $|\,\theta(\tau,\ a)\,|^2$ over all values of its parameters must be limited to $0 < a < \infty$. Using Eq. (12), we have

$$\begin{aligned}V &\stackrel{\Delta}{=} \int_0^\infty\int_{-\infty}^\infty |\,\theta(\tau,\ a)\,|^2\ d\tau\, da = \int_0^\infty\int_{-\infty}^\infty\iint aS(af)\, S^*(f)\, S^*(a\nu)\, S(\nu)\exp[2\pi i(f - \nu)\tau]\ df\, d\nu\, d\tau\, da\\ &= \int_0^\infty\int_{-\infty}^\infty a|S(af)|^2|\, S(f)|^2\ df\, da = \int_{-\infty}^\infty |\, S(f)|^2\int_0^\infty a|\, S(af)\,|^2\ da\, df\\ &= P\int_{-\infty}^\infty \frac{1}{f^2}|\, S(f)\,|^2\int_0^\infty \mu||S(\mu)\,|^2\ d\mu\, df.\end{aligned} \tag{14}$$

P indicates the Cauchy principal value of the integral. In this case

$$P\int_{-\infty}^{\infty} \frac{|S(f)|^2}{f^2} df = \lim_{\epsilon \to 0} \left[\int_{-\infty}^{-\epsilon} \frac{|S(f)|^2}{f^2} df + \int_{\epsilon}^{\infty} \frac{|S(f)|^2}{f^2} df \right].$$

If this limit does not exist (as will be the case if $S(f)$ is continuous and $S(0) \neq 0$), then the volume under the wideband ambiguity surface is infinite. If the Cauchy principal value exists, the volume will depend on the signal waveform, in contrast with the narrowband case.

If $s(t)$ represents a high-frequency narrowband signal, so that $S(f)$ vanishes outside of a small region near a center frequency f_0, then

$$V \approx \frac{1}{f_0^2} \int_{-\infty}^{\infty} |S(f)|^2 df \, f_0 \int_0^{\infty} |S(f)|^2 \, df = \frac{1}{2 f_0},$$

a constant, independent of the signal waveform†.

WIDEBAND CROSS-AMBIGUITY FUNCTIONS

As in the case of the narrowband function, the wideband autoambiguity function may be considered a special case of a cross-ambiguity function. Many of the properties of the special case have analogs in the more general case.

We define the wideband generalized cross-correlation function of $s_1(t)$ and $s_2(t)$, or the theta function for short, as

$$\theta(s_1, s_2; \tau, a) \triangleq \theta_{s_1, s_2}(\tau, a) \triangleq \sqrt{a} \int s_1(t) \, s_2^*(at - \tau) \, dt. \tag{15}$$

The wideband cross-ambiguity function is

$$|\theta(s_1, s_2; \tau, a)|^2 = a \left| \int s_1(t) \, s_2^*(at - \tau) \, dt \right|^2. \tag{16}$$

Alternatively, as in Eq. (12),

$$\theta(s_1, s_2; \tau, a) = \sqrt{a} \int S_1(af) \, S_2^*(f) \exp(2\pi i f \tau) \, df. \tag{17}$$

The following two theorems and corollaries are due to Speiser (6).

<u>Theorem 2</u>

$$\int \theta(s_1, s_2; \tau, a) \exp(-2\pi i \nu \tau) \, dt = \sqrt{a} \, S_1(a\nu) \, S_2^*(\nu), \ \forall \nu \tag{18}$$

<u>Proof</u>

$$\begin{aligned} \int \theta(s_1, s_2; \tau, a) \exp(-2\pi i \nu \tau) \, dt &= \sqrt{a} \iint s_1(t) \, s_2^*(at - \tau) \exp(-2\pi i \nu \tau) \, dt d\tau \\ &= \sqrt{a} \iint s_1(t) \, s_2^*(\eta) \exp[-2\pi i \nu (at - \eta)] \, dt d\eta \\ &= \sqrt{a} \int s_1(t) \exp(-2\pi i \nu a t) \, dt \int s_2^*(\eta) \exp(2\pi i \nu \eta) \, d\eta \\ &= \sqrt{a} \, S_1(a\nu) \, S_2^*(\nu), \ \forall \nu \end{aligned}$$

†The constant is not unity, as in the narrowband case, since the integration is now over $0 < f < \infty$ rather than $-\infty < \phi < \infty$ where $\phi = (a - 1) f$

Corollary 1

Letting $\nu = 0$ in Eq. (18), we get

$$\int \theta(s_1, s_2; \tau, a)\, d\tau = \sqrt{a}\, S_1(0) S_2^*(0).$$

Corollary 2 (Uniqueness theorem)

If $\theta(x, x; \tau, a) = \theta(y, y; \tau, a)$, then $x(t) = cy(t)$ almost everywhere (a.e.), where c is a constant and $|c| = 1$.

Proof

By Theorem 2, we have

$$X(a\nu) X^*(\nu) = Y(a\nu) Y^*(\nu). \tag{19}$$

$$\text{For } \nu = \nu_0, \text{ if } X^*(\nu_0) \neq 0,$$

$$X(a\nu_0) = \frac{Y^*(\nu_0)}{X^*(\nu_0)} Y(a\nu_0) = cY(a\nu_0), \ \forall a,$$

where c is a constant. Thus, $|X(a\nu_0)|^2, = |c|^2 |Y(a\nu_0)|^2, \forall a$, while from Eq. (19) $|X(\nu)|^2 = |Y(\nu)|^2, \forall \nu$; hence, $|c| = 1$. Since two functions having the same Fourier transform are equal a.e., $x(t) = cy(t)$ a.e.

Theorem 3 (Convolution Theorem)

$$\text{Let } u_1(t) = v_1(t) * w_1(t), \ u_2(t) = v_2(t) * w_2(t)$$

Then

$$\begin{aligned} \sqrt{a}\,\theta(u_1, u_2; \tau, a) &= \theta(v_1, v_2; \tau, \tau) \underset{\tau}{*} \theta(w_1, w_2; \tau, a) \\ &= \theta(v_1, w_2; \tau, a) \underset{\tau}{*} \theta(w_1, v_2; \tau, a), \end{aligned} \tag{20}$$

where $\underset{\tau}{*}$ indicates convolution with respect to τ.

Proof

$$\begin{aligned} \sqrt{a}\theta(u_1, u_2; \tau, a) &= a \int u_1(t) u_2^*(at - \tau)\, dt \\ &= a \iiint v_1(\eta) w_1(t - \eta) v_2^*(\zeta) w_2^*(at - \tau - \zeta)\, d\eta\, d\zeta\, dt \\ &= a \iiint v_1(\eta) v_2^*(\zeta) w_1(\nu) w_2^*(a\nu + a\eta - \tau - \zeta)\, d\nu\, d\eta\, d\zeta \\ &= \sqrt{a} \iint v_1(\eta) v_2^*(\zeta) \theta(w_1, w_2; \tau + \zeta - a\eta, a)\, d\eta\, d\zeta \\ &= \sqrt{a} \iint \theta(w_1, w_2; \xi, a) v_1(\eta) v_2^*(\xi + a\eta - \tau)\, d\eta\, d\xi \\ &= \int \theta(w_1, w_2; \xi, a) \theta(v_1, v_2; \tau - \xi, a)\, d\xi \\ &= \theta(w_1, w_2; \tau, a) \underset{\tau}{*} \theta(v_1, v_2; \tau, a). \end{aligned} \tag{21}$$

Similarly, it follows from Eq. (21) that

$$\sqrt{a}\,\theta(u_1,\ u_2;\tau,\ a) = \theta(v_1,\ w_2;\tau,\ a) \underset{\tau}{*} \theta(w_1,\ v_2;\tau,\ a)$$

ACCELERATING TARGETS

The wideband ambiguity function defined by Eq. (4) is valid only for targets moving with constant radial velocity. A generalization of the ambiguity function for accelerating targets was made by Kelly and Wishner (4). An additional modification is required. As before we let $s_1(t)$ represent a signal transmitted with a propagation velocity c, assumed constant. The echo is again represented by $s_2(t) = As_1[t - T(t)]$, where $T(t)$ is the total delay, the time required for the signal to reach the target and return. Thus, if $r(t)$ is the range of the target at any time t, Kelly and Wishner define the functional relationship

$$cT(t)/2 = r[t - T(t)/2]. \tag{22}$$

Expanding $T(t)$ in a Taylor series about $t = t_0$, we have

$$T(t) = T(t_0) + T'(t_0)(t - t_0) + \frac{T''(t_0)}{2!}(t - t_0)^2 + \dots,$$

where t_0 is arbitrary. It is convenient to choose t_0 as the time of reception of a signal transmitted at time $t = 0$. That is, the delay of the signal received at $t = t_0$ is just t_0: $T(t_0) = t_0$. Then

$$T(t) = t_0 + \delta(t - t_0) + \epsilon(t - t_0)^2 + \dots, \tag{23}$$

where $\delta \triangleq T'(t_0)$ and $\epsilon \triangleq T''(t_0)/2$.

Differentiating Eq. (22) yields

$$T'(t) = \frac{2}{c} r'[t - T(t)/2][1 - T'(t)/2]$$

$$= \frac{2r'[t - T(t)/2]}{c + r'[t - T(t)/2]} = \frac{2v[t - T(t)/2]}{c + v[t - T(t)/2}, $$

where $v(t) \triangleq r'(t)$.

$$\text{Thus,}\ \delta = \frac{2v[t_0 - t_0/2]}{c + v[t_0 - t_0/2]} = \frac{2v(t_0/2)}{c + v(t_0/2)} \approx 2v(t_0/2)/c.$$

Differentiating again,

$$T''(t) = \frac{\{c + v[t - T(t)/2]\}\, 2v'[t - T(t)/2][1 - T'(t)/2] - 2v[t - T(t)/2]\, v'[t - T(t)/2][1 - T'(t)/2]}{\{c + v[t - T(t)/2]\}^2}$$

$$= \frac{2cv'[t - T(t)/2][1 - T'(t)/2]}{\{c + v[t - T(t)/2]\}^2} = \frac{2cv'[t - T(t)/2]\left\{\dfrac{c}{c + v[t - T(t)/2]}\right\}}{\{c + v[t - T(t)/2]\}^2}$$

$$= \frac{2c^2 a[t - T(t)/2]}{\{c + v[t - T(t)/2]\}^3},$$

where $a(t) = v'(t) = r''(t)$.

$$\text{Thus, } \epsilon \triangleq T''(t_0)/2 = \frac{c^2 a(t_0/2)}{[c + v(t_0/2)]^3} \approx a(t_0/2)/c.$$

From Eq. (23),

$$\begin{aligned} t - T(t) &= t - t_0 - \delta(t - t_0) - \epsilon(t - t_0)^2 - \dots \\ &= (1 - \delta)\,(t - t_0) - \epsilon(t - t_0)^2 - \dots \\ &= \alpha(t - t_0) - \epsilon(t - t_0)^2 - \dots, \end{aligned}$$

$$\text{where } \alpha = 1 - \delta = 1 - \frac{2v(t_0/2)}{c + v(t_0/2)} = \frac{c - v(t_0/2)}{c + v(t_0/2)} \approx 1 - \frac{2v(t_0/2)}{c}$$

is the "Doppler stretch factor," as before.

For times $t \approx t_0$, we retain only terms of $o(t - t_0)^2$.

$$\text{Then } s_2(t) = As_1[t - T(t)] = As_1[\alpha(t - t_0) - \epsilon(t - t_0)^2].$$

Using Eq. (2) and dropping the subscripts, we achieve maximum resolution by minimizing the absolute value of

$$Q(t_0, \alpha, \epsilon) \triangleq \chi_{ss} \triangleq \sqrt{\alpha} \int s(t)\, s^*[\alpha(t - t_0) - \epsilon(t - t_0)^2]\; dt. \tag{24}$$

The ambiguity function for accelerating targets is

$$|Q(t_0, \alpha, \epsilon)|^2 = \alpha \left| \int s(t) s^*[\alpha(t - t_0) - \epsilon(t - t_0)^2]\; dt \right|^2. \tag{25}$$

For constant velocity targets, $\epsilon = a = 0$, and

$$Q(t_0, \alpha, 0) = \theta(\tau, \alpha) = \sqrt{\alpha} \int s(t)\, s^*(\alpha t - \tau)\; dt,$$

in agreement with Eq. (3), where $\tau = \alpha t_0 = T(0)$ is the signal delay at $t = 0$.

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Appendix

UNCERTAINTY RELATION

Borrowing from quantum mechanics, Gabor† derived an uncertainty relation for the signal duration-signal bandwidth product. This relation may be obtained easily from Eqs. (7) and (8). We have

$$\mathfrak{I}^2\beta^2 = 4\pi^2 \int t^2 |u(t)|^2 \, dt \int |u'(t)|^2 \, dt.$$

Reversing the procedure used in obtaining Eq. (6), the Schwarz inequality yields

$$\mathfrak{I}^2\beta^2 \geq 4\pi^2 |\int tu(t)u'^*(t) \, dt|^2 = 4\pi^2 |A|^2 = 4\pi^2 |a + ib|^2,$$

$$\text{where } A \triangleq a + ib \triangleq \int tu(t)u'^*(t) \, dt.$$

For any complex number $A = a + ib$,

$$|A|^2 = a^2 + b^2 \geq a^2 = (A + A^*)^2/4;$$

$$\text{therefore, } \mathfrak{I}^2\beta^2 \geq \pi^2 [\int t(u \frac{du^*}{dt} + u^* \frac{du}{dt}) \, dt]^2 = \pi^2 [\int t \frac{d}{dt}(uu^*) \, dt]^2 .$$

Integrating by parts, this becomes

$$\mathfrak{I}^2\beta^2 \geq \pi^2 [t|u(t)|^2 |_{-\infty}^{\infty} - \int |u(t)^2 \, dt]^2 = \pi^2,$$

if $t|u(t)|^2$ vanishes at $t = \pm\infty$. Thus,

$$\mathfrak{I}\beta \geq \pi. \qquad \text{(A1)}$$

Equality holds if and only if $b = 0$ and $u'(t) = k' \, tu(t)$; i.e., $u'(t)/u(t) = k't$. From this we get $\ln u(t) = k't^2/2 + \ln k$ or $u(t) = k \exp(k't^2/2 = k \exp(-at^2)$, where $a > 0$ to assure that $u(\pm\infty) = 0$. Thus, the waveform with minimum time-bandwidth product (as defined here) is a Gaussian pulse.

Woodward‡ points out that large intervals of time and frequency are not incompatible. Thus, the uncertainty relation, Eq. (A1), does not imply any restrictions on accuracy of range and velocity measurements. It was the limitation on resolution which lead Woodward to develop the ambiguity function.

†D. Gabor, "Theory of Communication," J. Inst. Elec. Engr. (Pt. III) 93:429-441 (1946).
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13. ABSTRACT

The wideband ambiguity function considers the actual time compression or expansion of the Doppler effect, rather than the frequency-shift approximation of the narrowband function. The wideband ambiguity function has been derived and its properties reviewed, including comparisons with those of the narrowband function. The significance of the difference between the two functions can be shown to depend on the product of signal duration, signal bandwidth, and target velocity.

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