Doppler Resilient Waveforms with Perfect Autocorrelation

Ali Pezeshki,* *Member, IEEE,* A. Robert Calderbank, *Fellow, IEEE,*
William Moran, *Member, IEEE,* and Stephen D. Howard

Submitted to IEEE Transactions on Information Theory
March 1, 2007

Abstract

We describe a method of constructing a sequence of phase coded waveforms with perfect autocorrelation in the presence of Doppler shift. The constituent waveforms are Golay complementary pairs which have perfect autocorrelation at zero Doppler but are sensitive to nonzero Doppler shifts. We extend this construction to multiple dimensions, in particular to radar polarimetry, where the two dimensions are realized by orthogonal polarizations. Here we determine a sequence of two-by-two Alamouti matrices where the entries involve Golay pairs and for which the sum of the matrix-valued ambiguity functions vanish at small Doppler shifts. The Prouhet-Thue-Morse sequence plays a key role in the construction of Doppler resilient sequences of Golay pairs.

Index Terms

Doppler resilient waveforms, Golay complementary sequences, perfect autocorrelation waveforms, Prouhet-Thue-Morse sequence, radar polarimetry.

This work was supported in part by the DARPA Waveforms for Active Sensing Program under contracts FA8750-05-2-0285 and N00173-06-1-G006.

A. Pezeshki (corresponding author) and A. R. Calderbank are with The Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544, USA (emails: pezeshki@math.princeton.edu; calderbk@math.princeton.edu).

W. Moran is with the Department of Electrical Engineering and Computer Science, University of Melbourne, Melbourne 03010, Australia (e-mail: b.moran@ee.unimelb.edu.au).

S. D. Howard is with the Defence Science and Technology Organisation, Edinburgh 05010, Australia.

I. INTRODUCTION

The value of perfect autocorrelation sequences in radar imaging is that their impulse-like autocorrelation function can enable enhanced range resolution (e.g. see [1]–[6]). An important class of perfect autocorrelation sequences are complementary sequences introduced by Golay [6]. Golay complementary sequences have the property that the sum of their autocorrelation functions vanishes at all (integer) delays other than zero. This means that the sum of the ambiguity functions (composite ambiguity function) of Golay complementary sequences is sidelobe free along the zero-Doppler axis, making them ideal for range imaging.

The concept of complementary sequences was generalized to multiphase (or polyphase) sequences by Heimiller [7], Frank *et al.* [8]–[10], and Sivaswami [11], and to multiple complementary codes by Tseng and Liu [12]. Over the past five decades, the use of complementary and polyphase sequences (and related codes) have been widely explored for radar imaging, e.g. see [1]–[18]. Recently, Deng [19] and Khan *et al.* [20] extended the use of polyphase sequences to orthogonal netted radar (a special case of MIMO radar), and Howard *et al.* [21] and Calderbank *et al.* [22] combined Golay complementary sequences with Alamouti signal processing to enable pulse compression for multi-channel and fully polarimetric radar systems. Golay complementary sequences have also been advocated for the next generation guided radar (GUIDAR) systems [23].

Despite the attention they received from the radar engineering community, complementary and polyphase sequences were somewhat ignored by communication engineers for many years, although their autocorrelation functions have as low sidelobes as the popular pseudo noise (PN) sequences. In fact, up until 1990, there were only a few articles on the use of complementary and polyphase sequences in communications, among which are the early work by Reed and Zetterberg [24] and the introduction of orthogonal complementary codes for synchronous spread spectrum multiuser communications by Suehiro and Hatori [25]. In 1990's, some researchers including Wilkinson and Jones [26], van Nee [27], and Ochiai and Imai [28] explored the use of Golay complementary sequences as codewords for OFDM, due to their small peak-to-mean envelope power ratio (PMEPR). However, the major advances in this context are due to Davis and Jedwab [29] and Paterson [30], who derived tight bounds for the PMEPR of Golay complementary sequences and related codes from cosets of the generalized first-order Reed-Muller code. Construction of low PMEPR codes from cosets of the generalized first-order Reed-Muller code has also been considered by Schmidt [31] and Schmidt and Finger [32]. Complementary codes have also been employed as pilot signals for channel estimation in OFDM systems [33].

Recently, complementary and polyphase codes (in particular orthogonal complementary codes) have been advocated by Chen *et al.* [34],[35] and Tseng and Bell [36] for enabling interference-free (both multipath and multi-access) multicarrier CDMA. Other work in this context include the extension of complementary codes using the Zadoff-Chu sequence by Lu and Dubey [37] and cyclic shifted orthogonal complementary codes by Park and Jim [38]. In [39], orthogonal complementary codes have been used in the design of access-request packets for contention resolution in random-access wireless networks.

Despite their many intriguing properties and recent theoretical advances, in practice a major barrier exists in adoption of complementary sequences for radar and communications; the perfect auto-correlation property of these sequences is extremely sensitive to Doppler shift. Although the shape of the composite ambiguity function of complementary sequences is ideal along the zero-Doppler axis, off the zero-Doppler axis it has large sidelobes in delay, which prevent unambiguous range imaging in radar or reliable detection in communications. Most generalizations of complementary sequences, including multiple complementary sequences and polyphase sequences suffer from the same problem to some degree. Examples of polyphase sequences that exhibit some tolerance to Doppler are Frank sequences [9], $P1$, $P2$, $P3$, and $P4$ sequences [18], PX sequences [40], and $P(n, k)$ sequences [41],[42]. Sivaswami [43] has also proposed a class of near complementary codes, called subcomplementary codes, which exhibit some tolerance to Doppler shift. Subcomplementary codes consist of a set of N length-N sequences that are phase-modulated by a binary Hadamard matrix. The necessary and sufficient conditions for a set of phase-modulated sequences to be subcomplementary have been derived by Guey and Bell in [44]. The design of Doppler tolerant polyphase sequences has also been considered for MIMO radar. In [20], Khan *et al.* have used a harmonic phase structural constraint along with a numerical optimization method to design a set of polyphase sequences with resilience to Doppler shifts for orthogonal netted radar.

In this paper, we present a novel and systematic way of designing a Doppler resilient sequence of Golay complementary waveforms for radar, for which the composite ambiguity function maintains ideal shape at small Doppler shifts. The idea is to determine a sequence of Golay pairs that annihilates the low-order terms of the Taylor expansion (around zero Doppler) of the composite ambiguity function. It turns out that the Prouhet-Thue-Morse sequence [45]-[48] plays a key role in determining the sequence of Doppler resilient Golay pairs. We then extend our analysis to the design of a Doppler resilient sequence of Alamouti waveform matrices of Golay pairs, for which the sum of the matrix-valued ambiguity functions vanishes at small Doppler shifts. Alamouti matrices of Golay waveforms have recently been shown [21],[22] to be useful for instantaneous radar polarimetry, which has the potential to significantly increase the performance of fully polarimetric radar systems, without increasing the receiver signal processing complexity beyond that of single channel matched filtering. Again, the Prouhet-Thue-Morse sequence plays a key role in determining the Doppler resilient sequence of Golay pairs. Finally, numerical examples are presented to demonstrate the perfect autocorrelation properties of Doppler resilient Golay pairs at small Doppler shifts.

II. Golay Complementary Sequences

Definition 1: Two length L unimodular sequences of complex numbers $x[l]$ and $y[l]$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$\mathrm{corr}_k(x[l]) + \mathrm{corr}_k(y[l]) = 2L\delta_{k,0}, \quad \text{for } k = -(L-1), \cdots, (L-1), \tag{1}$$

where $\mathrm{corr}_k(x[l])$ is the autocorrelation of $x[l]$ at lag k and $\delta_{k,0}$ is the Kronecker delta function.

Let $X(z) = \mathcal{Z}\{x[l]\}$ and $Y(z) = \mathcal{Z}\{y[l]\}$ be the z-transforms of $x[l]$ and $y[l]$ so that

$$\begin{aligned} X(z) &= x[0] + x[1]z^{-1} + \ldots + x[L-1]z^{-(L-1)} \\ Y(z) &= y[0] + y[1]z^{-1} + \ldots + y[L-1]z^{-(L-1)}. \end{aligned} \tag{2}$$

Then, $x[l]$ and $y[l]$ (or alternatively $X(z)$ and $Y(z)$) are Golay complementary if $X(z)$ and $Y(z)$ satisfy

$$X(z)\widetilde{X}(z) + Y(z)\widetilde{Y}(z) = 2L \tag{3}$$

or equivalently

$$\|X(z)\|^2 + \|Y(z)\|^2 = 2L, \tag{4}$$

where $\widetilde{X}(z) = X^*(1/z^*)$ and $\widetilde{Y}(z) = Y^*(1/z^*)$ are the z-transforms of $\widetilde{x}[l] = x^*[-l]$ and $\widetilde{y}[l] = y^*[-l]$, the time reversed complex conjugates of $x[l]$ and $y[l]$.

Henceforth we drop the discrete time index l from $x[l]$ and $y[l]$ and simply use x and y. We use the notation (x, y) whenever x and y are Golay complementary and call (x, y) a Golay pair. From (3) it follows that if (x, y) is a Golay pair then $(\pm x, \pm\widetilde{y})$, $(\pm\widetilde{x}, \pm y)$, and $(\pm\widetilde{x}, \pm\widetilde{y})$ are also Golay pairs.

A. Golay Pairs for Radar Detection

Consider a single transmitter/single receiver radar system. Suppose Golay pairs $(x_0, x_1), (x_2, x_3), \ldots,$ (x_{N-2}, x_{N-1}) are transmitted over N pulse repetition intervals (PRIs) to interrogate a radar scene containing a stationary (relative to the transmitter and receiver) point target. Let $R_n(z) = \mathcal{Z}\{r_n[k]\}$ denote the z-transform of the radar return associated with the nth PRI. Then, the radar measurement equation can be written (in z-domain) as

$$\underbrace{[R_0(z), \ldots, R_{N-1}(z)]}_{\underline{\mathbf{r}}^T(z)} = hz^{-d_0} \underbrace{[X_0(z), \ldots, X_{N-1}(z)]}_{\underline{\mathbf{x}}^T(z)} + \underbrace{[W_0(z), \ldots, W_{N-1}(z)]}_{\underline{\mathbf{w}}^T(z)} \tag{5}$$

where the delay d_0 in z^{-d_0} depends on the target range r_0 and is given by $d_0 = [2r_0/(ct_0)]$, where t_0 is the "chip" interval (time interval between two consecutive values in $x[l]$ or $y[l]$), c is the speed of light, and $[a]$ denotes the integer part of a. Without loss of generality, from hereon we assume that $d_0 = 0$, centering the delay axis at the target location. The scalar h is the target scattering coefficient, which we assume to be proper complex normal with zero mean and variance $2\sigma_h^2$, but fixed over the N RPIs. Elements of $\underline{\mathbf{w}}^T(z)$ are z-transforms of iid samples of proper complex white Gaussian noises with variance $2\sigma_w^2$.

If we process the radar return vector $\underline{\mathbf{r}}^T(z)$ by a receiver vector of the form

$$\underline{\widetilde{\mathbf{x}}}(z) = [\widetilde{X}_0(z), \ldots, \widetilde{X}_{N-1}(z)]^T \tag{6}$$

then the receiver output will be

$$\begin{aligned} U(z) = \mathbf{r}^T(z)\underline{\widetilde{\mathbf{x}}}(z) &= h\underline{\mathbf{x}}^T(z)\underline{\widetilde{\mathbf{x}}}(z) + \underline{\mathbf{w}}^T(z)\underline{\widetilde{\mathbf{x}}}(z) \\ &= NLh + \underline{\mathbf{w}}^T(z)\underline{\widetilde{\mathbf{x}}}(z), \end{aligned} \tag{7}$$

where the second equality follows by replacing $\underline{\mathbf{x}}^T(z)\underline{\widetilde{\mathbf{x}}}(z)$ with

$$\underline{\mathbf{x}}^T(z)\underline{\widetilde{\mathbf{x}}}(z) = (\|X_0(z)\|^2 + \|X_1(z)\|^2) + \ldots + (\|X_{N-2}(z)\|^2 + \|X_{N-1}(z)\|^2) = NL. \tag{8}$$

The term $\underline{\mathbf{x}}^T(z)\underline{\widetilde{\mathbf{x}}}(z)$ is the z-transform of the composite ambiguity function of Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ along the zero-Doppler axis. We notice that $\underline{\mathbf{x}}^T(z)\underline{\widetilde{\mathbf{x}}}(z)$ is a constant, which means that the composite ambiguity function of $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ vanishes at all (integer) delays along the zero-Doppler axis.

Transforming (7) back to the time domain, we have

$$u[k] = \mathcal{Z}^{-1}\{U(z)\} = NLh\delta_{k,0} + n[k], \tag{9}$$

where $n[k]$ is a proper complex white Gaussian noise with variance $2\sigma_n^2 = (NL)2\sigma_w^2$. This shows that detecting a stationary point in range amounts to the following Gaussian hypothesis test

$$u[k] = \begin{cases} n \sim CN[0, 2\sigma_n^2] & : \mathrm{H}_0 \\ NLh + n \sim CN[0, (2N^2L^2\sigma_h^2 + 2\sigma_n^2)] & : \mathrm{H}_1 \end{cases} \tag{10}$$

where $CN[0, 2\sigma_n^2]$ denotes the proper complex normal distribution with mean zero and variance $2\sigma_n^2$.

Remark 1: In the above analysis, the radar return associated with each PRI is processed separately at the receiver, that is each radar return is correlated with its corresponding waveform and then all the correlator outputs are added together. Hence the receiver output (in time domain) is

$$\begin{aligned} u[k] &= \sum_{n=0}^{N-1} \mathrm{xcorr}_k(r_n[k'], x_n[k']) \\ &= h \sum_{n=0}^{N-1} \mathrm{corr}_k(x_n[k']) + n[k] \\ &= NLh\delta_{k,0} + n[k], \end{aligned} \tag{11}$$

where $\mathrm{xcorr}_k(r_n[k'], x_n[k'])$ is the cross-correlation between $r_n[k']$ and $x_n[k']$ at lag k. If we want to process all the PRIs together then we must correlate the augmented radar return $r_a[k]$,

$$r_a[k] = r_0[k] + r_1[k - D] + \ldots + r_{N-1}[k - (N-1)D] \tag{12}$$

with the augmented waveform $x_a[k]$,

$$x_a[k] = x_0[k] + x_1[k - D] + \ldots + x_{N-1}[k - (N-1)D], \tag{13}$$

where D is the delay associated with a PRI. The receiver output in this case is

$$\begin{aligned} u_a[k] &= \mathrm{xcorr}_k(r_a[k'], x_a[k']) \\ &= h \sum_{n=0}^{N-1} \mathrm{corr}_k(x_n[k']) + h \sum_{\substack{n'=0 \\ n' \neq n''}}^{N-1} \sum_{n''=0}^{N-1} \mathrm{xcorr}_k(x_{n'}[k' - n'D], x_{n''}[k' - n''D]) + n_a[k] \\ &= NLh\delta_{k,0} + h \sum_{\substack{n'=0 \\ n' \neq n''}}^{N-1} \sum_{n''=0}^{N-1} \mathrm{xcorr}_k(x_{n'}[k' - n'D], x_{n''}[k' - n''D]) + n_a[k], \end{aligned} \tag{14}$$

where $n_a[k]$ is a noise term. The cross terms $\mathrm{xcorr}_k(x_{n'}[k'-n'D], x_{n''}[k'-n''D])$ result in range sidelobes whose peaks are offset by integer multiples of D from the origin $k = 0$. Thus, by processing each radar return separately as in (11) we can avoid range sidelobes caused by cross-correlations between different waveforms. However, the Doppler resolution will be limited by the time duration of a single waveform, whereas in the case where all the returns are processed together the Doppler resolution is enhanced due to having a longer transmit pulse.

B. Effect of Doppler on Golay Pairs

We now consider the case where the target moves at a constant speed, causing a Doppler shift of θ [rad] between two consecutive PRIs. We assume that the radar PRI is short enough that during the N PRIs where the Golay pairs are transmitted the target range remains approximately the same. Then the composite radar measurement is given by

$$\underline{\mathbf{r}}^T(z, \theta) = h\underline{\mathbf{x}}^T(z)\mathbf{D}(\theta) + \underline{\mathbf{w}}^T(z), \tag{15}$$

where $\mathbf{D}(\theta)$ is the following diagonal Doppler modulation matrix:

$$\mathbf{D}(\theta) = \mathrm{diag}(1, e^{j\theta}, \ldots, e^{j(N-1)\theta}). \tag{16}$$

If we now process the radar measurement vector $\underline{\mathbf{r}}^T(z,\theta)$ using the receiver vector $\underline{\tilde{\mathbf{x}}}(z)$ the receiver output will be

$$U(z,\theta) = \underline{\mathbf{r}}^T(z,\theta)\underline{\tilde{\mathbf{x}}}(z) = hG(z,\theta) + \underline{\mathbf{w}}^T(z)\underline{\tilde{\mathbf{x}}}(z), \tag{17}$$

where $G(z,\theta)$ is the z-transform of the composite ambiguity function of $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$, $\zeta(z,\theta)$
and is given by

$$G(z,\theta) = \underline{\mathbf{x}}^T(z)\mathbf{D}(\theta)\underline{\tilde{\mathbf{x}}}(z) = \|X_0(z)\|^2 + e^{j\theta}\|X_1(z)\|^2 + \ldots + e^{j(N-1)\theta}\|X_{N-1}(z)\|^2. \tag{18}$$

We notice that off the zero-Doppler axis ($\theta \neq 0$) the composite ambiguity function $G(z,\theta)$ is not sidelobe-free at integer delays. In fact, even small Doppler shifts can result in large sidelobes at integer delays.

One way to solve this problem is to use a bank of Doppler filters to estimate the unknown Doppler shift θ and then compensate for the Doppler effect by post-multiplying (15) by $\mathbf{D}^H(\theta)$ (where H denotes Hermitian transpose) prior to applying $\underline{\tilde{\mathbf{x}}}(z)$. However, since even a slight mismatch in Doppler can result in large sidelobes, we have to cover the possible Doppler range at a fine resolution, which requires the use of many Doppler filters. This motivates the question of whether it is possible to design *Doppler resilient* Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ so that $G(z,\theta) = \sum_{n=0}^{N-1} e^{jn\theta}\|X_n(z)\|^2 \approx \alpha z^0$, where α is constant, for a reasonable range of Doppler shifts θ. We are looking to construct the Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ so that $G(z,\theta)$ (which is a two-sided polynomial of degree $L-1$ in z^{-1}) vanishes at every delay but zero.

III. Doppler Resilient Golay Pairs

In this section we consider the design of Doppler resilient sequences of Golay pairs. More precisely, we describe how to select Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ so that in the Taylor expansion of $G(z,\theta)$ around $\theta = 0$ the coefficients of all terms up to a certain order, say M, vanish at all nonzero delays.

Consider the Taylor expansion of $G(z,\theta)$ around $\theta = 0$, i.e.,

$$G(z,\theta) = \sum_{m=0}^{\infty} C_m(z)\theta^m, \tag{19}$$

where

$$C_m(z) = \sum_{n=0}^{N-1} n^m \|X_n(z)\|^2, \qquad \text{for} \;\; m = 0, 1, 2, 3, \ldots \tag{20}$$

In general, the coefficients $C_m(z)$, $m = 1, 2, 3, \ldots$ are two-sided polynomials in z^{-1} of the form

$$C_m(z) = \sum_{l=-(L-1)}^{L-1} c_{m,l} z^{-l}, \qquad m = 1, 2, 3, \ldots \tag{21}$$

For instance, the first coefficient $C_1(z)$ is

$$C_1(z) = 0\|X_0(z)\|^2 + 1\|X_1(z)\|^2 + 2\|X_2(z)\|^2 + \ldots + (N-1)\|X_{N-1}(z)\|^2. \tag{22}$$

Noting that $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ are Golay pairs we can simplify $C_1(z)$ as

$$C_1(z) = N(N-2)L/2 + \|X_1(z)\|^2 + \|X_3(z)\|^2 + \ldots + \|X_{N-1}(z)\|^2. \tag{23}$$

Each term of the form $\|X_{2k+1}(z)\|^2 = X_{2k+1}(z)X^*_{2k+1}(1/z^*)$ is a two-sided polynomial of degree $L-1$ in the delay operator z^{-1}, which can not be matched with any of the other terms, as we have already taken into account all the Golay pairs. Consequently, $C_1(z)$ is a two-sided polynomial in z^{-1} of the form $C_1(z) = \sum_{l=-(L-1)}^{L-1} c_{1,l} z^{-l}$.

We wish to design the Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ so that $c_{1,l}$ vanish for all nonzero l. More generally, we wish to design $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ so that in the Taylor expansion in (19) the coefficients of all the terms up to a given order M vanish at all nonzero delays, i.e. $c_{m,l} = 0$, for all m $(1 \leq m \leq M)$ and for all nonzero l. Although not necessary, we continue to carry the term $0^m\|X_0(z)\|^2$ in writing $C_m(z)$ for reasons that will become clear. We note that there is no need to consider the zero-order term, as $C_0(z) = NL$.

A. The Requirement that $C_1(z)$ Vanish at All Nonzero Delays

To provide intuition, we first consider the case $N = 2^2 = 4$, where Golay pairs (x_0, x_1) and (x_2, x_3) are transmitted over four PRIs. Then, as the following calculation shows, $C_1(z)$ will vanish at all nonzero delays if the Golay pairs (x_0, x_1) and (x_2, x_3) are selected such that (x_1, x_3) is also a Golay pair:

$$C_1(z) = \underbrace{\underbrace{0\|X_0(z)\|^2 + \|X_1(z)\|^2}_{1\|X_1(z)\|^2} + \underbrace{2\|X_2(z)\|^2 + 3\|X_3(z)\|^2}_{2\times 2L + 1\|X_3(z)\|^2}}_{3\times 2L} \tag{24}$$

The trick is to break 3 into $2+1$, and then pair the extra $\|X_3(z)\|^2$ with $\|X_1(z)\|^2$. Note that it is easy to choose the pairs (x_0, x_1) and (x_2, x_3) such that (x_1, x_3) is also a Golay pair. For example, let (x, y) be an arbitrary Golay pair, then $(x_0 = x, x_1 = y)$, $(x_2 = -\widetilde{y}, x_3 = \widetilde{x})$, and $(x_1 = y, x_3 = \widetilde{x})$ are all Golay pairs. Other combinations of $\pm x$, $\pm\widetilde{x}$, $\pm y$, and $\pm\widetilde{y}$ are also possible. The calculation in (24) shows that it is possible to make $C_1(z)$ $(M = 1)$ vanish at all nonzero delays with $N = 2^{1+1}$ Golay sequences $x_0, \ldots, x_3$.

B. The Requirement that $C_1(z)$ and $C_2(z)$ Vanish at All Nonzero Delays

It is easy to see that when $N = 4$ it is not possible to force $C_2(z)$ $(M = 2)$ to zero at all nonzero delays. However, this is possible when $N = 2^{2+1} = 8$. As the following calculations show, we can make both $C_1(z)$ and $C_2(z)$ vanish at all nonzero l if we select the Golay pairs $(x_0, x_1), \ldots, (x_6, x_7)$ such that (x_1, x_3), (x_5, x_7), and (x_3, x_7) are also Golay pairs.[1]

Making $C_1(z)$ vanish:

$$
\begin{aligned}
&C_1(z) = \\
&\underbrace{0\|X_0\|^2 + 1\|X_1\|^2}_{} + \underbrace{2\|X_2\|^2 + 3\|X_3\|^2}_{2 \times 2L +} + \underbrace{4\|X_4\|^2 + 5\|X_5\|^2}_{4 \times 2L +} + \underbrace{6\|X_6\|^2 + 7\|X_7\|^2}_{6 \times 2L +} \\
&\underbrace{[(1-0)=1]\|X_1\|^2 \quad [(3-2)=1]\|X_3\|^2}_{3 \times 2L} \quad \underbrace{[(5-4)=1]\|X_5\|^2 \quad [(7-6)=1]\|X_7\|^2}_{11 \times 2L}
\end{aligned}
\tag{25}
$$

Making $C_2(z)$ vanish:

$$
\begin{aligned}
&C_2(z) = \\
&\underbrace{0^2\|X_0\|^2 + 1^2\|X_1\|^2}_{} + \underbrace{2^2\|X_2\|^2 + 3^2\|X_3\|^2}_{4 \times 2L +} + \underbrace{4^2\|X_4\|^2 + 5^2\|X_5\|^2}_{16 \times 2L +} + \underbrace{6^2\|X_6\|^2 + 7^2\|X_7\|^2}_{36 \times 2L +} \\
&\underbrace{[(1^2 - 0^2) = 1]\|X_1\|^2 \quad [(3^2 - 2^2) = 5]\|X_3\|^2}_{5 \times 2L +} \quad \underbrace{[(5^2 - 4^2) = 9]\|X_5\|^2 \quad [(7^2 - 6^2) = 13]\|X_7\|^2}_{61 \times 2L +} \\
&\underbrace{[(3^2 - 2^2 - 1^2 + 0^2) = 4]\|X_3\|^2 \qquad [(7^2 - 6^2 - 5^2 + 4^2) = 4]\|X_7\|^2}_{[(0^2 + 1^2 + 2^2 + \ldots + 7^2) = 70] \times 2L}
\end{aligned}
\tag{26}
$$

Note that it is easy to select the Golay pairs $(x_0, x_1), \ldots, (x_6, x_7)$ such that (x_1, x_3), (x_5, x_7), and (x_3, x_7) are also Golay pairs. For example, $(x_0 = x, x_1 = y)$, $(x_2 = -\widetilde{y}, x_3 = \widetilde{x})$, $(x_4 = -\widetilde{y}, x_5 = \widetilde{x})$, and $(x_6 = x, x_7 = y)$, where (x, y) is an arbitrary Golay pair, satisfy all the extra Golay pair conditions.

[1]In writing (25) and (26) we have dropped the argument z for simplicity.

We notice that what allows us to make both $C_1(z)$ and $C_2(z)$ vanish at all nonzero l is the identity

$$3^m - 2^m - 1^m + 0^m = 7^m - 6^m - 5^m + 4^m, \quad \text{for} \quad m = 1, 2 \tag{27}$$

or alternatively

$$(0^m + 3^m + 5^m + 6^m) - (1^m + 2^m + 4^m + 7^m) = 0, \quad \text{for} \quad m = 1, 2, \tag{28}$$

where $m = 1$ and $m = 2$ correspond to the calculations for $C_1(z)$ and $C_2(z)$, respectively. In other words, the reason $C_1(z)$ and $C_2(z)$ can be forced to zero at all nonzero delays is that the set $\mathbb{S} = \{0, 1, \ldots, 7\}$ can be partitioned into two disjoint subsets $\mathbb{S}_0 = \{0, 3, 5, 6\}$ and $\mathbb{S}_1 = \{1, 2, 4, 7\}$ whose elements satisfy (28). This is a special case of the Prouhet (or Prouhet-Tarry-Escott) problem [48],[49] which we will discuss in more detail later in this section. But for now we just note that $\mathbb{S}_0$ is the set of all numbers in $\mathbb{S}$ that correspond to the zeros in the length-8 Prouhet-Thue-Morse sequence (PTM) [45]-[48]

$$(s_k)_{k=0}^{7} \;=\; 0\;1\;1\;0\;1\;0\;0\;1, \tag{29}$$

and $\mathbb{S}_1$ is the set of all numbers in $\mathbb{S}$ that correspond to the ones in $(s_k)_{k=0}^{7}$.

A key observation here is that the extra Golay pair conditions we had to introduce to make $C_1(z)$ and $C_2(z)$ vanish at all nonzero l are all associated with pairs of the form (x_p, x_q) where p and q are odd, and $p \in \mathbb{S}_0$ and $q \in \mathbb{S}_1$. This suggests a close connection between the Prouhet-Thue-Morse sequence and the way Golay sequences $x_0, x_1, \ldots, x_{N-1}$ must be paired.

C. The Requirement that $C_1(z)$ Through $C_M(z)$ Vanish

We now address the general problem of selecting the Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ to make $C_m(z)$, $m = 1, 2, \ldots, M$ vanish at all nonzero delays. We begin with some definitions and results related to the Prouhet-Thue-Morse sequence.

Definition 2.[45]-[48] The Prouhet-Thue-Morse (PTM) sequence $\mathcal{S} = (s_k)_{k\geq 0}$ over $\{0, 1\}$ is defined by the following recursions:

1) $s_0 = 0$
2) $s_{2k} = s_k$
3) $s_{2k+1} = \overline{s}_k = 1 - s_k$

for all $k > 0$, where $\overline{s} = 1 - s$ denotes the binary complement of $s \in \{0, 1\}$.

For example, the PTM sequence of length 32 is

$$\mathcal{S} = (s_k)_{k=0}^{31} \;=\; 0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,1\,0\,0\,1\,0\,1\,1\,0\,0\,1\,1\,0\,1\,0\,0\,1. \tag{30}$$

Prouhet's problem.[48],[49] Let $\mathbb{S} = \{0, 1, \ldots, N-1\}$ be the set of all integers between 0 and $N-1$. The Prouhet's problem (or Prouhet-Tarry-Escott problem) is the following. Given M, is it possible to partition $\mathbb{S}$ into two disjoint subsets $\mathbb{S}_0$ and $\mathbb{S}_1$ such that $\sum_{p\in\mathbb{S}_0} p^m = \sum_{q\in\mathbb{S}_1} q^m$ for all $0 \le m \le M$? Prouhet's proved that this is possible when $N = 2^{M+1}$ and that the partitions are identified by the PTM sequence.

Theorem 1 (Prouhet).[48],[49] Let $\mathcal{S} = (s_k)_{k\geq 0}$ be the PTM sequence. Define

$$\begin{aligned} \mathbb{S}_0 &= \{p \in \mathbb{S} = \{0, 1, 2, \ldots, 2^{M+1}-1\} |\ s_p = 0\} \\ \mathbb{S}_1 &= \{q \in \mathbb{S} = \{0, 1, 2, \ldots, 2^{M+1}-1\} |\ s_q = 1\} \end{aligned} \tag{31}$$

Then, for any m with $0 \le m \le M$ we have

$$\sum_{p\in\mathbb{S}_0} p^m = \sum_{q\in\mathbb{S}_1} q^m. \tag{32}$$

Lemma 1. Let $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$, $N = 2^{M+1}$ be Golay pairs. Let $\mathbb{X}_0 = \{x_p | p \in \mathbb{S}_0\}$ and $\mathbb{X}_1 = \{x_q | q \in \mathbb{S}_q\}$. Then, neither $\mathbb{X}_0$ nor $\mathbb{X}_1$ contains any of the Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$.

Proof: The Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ are of the form (x_{2k}, x_{2k+1}), where $k = 0, 1, \ldots, N/2-1$. From the definition of the PTM sequence we have $s_{2k+1} = \overline{s}_k = \overline{s}_{2k}$. Therefore, x_{2k} and x_{2k+1} cannot be in the same set.□

Lemma 2. Assume that the Golay pairs (x_0, x_1), …, (x_{N-2}, x_{N-1}), $N = 2^{M+1}$ are such that all pairs of the form $(x_{2k'+1} \in \mathbb{X}_0, x_{2k''+1} \in \mathbb{X}_1)$, i.e., all pairs of the form $(x_{2k'+1}, x_{2k''+1})$ with $2k'+1 \in \mathbb{S}_0$ and $2k''+1 \in \mathbb{S}_1$, are also Golay complementary. Then,

$$\|X_p(z)\|^2 = \|X_{p'}(z)\|^2 \quad \text{and} \quad \|X_q(z)\|^2 = \|X_{q'}(z)\|^2 \tag{33}$$

for all $p, p' \in \mathbb{S}_0$ (i.e. for all $x_p, x_{p'} \in \mathbb{X}_0$) and for all $q, q' \in \mathbb{S}_1$ (i.e. for all $x_q, x_{q'} \in \mathbb{X}_1$), and all pairs of the form $(x_p \in \mathbb{X}_0, x_q \in \mathbb{X}_1)$, i.e. all pairs (x_p, x_q) with $p \in \mathbb{S}_0$ and $q \in \mathbb{S}_1$, are Golay complementary.

Proof: Assume $p = 2k$ is even and $p \in \mathbb{S}_0$. Then $q = 2k+1$ is odd and $q \in \mathbb{S}_1$. We know that the pair $(x_{p=2k} \in \mathbb{X}_0, x_{q=2k+1} \in \mathbb{X}_1)$ is Golay complementary, as all the original Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ are of the form (x_{2k}, x_{2k+1}), hence

$$\|X_p(z)\|^2 + \|X_q(z)\|^2 = 2L. \tag{34}$$

Let $p' \in \mathbb{S}_0$ and assume p' is odd. Then, since $q = 2k+1 \in \mathbb{S}_1$ and all pairs of the form $(x_{2k'+1} \in \mathbb{X}_0, x_{2k''+1} \in \mathbb{X}_1)$ are Golay complementary (from our assumption), we have

$$\|X_{p'}(z)\|^2 + \|X_q(z)\|^2 = 2L. \tag{35}$$

Subtracting (35) from (34) gives

$$\|X_p(z)\|^2 = \|X_{p'}(z)\|^2. \tag{36}$$

Since (36) is true for any even $p \in \mathbb{S}_0$ and any odd $p' \in \mathbb{S}_0$ it must be true for any $p, p' \in \mathbb{S}_0$, or equivalently any $x_p, x_{p'} \in \mathbb{X}_0$. Similarly, we can prove that $\|X_q(z)\|^2 = \|X_{q'}(z)\|^2$ for all $x_q, x_{q'} \in \mathbb{X}_1$. Since at least one element from $\mathbb{X}_0$ forms a pair with one element in $\mathbb{X}_1$ (e.g. (x_0, x_1)) then all pairs of the form $(x_p \in \mathbb{X}_0, x_q \in \mathbb{X}_1)$ must be Golay complementary.□

Remark 2: We note that to construct Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$, $N = 2^{M+1}$ that satisfy the conditions of Lemma 2 we can consider an arbitrary Golay pair (x, y) and then arbitrarily choose $x_p \in \mathbb{X}_0$ from the set $\{x, -x, \widetilde{x}, -\widetilde{x}\}$ and $x_q \in \mathbb{X}_1$ from the set $\{y, -y, \widetilde{y}, -\widetilde{y}\}$, for any $p \in \mathbb{S}_0$ and any $q \in \mathbb{S}_1$.

We now present the main result of this section by stating the following theorem.

Theorem 2. The coefficients $C_1(z), \ldots, C_M(z)$ in the Taylor expansion (19) will vanish at all nonzero delays if the Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$, $N = 2^{M+1}$ are selected such that all pairs (x_p, x_q) where p and q are odd and $p \in \mathbb{S}_0$ and $q \in \mathbb{S}_1$ are also Golay complementary.

Proof: From Lemma 2, we have $\|X_p(z)\|^2 = \|X_{p'}(z)\|^2$ for all $p, p' \in \mathbb{S}_0$ and $\|X_q(z)\|^2 = \|X_{q'}(z)\|^2$ for all $q, q' \in \mathbb{S}_1$. Therefore, we can write $C_m(z)$ $(1 \leq m \leq M)$ as

$$C_m(z) = \sum_{n=0}^{N-1} n^m \|X_n(z)\|^2 = (\sum_{p \in \mathbb{S}_0} p^m)\|X_0(z)\|^2 + (\sum_{q \in \mathbb{S}_1} q^m)\|X_1(z)\|^2. \tag{37}$$

From the Prouhet theorem (Theorem 1), we have $\sum\limits_{p \in \mathbb{S}_0} p^m = \sum\limits_{q \in \mathbb{S}_1} q^m = \beta$, where β is constant. Therefore, we have

$$C_m(z) = \beta(\|X_0(z)\|^2 + \|X_1(z)\|^2) = 2\beta L. \tag{38}$$

IV. Doppler Resilient Golay Pairs for Fully Polarimetric Radar Systems

Fully polarimetric radar systems are capable of simultaneously transmitting and receiving on two orthogonal polarizations. The use of two orthogonal polarizations increases the degrees of freedom and can result in significant improvement in detection performance. Recently, Howard *et al.* [21] (also see [22]) proposed a novel approach to radar polarimetry that uses orthogonal polarization modes to provide essentially independent channels for viewing a target, and achieve diversity gain. Unlike conventional radar polarimetry, where polarized waveforms are transmitted sequentially and processed non-coherently, the approach in [21] allows for *instantaneous radar polarimetry*, where polarization modes are combined coherently on a pulse by pulse basis. Instantaneous radar polarimetry enables detection based on full polarimetric properties of the target and hence can provide better discrimination against clutter. When compared to a radar system with a singly-polarized transmitter and a singly-polarized receiver the instantaneous radar polarimetry can achieve the same detection performance (same false alarm and detection probabilities) with a substantially smaller transmit energy, or alternatively it can detect at substantially greater ranges for a given transmit energy [21].

A key ingredient of the approach in [21] is a unitary Alamouti matrix of Golay waveforms that has a perfect *matrix-valued ambiguity function* along the zero-Doppler axis. The unitary property of the waveform matrix allows for detection in range based on the full polarimetric properties of the target, without increasing the receiver signal processing complexity beyond that of single channel matched filtering. We show in this section that it is possible to design a sequence of Alamouti matrices of Golay waveforms, for which the sum of the matrix-valued ambiguity functions vanishes at all nonzero (integer) delays for small Doppler shifts.

Figure 1 shows the scattering model of the fully polarimetric radar system considered in [21], where h_{VH} denotes the scattering coefficient into the vertical polarization channel from a horizontally polarized incident field. Howard *et al.* employ Alamouti signal processing [50] to coordinate the transmission of $(N/2)$ Golay pairs $(x_0, x_1), \dots, (x_{N-2}, x_{N-1})$ over vertical and horizontal polarizations during N PRIs. The waveform matrix is of the form

$$\mathbf{X}(z) = \begin{pmatrix} X_0(z) & -\widetilde{X}_1(z) & \dots & X_{2k}(z) & -\widetilde{X}_{2k+1}(z) & \dots & X_{N-2}(z) & -\widetilde{X}_{N-1}(z) \\ X_1(z) & \widetilde{X}_0(z) & \dots & X_{2k+1}(z) & \widetilde{X}_{2k}(z) & \dots & X_{N-1}(z) & \widetilde{X}_{N-2}(z) \end{pmatrix}, \tag{39}$$

where different rows in $\mathbf{X}(z)$ correspond to vertical and horizontal polarizations, and different columns correspond to different time slots (PRIs).

The radar measurement matrix $\mathbf{R}(z)$ for this transmission scheme is given by

$$\mathbf{R}(z) = \mathbf{H}\mathbf{X}(z)\mathbf{D}(\theta) + \mathbf{W}(z), \tag{40}$$

where $\mathbf{H}$ is the 2 by 2 target scattering matrix, with entries h_{VV}, h_{VH}, h_{HV}, and h_{HH}, $\mathbf{W}(z)$ is a 2 by N noise matrix with entries that are iid proper complex normal with zero mean and variance $2\sigma_w^2$, and $\mathbf{D}(\theta)$ is the diagonal Doppler modulation matrix introduced in (16).

If we process $\mathbf{R}(z)$ with a receiver matrix $\widetilde{\mathbf{X}}(z)$ of the form

$$\widetilde{\mathbf{X}}(z) = \begin{pmatrix} \widetilde{X}_0(z) & -X_1(z) & \dots & \widetilde{X}_{2k}(z) & -X_{2k+1}(z) & \dots & \widetilde{X}_{N-2}(z) & -X_{N-1}(z) \\ \widetilde{X}_1(z) & X_0(z) & \dots & \widetilde{X}_{2k+1}(z) & X_{2k}(z) & \dots & \widetilde{X}_{N-1}(z) & X_{N-2}(z) \end{pmatrix}^T \tag{41}$$

then the receiver output will be

$$\mathbf{U}(z,\theta) = \mathbf{R}(z)\widetilde{\mathbf{X}}(z) = \mathbf{H}\mathbf{G}(z,\theta) + \mathbf{W}(z)\widetilde{\mathbf{X}}(z), \tag{42}$$

where the term $\mathbf{G}(z,\theta) = \mathbf{X}(z)\mathbf{D}(\theta)\widetilde{\mathbf{X}}(z)$ can be viewed as the z-transform of a *matrix-valued ambiguity function* for $\mathbf{X}(z)$. Along the zero-Doppler axis, where $\mathbf{D}(\theta = 0) = \mathbf{I}$, due to the interplay between

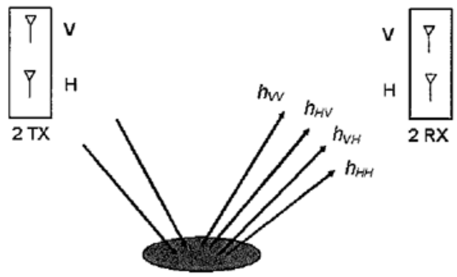


Fig. 1. Scattering model for a fully polarimetric radar system, with a dually-polarized transmit and a dually-polarized receive antenna.

Alamouti signal processing and the Golay property, the term $\mathbf{G}(z,\theta)$ reduces to

$$\mathbf{G}(z,0)=\mathbf{X}(z)\widetilde{\mathbf{X}}(z)=\begin{pmatrix}\sum_{n=0}^{N-1}\|X_n(z)\|^2=NL & \sum_{k=0}^{N/2-1}(1-1)X_{2k}(z)\widetilde{X}_{2k+1}(z)=0\\ \sum_{k=0}^{N/2-1}(1-1)\widetilde{X}_{2k}(z)X_{2k+1}(z)=0 & \sum_{n=0}^{N-1}\|X_n(z)\|^2=NL\end{pmatrix}. \tag{43}$$

This shows that $\mathbf{X}(z)$ has a perfect matrix-valued ambiguity function along the zero-Doppler axis; that is along the zero-Doppler axis $G(z,\theta)$ vanishes at all nonzero (integer) delays, and is unitary at zero-delay. A consequence of (43) is that detecting a point target in range reduces to a simple Gaussian hypothesis test, for which the likelihood ratio detector is the same as an energy detector. However, off the zero-Doppler axis the property in (43) no longer holds, and the elements of the matrix-valued ambiguity function $G(z,\theta)$ can have large sidelobes, even at small Doppler shifts.

We consider how the Golay pairs $(x_0,x_1),\ldots,(x_{N-2},x_{N-1})$ must be selected so that for small Doppler shifts we have

$$\mathbf{G}(z,\theta)=\mathbf{X}(z)\mathbf{D}(\theta)\widetilde{\mathbf{X}}(z)=\begin{pmatrix}G_1(z,\theta) & G_2(z,\theta)\\ \widetilde{G}_2(z,\theta) & G_1(z,\theta)\end{pmatrix}\approx\begin{pmatrix}NL & 0\\ 0 & NL\end{pmatrix}, \tag{44}$$

where

$$\begin{aligned}G_1(z,\theta)&=\sum_{n=0}^{N-1}e^{jn\theta}\|X_n(z)\|^2\\ &=\|X_0(z)\|^2+e^{j\theta}\|X_1(z)\|^2+\ldots+e^{j(N-1)\theta}\|X_{N-1}(z)\|^2\end{aligned} \tag{45}$$

and

$$\begin{aligned}G_2(z,\theta)&=\sum_{k=0}^{N/2-1}(e^{j2k\theta}-e^{j(2k+1)\theta})X_{2k}(z)\widetilde{X}_{2k+1}(z)\\ &=(1-e^{j\theta})X_0(z)\widetilde{X}_1(z)+\ldots+(e^{j(N-2)\theta}-e^{j(N-1)\theta})X_{N-2}(z)\widetilde{X}_{N-1}(z).\end{aligned} \tag{46}$$

The diagonal term of $\mathbf{G}(z,\theta)$, i.e., $G_1(z,\theta)$, is equal to the single channel composite ambiguity function $G(z,\theta)$ in (18). Therefore, we can use Theorem 2 to design the Golay pairs $(x_0,x_1),\ldots,(x_{N-2},x_{N-1})$, $N=2^{M+1}$ such that in the Taylor expansion (19) the coefficients $C_m(z)$, $m=1,2,\ldots,M$ vanish at all nonzero delays. Thus, from now on we only discuss how the off-diagonal term $G_2(z,\theta)$ can be forced to zero for small Doppler shifts.

Consider the Taylor expansion of $G_2(z,\theta)$ around $\theta=0$, i.e.,

$$G_2(z,\theta)=\sum_{m=0}^{\infty} B_m(z)\theta^m, \tag{47}$$

where

$$B_m(z)=\sum_{k=0}^{N/2-1}((2k)^m-(2k+1)^m)X_{2k}(z)\widetilde{X}_{2k+1}(z), \qquad \text{for} \quad m=0,1,2,\ldots. \tag{48}$$

In general, the coefficients $B_m(z)$, $m=1,2,3,\ldots$ are two-sided polynomials in z^{-1} of the form

$$B_m(z)=\sum_{l=-(L-1)}^{L-1} b_{m,l}z^{-l}, \qquad m=1,2,3,\ldots \tag{49}$$

For instance, the first coefficient $B_1(z)$ is

$$B_1(z)=(0-1)X_0(z)\widetilde{X}_1(z)+\ldots+((N-2)-(N-1))X_{N-2}(z)\widetilde{X}_{N-1}(z). \tag{50}$$

Each term of the form $X_{2k}(z)\widetilde{X}_{2k+1}(z)$ in (50) is a two-sided polynomial of degree $L-1$ in z^{-1}, and since in general the terms $X_{2k}(z)\widetilde{X}_{2k+1}(z)$ for different values of k do not cancel each other, $B_1(z)$ is also a two-sided polynomial of degree $L-1$ in z^{-1}.

Suppose that the Golay pairs $(x_0,x_1),\ldots,(x_{N-2},x_{N-1})$, $N=2^{M+1}$ satisfy the conditions of Theorem 2 so that $C_1(z),\ldots,C_M(z)$ vanish at all nonzero delays. We wish to determine the extra conditions required for $(x_0,x_1),\ldots,(x_{N-2},x_{N-1})$ to force $B_1(z),\ldots,B_M(z)$ to zero at all delays. As we show, again the PTM sequence is the key to finding the zero-forcing conditions. The zero-order term $B_0(z)$ is always zero and hence we do not consider it in our discussion.

A. The Requirement that $B_1(z)$ Vanish

Again, to gain intuition, we first consider the case $N=2^2=4$. Then, as the following calculation shows, $B_1(z)$ will vanish if the Golay pairs (x_0,x_1) and (x_2,x_3) are selected so that $X_0(z)\widetilde{X}_1(z)=-X_2(z)\widetilde{X}_3(z)$:

$$B_1(z)=\underbrace{(0-1)X_0(z)\widetilde{X}_1(z)+\underbrace{(2-3)X_2(z)\widetilde{X}_3(z)}_{-(2-3)X_0(z)\widetilde{X}_1(z)}}_{0} \tag{51}$$

In summary, to make $C_1(z)$ vanish at all nonzero delays and to force $B_1(z)$ to zero at the same time, the Golay pairs (x_0, x_1) and (x_2, x_3) must be selected such that (x_1, x_3) is also a Golay pair and $X_0(z)\widetilde{X}_1(z) = -X_2(z)\widetilde{X}_3(z)$. If we let (x, y) be an arbitrary Golay pair then it is easy to see that $(x_0 = x, x_1 = y)$, $(x_2 = -\widetilde{y}, x_3 = \widetilde{x})$ satisfy these conditions (other choices are also possible). The Alamouti waveform matrix $\mathbf{X}(z)$ for this choice of Golay pairs is given by

$$\mathbf{X}(z) = \begin{pmatrix} X_0(z) = X(z) & -\widetilde{X}_1(z) = -\widetilde{Y}(z) & X_2(z) = -\widetilde{Y}(z) & -\widetilde{X}_3(z) = -X(z) \\ X_1(z) = Y(z) & \widetilde{X}_0(z) = \widetilde{X}(z) & X_3(z) = \widetilde{X}(z) & \widetilde{X}_2(z) = -Y(z) \end{pmatrix}. \tag{52}$$

B. The Requirement that $B_1(z)$ and $B_2(z)$ Vanish:

Let us now consider the case $N = 2^3 = 8$. Then, as the following calculations show, both $B_1(z)$ and $B_2(z)$ will vanish if we select $(x_0, x_1), \ldots, (x_6, x_7)$ such that $X_0(z)\widetilde{X}_1(z) = -X_2(z)\widetilde{X}_3(z) = -X_4(z)\widetilde{X}_5(z) = X_6(z)\widetilde{X}_7(z)$.

Making $B_1(z)$ vanish:

$$B_1(z) = \underbrace{\underbrace{(0-1)X_0\widetilde{X}_1}_{[(0-1)=-1]X_0\widetilde{X}_1} + \underbrace{(2-3)X_2\widetilde{X}_3}_{[-(2-3)=1]X_0\widetilde{X}_1} + \underbrace{(4-5)X_4\widetilde{X}_5}_{[-(4-5)=1]X_0\widetilde{X}_1} + \underbrace{(6-7)X_6\widetilde{X}_7}_{[(6-7)=-1]X_0\widetilde{X}_1}}_{0} \tag{53}$$

Making $B_2(z)$ vanish:

$$B_2(z) = \underbrace{\underbrace{\underbrace{(0^2-1^2)X_0\widetilde{X}_1}_{[(0^2-1^2)=-1]X_0\widetilde{X}_1} + \underbrace{(2^2-3^2)X_2\widetilde{X}_3}_{[-(2^2-3^2)=5]X_0\widetilde{X}_1}}_{[0^2-1^2-2^2+3^2=4]X_0\widetilde{X}_1} + \underbrace{\underbrace{(4^2-5^2)X_4\widetilde{X}_5}_{[-(4^2-5^2)=9]X_0\widetilde{X}_1} + \underbrace{(6^2-7^2)X_6\widetilde{X}_7}_{[(6^2-7^2)=-13]X_0\widetilde{X}_1}}_{[-4^2+5^2+6^2-7^2=-4]X_0\widetilde{X}_1}}_{0} \tag{54}$$

In writing (53) and (54) we have dropped the argument z for simplicity.

In summary, to make $C_1(z)$ and $C_2(z)$ vanish at all nonzero delays and to force $B_1(z)$ and $B_2(z)$ to zero at the same time, the Golay pairs $(x_0, x_1), \ldots, (x_6, x_7)$ must satisfy the conditions of Theorem 2, and the within-pair cross-spectral densities must satisfy

$$X_0(z)\widetilde{X}_1(z) = -X_2(z)\widetilde{X}_3(z) = -X_4(z)\widetilde{X}_5(z) = X_6(z)\widetilde{X}_7(z). \tag{55}$$

It is easy to see that the Golay pairs in the following waveform matrix $\mathbf{X}(z)$ satisfy all these conditions:

$$\mathbf{X}(z) = \begin{pmatrix}\mathbf{X}_0(z) & \mathbf{X}_1(z) & \mathbf{X}_1(z) & \mathbf{X}_0(z)\end{pmatrix}, \tag{56}$$

where $\mathbf{X}_0(z)$ and $\mathbf{X}_1(z)$ are given by

$$\mathbf{X}_0(z) = \begin{pmatrix} X(z) & -\widetilde{Y}(z) \\ Y(z) & \widetilde{X}(z) \end{pmatrix} \quad \text{and} \quad \mathbf{X}_1(z) = \begin{pmatrix} -\widetilde{Y}(z) & -X(z) \\ \widetilde{X}(z) & -Y(z) \end{pmatrix}, \tag{57}$$

and (x, y) is an arbitrary Golay pair.

The trick in forcing $B_1(z)$ and $B_2(z)$ to zero is to cleverly select the signs of the cross-correlation functions (cross-spectral densities) between the two sequences in every Golay pair relative to the cross-correlation function (cross-spectral density) for x_0 and x_1. If we let 0 and 1 correspond to the positive and negative signs respectively, we observe that the sequence of signs in (55) corresponds to the length-4 PTM sequence. In the next section, we show that the PTM sequence is in fact the right sequence for specifying the relative signs of the cross-correlation functions between the Golay sequences in each Golay pair.

Remark 3: Representing $\mathbf{X}_0(z)$ and $\mathbf{X}_1(z)$ by 0 and 1 respectively, we notice that the placements of $\mathbf{X}_0(z)$ and $\mathbf{X}_1(z)$ in $\mathbf{X}(z)$ are also determined by the length-4 PTM sequence.

C. The Requirement that $B_1(z)$ Through $B_M(z)$ Vanish

We now consider the general case $N = 2^{M+1}$ where Golay pairs (x_0, x_1), …, (x_{N-2}, x_{N-1}) are used to construct a Doppler resilient waveform matrix $\mathbf{X}(z)$. We have the following theorem.

Theorem 3: Let $N = 2^{M+1}$ and let (x_0, x_1), …, (x_{N-2}, x_{N-1}) be Golay pairs. Then, for any m between 1 and M, $B_m(z)$ will vanish at all delays if

$$X_{2k}(z)\widetilde{X}_{2k+1}(z) = (-1)^{s_k} X_0(z)\widetilde{X}_1(z), \qquad \text{for all } k,\ 0 \le k \le N/2 - 1, \tag{58}$$

where s_k is the kth element in the PTM sequence.

Proof: For any m $(1 \le m \le M)$, $B_m(z)$ may be written as

$$\begin{aligned} B_m(z) &= \sum_{k=0}^{N/2-1} ((2k)^m - (2k+1)^m) X_{2k}(z)\widetilde{X}_{2k+1}(z) \\ &= \left[\sum_{k=0}^{N/2-1} (-1)^{s_k}((2k)^m - (2k+1)^m)\right] X_0(z)\widetilde{X}_1(z), \end{aligned} \tag{59}$$

where the second equality in (59) follows by replacing $X_{2k}(z)\widetilde{X}_{2k+1}(z)$ with $(-1)^{s_k}X_0(z)\widetilde{X}_1(z)$. Since in the PTM sequence $s_k = s_{2k} = \overline{s}_{2k+1}$, we can rewrite (59) as

$$\begin{aligned} B_m(z) &= \left[\sum_{k=0}^{N/2-1}(-1)^{s_{2k}}(2k)^m - (-1)^{\overline{s}_{2k+1}}(2k+1)^m\right] X_0(z)\widetilde{X}_1(z) \\ &= \left[\sum_{k=0}^{N/2-1}(-1)^{s_{2k}}(2k)^m + (-1)^{s_{2k+1}}(2k+1)^m\right] X_0(z)\widetilde{X}_1(z) \\ &= \left[\sum_{k=0}^{N-1}(-1)^{s_k}k^m\right] X_0(z)\widetilde{X}_1(z). \end{aligned} \tag{60}$$

However, from the Prouhet theorem (Theorem 1), it is easy to see that $\sum_{k=0}^{N-1}(-1)^{s_k}k^m = 0$, and therefore $B_m(z) = 0$.□

Finally, we note that it is always possible to find Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ that satisfy the conditions of both Theorem 2 and Theorem 3. Suppose $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ are built from an arbitrary Golay pair (x, y) (as explained in Section III) to satisfy the conditions of Theorem 2. Then, we can apply the time reverse operator and change the sign of the elements within the pairs to satisfy the conditions of Theorem 3, as the Golay property is invariant to time reversal and changes in the signs of the Golay sequences within a pair.

V. Numerical Examples

In this section, we present numerical examples to verify the results of Sections III and IV and compare our Doppler resilient design to a conventional scheme, where the same Golay pair is repeated.

A. Single Channel Radar System

We first consider the case of a single channel radar system. In this case, the composite ambiguity function $G(z, \theta)$ is given by (18) and has a Taylor expansion of the form (19). Following Theorem 2, we coordinate the transmission of eight Golay pairs $(x_0, x_1), \ldots, (x_{14}, x_{15})$ over $N = 16$ PRIs to make the Taylor expansion coefficients $C_1(z), \ldots, C_3(z)$ $(M = 3)$ vanish at all nonzero delays. Starting from a Golay pair (x, y), it is easy to verify that the eight Golay pairs in the following waveform vector $\underline{\mathbf{x}}^T(z)$ satisfy the conditions of Theorem 2:

$$\underline{\mathbf{x}}^T(z) = \left(\underline{\mathbf{x}}_0^T(z) \quad \underline{\mathbf{x}}_1^T(z) \quad \underline{\mathbf{x}}_1^T(z) \quad \underline{\mathbf{x}}_0^T(z) \quad \underline{\mathbf{x}}_1^T(z) \quad \underline{\mathbf{x}}_0^T(z) \quad \underline{\mathbf{x}}_0^T(z) \quad \underline{\mathbf{x}}_1^T(z)\right), \tag{61}$$

where $\underline{\mathbf{x}}_0^T(z) = [X(z) \quad Y(z)]$ and $\underline{\mathbf{x}}_1^T(z) = [-\widetilde{Y}(z) \quad \widetilde{X}(z)]$.

Remark 4: Representing $\underline{\mathbf{x}}_0^T(z)$ and $\underline{\mathbf{x}}_1^T(z)$ by 0 and 1 respectively, we notice that the placements of $\underline{\mathbf{x}}_0^T(z)$ and $\underline{\mathbf{x}}_1^T(z)$ in $\underline{\mathbf{x}}^T(z)$ are determined by the length-8 PTM sequence.

We compare the Doppler resilient transmission scheme in (61) with a conventional transmission scheme, where the same Golay pair ($x_0 = x, x_1 = y$) is transmitted during all PRIs, resulting in a waveform vector $\underline{\mathbf{x}}_c^T(z)$ of the form

$$\underline{\mathbf{x}}_c^T(z) = \left(\underline{\mathbf{x}}_0^T(z) \quad \underline{\mathbf{x}}_0^T(z) \quad \underline{\mathbf{x}}_0^T(z) \quad \underline{\mathbf{x}}_0^T(z) \quad \underline{\mathbf{x}}_0^T(z) \quad \underline{\mathbf{x}}_0^T(z) \quad \underline{\mathbf{x}}_0^T(z) \quad \underline{\mathbf{x}}_0^T(z) \right), \tag{62}$$

with the composite ambiguity function

$$\begin{aligned} G_c(z,\theta) &= \underline{\mathbf{x}}_c^T(z)\mathbf{D}(\theta)\underline{\widetilde{\mathbf{x}}}_c(z) \\ &= \|X_0(z)\|^2 + e^{j\theta}\|X_1(z)\|^2 + \ldots + e^{j(N-2)\theta}\|X_0(z)\|^2 + e^{j(N-1)\theta}\|X_1(z)\|^2. \end{aligned} \tag{63}$$

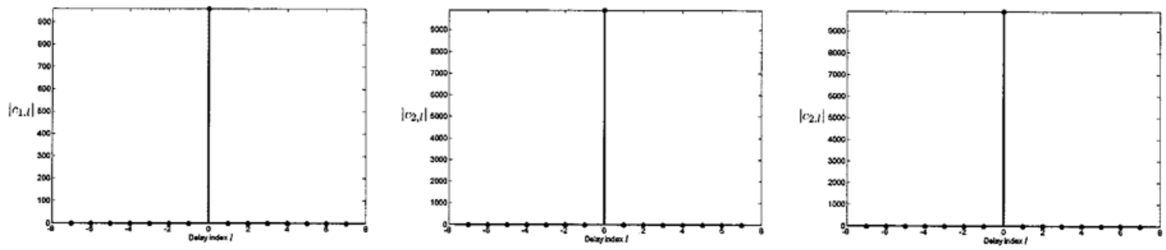
The pair (x, y) used in constructing $\underline{\mathbf{x}}^T(z)$ and $\underline{\mathbf{x}}_c^T(z)$ can be any Golay pair. Here, we choose (x, y) to be the following length-8 ($L = 8$) Golay pair:

$$\begin{aligned} x[l] &= \{1, 1, -1, 1, 1, 1, 1, -1\} &\iff& X(z) = 1 + z^{-1} - z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} - z^{-7} \\ y[l] &= \{-1, -1, 1, -1, 1, 1, 1, -1\} &\iff& Y(z) = -1 - z^{-1} + z^{-2} - z^{-3} + z^{-4} + z^{-5} + z^{-6} - z^{-7} \end{aligned} \tag{64}$$

Referring to the Taylor expansion of $G(z, \theta)$ in (19), the coefficients $C_1(z)$, $C_2(z)$, and $C_3(z)$ are each two-sided polynomials of degree $L - 1 = 7$ in z^{-1} of the form (21). Figures 2(a)-(c) show the plots of the magnitudes of the coefficients $c_{m,l}$, $m = 1, 2, 3$ of these polynomials versus delay index l. The plots show that $C_1(z)$, $C_2(z)$, and $C_3(z)$ indeed vanish at all nonzero delays.

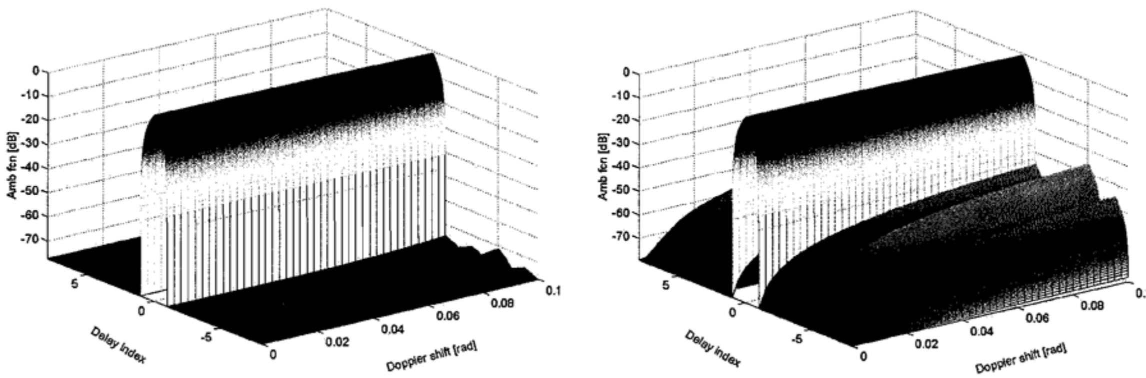
Figures 3(a),(b) show the plots of the composite ambiguity functions $G(z, \theta)$ and $G_c(z, \theta)$ versus delay index l and Doppler shift θ. Comparison of $G(z, \theta)$ and $G_c(z, \theta)$ at Doppler shifts $\theta = 0.025$ rad, $\theta = 0.05$ rad, and $\theta = 0.075$ rad is provided in Figs. 4(a)-(c), where the solid lines correspond to $G(z, \theta)$ (Doppler resilient scheme) and the dashed lines correspond to $G_c(z, \theta)$ (conventional scheme). We notice that the peaks of the range sidelobes of $G(z, \theta)$ are at least 24 dB (for $\theta = 0.025$ rad), 28 dB (for $\theta = 0.05$ rad), and 29 dB (for $\theta = 0.075$ rad) smaller than those of $G_c(z, \theta)$. These plots clearly show the Doppler resilience of the waveform vector in (61).

Remark 5: For a radar with carrier frequency $f_0 = 2.5$ GHz and PRI= 100 μsec, the Doppler shift range of 0 to 0.05 rad (0.075 rad) corresponds to a maximum target speed of $V \approx 35$ kmph (50 kmph). To cover a larger speed range we can use our design with a bank of Doppler filters to provide Doppler resilience within an interval around the Doppler frequency associated with each filter.



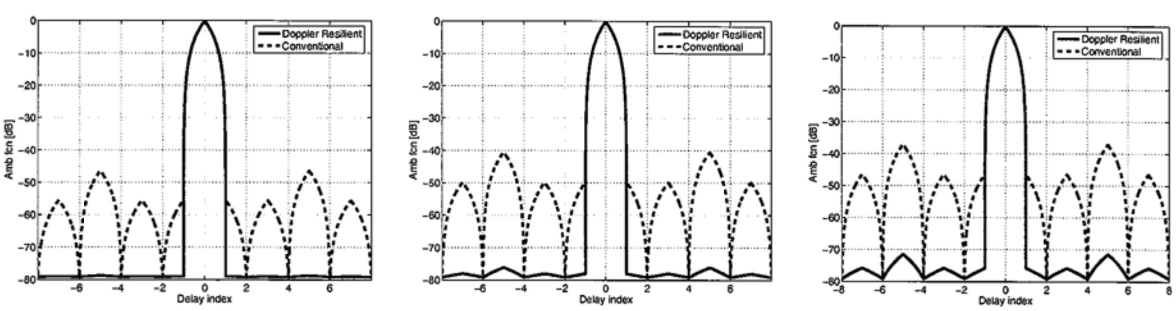
(a) $|c_{1,l}|$ (b) $|c_{2,l}|$ (c) $|c_{3,l}|$

Fig. 2. The plots of the magnitudes of the coefficients $c_{m,l}$ of two-sided polynomials $C_m(z) = \sum_{l=-(L-1)}^{L-1} c_{m,l} z^{-l}$, $m = 1, 2, 3$ versus delay index l: (a) $|c_{1,l}|$, (b) $|c_{2,l}|$, and (c) $|c_{3,l}|$.



(a) $G(z, \theta)$ (b) $G_c(z, \theta)$

Fig. 3. (a) The plot of the composite ambiguity function $G(z, \theta)$ (corresponding to the Doppler resilient transmission scheme) versus delay index l and Doppler shift θ, (b) the plot of the composite ambiguity function $G_c(z, \theta)$ (corresponding to the conventional transmission scheme) versus delay index l and Doppler shift θ.



(a) $\theta = 0.025$ rad (b) $\theta = 0.05$ rad (c) $\theta = 0.075$ rad

Fig. 4. Comparison of the composite ambiguity functions $G(z, \theta)$ and $G_c(z, \theta)$ at Doppler shifts (a) $\theta = 0.025$ rad, (b) $\theta = 0.05$ rad, and (c) $\theta = 0.075$ rad.

B. Fully Polarimetric Radar System

We now consider the matrix-valued composite ambiguity function $\mathbf{G}(z,\theta)$ in (44), corresponding to the fully polarimetric radar system described in Section IV. Following Theorems 2 and 3, we coordinate the transmission of eight Golay pairs $(x_0, x_1), \ldots, (x_{14}, x_{15})$ across vertical and horizontal polarizations and over $N = 16$ PRIs, so that in the Taylor expansions of $G_1(z,\theta)$ (the diagonal element of $\mathbf{G}(z,\theta)$) and $G_2(z,\theta)$ (the off-diagonal element of $\mathbf{G}(z,\theta)$) the coefficients $C_1(z)$, $C_2(z)$, and $C_3(z)$ vanish at all nonzero delays and $B_1(z)$, $B_2(z)$, and $B_3(z)$ vanish at all delays. Letting $\mathbf{X}_0(z)$ and $\mathbf{X}_1(z)$ be the Alamouti matrices in (57), then it is easy to check that the Golay pairs in the following waveform matrix $\mathbf{X}(z)$ satisfy all the conditions of Theorems 2 and 3:

$$\mathbf{X}(z) = \begin{pmatrix}\mathbf{X}_0(z) & \mathbf{X}_1(z) & \mathbf{X}_1(z) & \mathbf{X}_0(z) & \mathbf{X}_1(z) & \mathbf{X}_0(z) & \mathbf{X}_0(z) & \mathbf{X}_1(z)\end{pmatrix}. \tag{65}$$

Remark 6: Representing $\mathbf{X}_0(z)$ and $\mathbf{X}_1(z)$ by 0 and 1 respectively, we notice that the placements of $\mathbf{X}_0(z)$ and $\mathbf{X}_1(z)$ in $\mathbf{X}(z)$ are determined by the length-8 PTM sequence.

We compare the Doppler resilient transmission scheme in (65) with a conventional transmission scheme, where the Alamouti waveform matrix built from a single Golay pair $(x_0 = x, x_1 = y)$ is repeated and the waveform matrix is of the form

$$\mathbf{X}_c(z) = \begin{pmatrix}\mathbf{X}_0(z) & \mathbf{X}_0(z) & \mathbf{X}_0(z) & \mathbf{X}_0(z) & \mathbf{X}_0(z) & \mathbf{X}_0(z) & \mathbf{X}_0(z) & \mathbf{X}_0(z)\end{pmatrix}. \tag{66}$$

The matrix-valued composite ambiguity function of $\mathbf{X}_c(z)$ is given by

$$\mathbf{G}_c(z,\theta) = \mathbf{X}_c(z)\mathbf{D}(\theta)\widetilde{\mathbf{X}}_c(z) = \begin{pmatrix} G_{c1}(z,\theta) & G_{c2}(z,\theta) \\ \widetilde{G}_{c2}(z,\theta) & G_{c1}(z,\theta)\end{pmatrix}, \tag{67}$$

where

$$G_{c1}(z,\theta) = \|X_0(z)\|^2 + e^{j\theta}\|X_1(z)\|^2 + \ldots + e^{j(N-2)\theta}\|X_0(z)\|^2 + e^{j(N-1)\theta}\|X_1(z)\|^2 \tag{68}$$

and

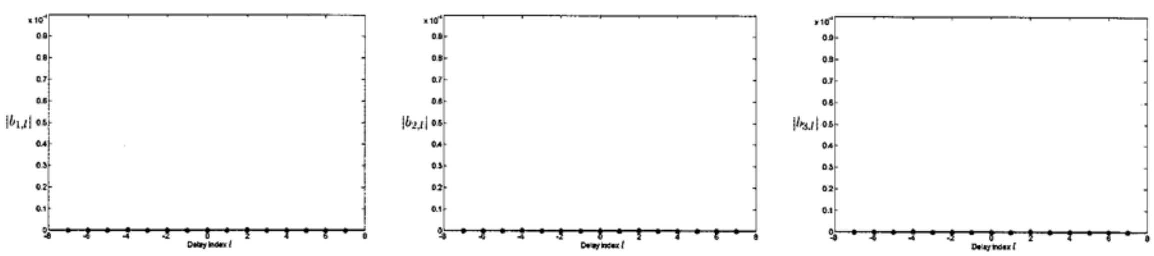
$$G_{c2}(z,\theta) = (1 - e^{j\theta} + \ldots + e^{j(N-2)\theta} - e^{j(N-1)\theta})X_0(z)\widetilde{X}_1(z). \tag{69}$$

The Golay pair (x, y) used in building both $\mathbf{X}(z)$ and $\mathbf{X}_c(z)$ is the length-8 Golay pair in (64).

We notice that the diagonal elements of $\mathbf{G}(z,\theta)$ and $\mathbf{G}_c(z,\theta)$, i.e., $G_1(z,\theta)$ and $G_{c1}(z,\theta)$, are equal to the single channel composite ambiguity functions $G(z,\theta)$ and $G_c(z,\theta)$, respectively. Therefore, the plots in Fig. 2 through Fig. 4 also apply for comparing $G_1(z,\theta)$ and $G_{c1}(z,\theta)$. Thus in this example, we only need to consider the off-diagonal elements of $\mathbf{G}(z,\theta)$ and $\mathbf{G}_c(z,\theta)$, i.e., $G_2(z,\theta)$ and $G_{c2}(z,\theta)$.

Referring to the Taylor expansion of $G_2(z,\theta)$ in (47), the coefficients $B_1(z)$, $B_2(z)$, and $B_3(z)$ are each two-sided polynomials of degree $L-1=7$ in z^{-1} of the form (49). Figures 5(a)-(c) show the plots of the magnitudes of the coefficients $b_{m,l}$, $m=1,2,3$ of these polynomials versus delay index l. We notice that $B_1(z)$, $B_2(z)$, and $B_3(z)$ indeed vanish at all delays.

Figures 6(a),(b) show the plots of the off-diagonal elements $G_2(z,\theta)$ and $G_{c2}(z,\theta)$ versus delay index l and Doppler shift θ. Comparison of $G_2(z,\theta)$ and $G_{c2}(z,\theta)$ at Doppler shifts $\theta=0.025$ rad, $\theta=0.05$ rad, and $\theta=0.075$ rad is provided in Figs. 7(a)-(c), where the solid lines correspond to $G_2(z,\theta)$ (Doppler resilient scheme) and the dashed lines correspond to $G_{c2}(z,\theta)$ (conventional scheme). We notice that the peaks of the range sidelobes of $G_2(z,\theta)$ are at least 24 dB (for $\theta=0.025$ rad), 12 dB (for $\theta=0.05$ rad), and 5 dB (for $\theta=0.075$ rad) smaller than those of $G_{c2}(z,\theta)$. These plots together with the plots in Figs. 4(a)-(c) (corresponding to the diagonal elements of $\mathbf{G}(z,\theta)$ and $\mathbf{G}_c(z,\theta)$) show the Doppler resilience of the waveform matrix in (65).

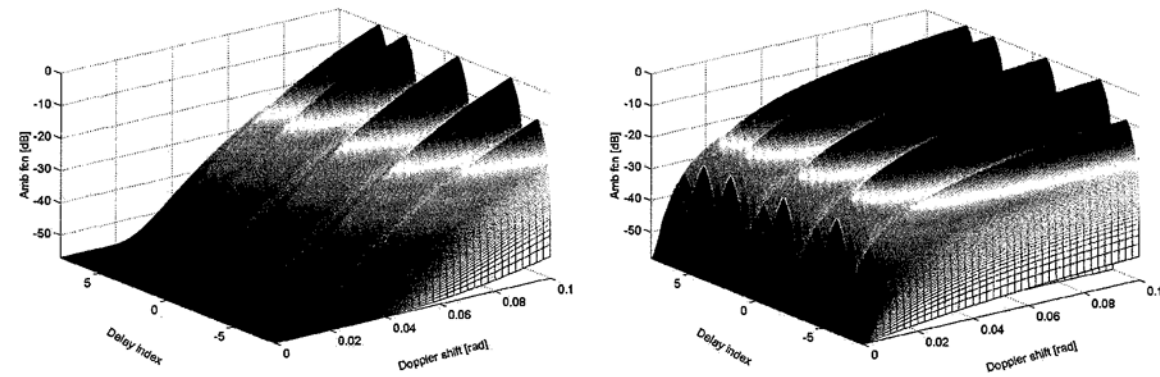


(a) $|b_{1,l}|$ (b) $|b_{2,l}|$ (c) $|b_{3,l}|$

Fig. 5. The plots of the magnitudes of the coefficients $b_{m,l}$ of two-sided polynomials $B_m(z)=\sum_{l=-(L-1)}^{L-1} b_{m,l}z^{-l}$, $m=1,2,3$ versus delay index l: (a) $|b_{1,l}|$, (b) $|b_{2,l}|$, and (c) $|b_{3,l}|$.

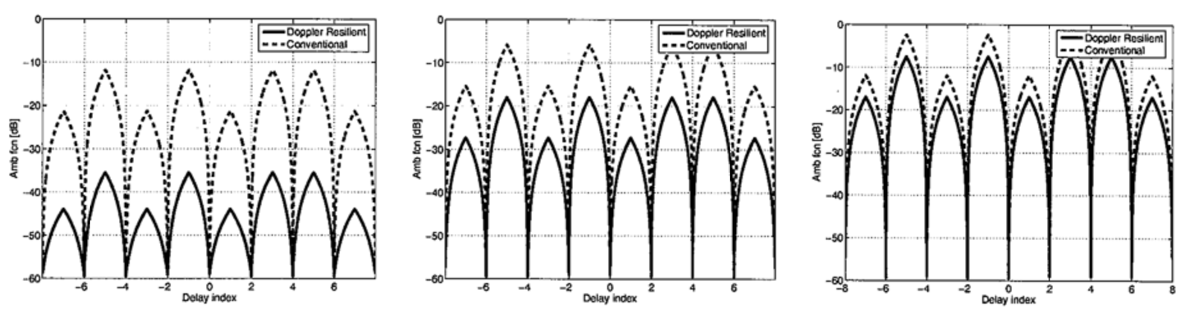
VI. Conclusions

We have constructed a Doppler resilient sequence of Golay complementary waveforms with perfect autocorrelation at small Doppler shifts, and extended our results to the design of Doppler resilient Alamouti waveform matrices of Golay pairs for instantaneous radar polarimetry. The main contribution is a method for selecting Golay complementary sequences to force the low-order terms of the Taylor expansion of a composite ambiguity function (or Doppler modulated autocorrelation sum) to zero. The Prouhet-Thue-Morse sequence was found to be the key to selecting the Doppler resilient Golay pairs. Numerical examples were presented, demonstrating the perfect correlation properties of Doppler resilient Golay pairs at small Doppler shifts.



(a) $G_2(z, \theta)$ (b) $G_{c2}(z, \theta)$

Fig. 6. (a) The plot of the off-diagonal element $G_2(z, \theta)$ (corresponding to the Doppler resilient transmission scheme) versus delay index l and Doppler shift θ, (b) the plot of the off-diagonal element $G_{c2}(z, \theta)$ (corresponding to the conventional transmission scheme) versus delay index l and Doppler shift θ.



(a) $\theta = 0.025$ rad (b) $\theta = 0.05$ rad (c) $\theta = 0.075$ rad

Fig. 7. Comparison of the off-diagonal elements $G_2(z, \theta)$ and $G_{c2}(z, \theta)$ of the matrix-valued ambiguity functions for the conventional and Doppler resilient schemes at Doppler shifts (a) $\theta = 0.025$ rad, (b) $\theta = 0.05$ rad, and (c) $\theta = 0.075$ rad.

Acknowledgments

The authors would like to thank Louis Scharf for his comments on Section II-A.

References

[1] R. J. Turyn, "Ambiguity functions of complementary sequences," *IEEE Trans. Inform. Theory*, vol. 9, pp. 46–47, Jan. 1963.

[2] D. F. Albanese and A. M. Klein, "Pseudorandom code waveform design for CW radar," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-15, no. 1, pp. 67–75, Jan. 1979.

[3] N. Levanon and A. Freedman, "Periodic ambiguity function of CW signals with perfect periodic autocorrelation," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 28, no. 2, pp. 387–394, Apr. 1992.

[4] N. Levanon, "CW alternatives to coherent pulse train-signals and processors," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 29, no. 1, pp. 250–254, Jan. 1993.

[5] N. Levanon and E. Mozeson, *Radar Signals.* New York: Wiley, 2004.

[6] M. J. E. Golay, "Complementary series," *IRE Trans. Inform. Theory*, vol. 7, no. 2, pp. 82–87, April 1961.

[7] R. C. Heimiller, "Phase shift pulse codes with good periodic correlation properties," *IRE Trans. Inform. Theory*, vol. IT-7, no. 4, pp. 254–257, Oct. 1961.

[8] R. L. Frank, S. A. Zadoff, and R. C. Heimiller, "Phase shift pulse codes with good periodic correlation properties," *IRE Trans. Inform. Theory (Correspondence)*, vol. IT-8, no. 6, pp. 381–382, Oct. 1962.

[9] R. L. Frank, "Polyphase codes with good nonperiodic correlation properties," *IEEE Trans. Inform. Theory*, vol. IT-9, no. 1, pp. 43–45, Jan. 1963.

[10] ——, "Polyphase complementary codes," *IEEE Trans. Inform. Theory*, vol. IT-26, no. 6, pp. 641–647, Nov. 1980.

[11] R. Sivaswami, "Multiphase complementary codes," *IEEE Trans. Inform. Theory*, vol. IT-24, no. 3, pp. 546–552, Sept. 1978.

[12] C. C. Tseng and C. L. Liu, "Complementary sets of sequences," *IEEE Trans. Inform. Theory*, vol. IT-18, no. 5, pp. 644–652, Sept. 1972.

[13] G. R. Welti, "Quaternary codes for pulsed radar," *IRE Trans. Inform. Theory*, vol. IT-6, no. 3, pp. 400–408, June 1960.

[14] Y. Taki, M. Miyakawa, M. Hatori, and S. Namba, "Even-shift orthogonal sequences," *IEEE Trans. Inform. Theory*, vol. IT-15, no. 2, pp. 295–300, Mar. 1969.

[15] K. Gerlach and F. F. Kretschmer, "General forms and properties of zero cross-correlation radar waveforms," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 28, no. 1, pp. 98–104, Jan. 1998.

[16] B. L. Lewis and F. F. Kretschmer, "A new class of polyphase pulse compression codes and techniques," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-17, no. 3, pp. 364–372, May 1981.

[17] ——, "Linear frequency modulaton derived polyphase pulse compression codes," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-18, no. 5, pp. 637–641, Sept. 1982.

[18] F. F. Kretschmer and B. L. Lewis, "Doppler properties of polyphase coded pulse-compression waveforms," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-19, no. 4, pp. 521–531, April 1983.

[19] H. Deng, "Polyphase code design for orthogonal netted radar systems," *IEEE Trans. Signal Process.*, vol. 52, no. 11, pp. 3126–3135, Nov. 2004.

[20] H. A. Khan, Y. Zhang, C. Ji, C. J. Stevens, D. J. Edwards, and D. O'Brien, "Optimizing polyphase sequences for orthogonal netted radar," *IEEE Signal Process. Lett.*, vol. 13, no. 10, pp. 589–592, Oct. 2006.

[21] S. D. Howard, A. R. Calderbank, and W. Moran, "A simple polarization diversity technique for radar detection," in *Proc. Second Int. Conf. Waveform Diversity and Design*, HI, Jan. 22-27 2006.

[22] A. R. Calderbank, S. D. Howard, W. Moran, A. Pezeshki, and M. Zoltowski, "Instantaneous radar polarimetry with multiple dually-polarized antennas," in *Conf. Rec. Fortieth Asilomar Conf. Signals, Syst., Comput.*, Pacific Grove, CA, Oct. 29-Nov. 1 2006.

[23] K. Harman and B. Hodgins, "Next generation of GUIDAR technology," *IEEE Aerosp. Electron. Syst. Mag.*, vol. 20, no. 5, pp. 16–26, Mar. 2005.

[24] I. S. Reed and L.-H. Zetterberg, "Communication with orthogonal polyphase signals over a noisy channel with Doppler frequency shift," *IEEE Trans. Commun. Technol.*, vol. 12, no. 4, pp. 116–118, Dec. 1964.

[25] N. Suehiro and M. Hatori, "N-shift cross-orthogonal sequences," *IEEE Trans. Inform. Theory*, vol. IT-34, no. 1, pp. 143–146, Jan. 1988.

[26] T. A. Wilkinson and A. E. Jones, "Minimization of the peak-to-mean envelope power ratio of multicarrier transmission scheme by block coding," in *IEEE Vehicular Technol. Conf. (VTC)*, Chicago, IL, Jul. 1995, pp. 825–829.

[27] R. J. van Nee, "OFDM codes for peak-to-average power reduction and error correction," in *IEEE Global Telecom. Conf. (GLOBECOM)*, London, UK, Nov. 1996, pp. 740–744.

[28] H. Ochiai and H. Imai, "Block coding scheme based on complementary sequences for multicarrier signals," *IEICE Trans. Fundamentals*, vol. E80-A, pp. 2136–2146, 1997.

[29] J. A. Davis and J. Jedwab, "Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed-Muller codes," *IEEE Trans. Inform. Theory*, vol. 45, no. 7, pp. 2397–2417, Nov. 1999.

[30] K. G. Paterson, "Generalized Reed-Muller codes and power control in OFDM modulation," *IEEE Trans. Inform. Theory*, vol. 46, no. 1, pp. 104–120, Jan. 2000.

[31] K. Schmidt, "On cosets of the generalized first-order Reed-Muller code with low PMEPR," *IEEE Trans. Inform. Theory*, vol. 52, no. 7, pp. 3220–3232, June 2006.

[32] K. Schmidt and A. Finger, "Constructions of complementary sequences for power-controlled OFDM transmission," in *Proc. Workshop on Coding and Cryptography (WCC) 2005 (Lecture Notes in Computer Science)*. Berlin, Germany: Springer-Verlag, 2006.

[33] M. Ku and C. Huang, "A complementary code pilot-based transmitter diversity technique for OFDM systems," *IEEE Trans. Wireless Commun.*, vol. 5, no. 3, pp. 504–508, Mar. 2006.

[34] H. H. Chen, J. F. Yeh, and N. Seuhiro, "A multi-carrier CDMA architecture based on orthogonal complementary codes for new generation of wideband wireless communications," *IEEE Commun. Mag.*, vol. 39, no. 10, pp. 126–135, Oct. 2001.

[35] H. H. Chen, H. W. Chiu, and M. Guizani, "Orthogonal complementary codes for interference-free CDMA technologies," *IEEE Wireless Commun. Mag.*, vol. 13, no. 1, pp. 68–79, Feb. 2006.

[36] S. M. Tseng and M. R. Bell, "Asynchronous multicarrier DS-CDMA using mutually orthogonal complementary sets of sequences," *IEEE Trans. Commun.*, vol. 48, no. 1, pp. 53–59, Jan. 2000.

[37] L. Lu and V. K. Dubey, "Extended orthogonal polyphase codes for multicarrier CDMA system," *IEEE Commun. Lett.*, vol. 8, no. 12, pp. 700–702, Dec. 2004.

[38] H. Park and J. Lim, "Cyclic shifted orthogonal complementary codes for multicarrier CDMA systems," *IEEE Commun. Lett.*, vol. 10, no. 6, pp. 1–3, June 2006.

[39] X. Li, "Contention resolution in random-access wireless networks based on orthogonal complementary codes," *IEEE Trans. Commun.*, vol. 52, no. 1, pp. 82–89, Jan. 2004.

[40] P. B. Rapajik and R. A. Kennedy, "Merit factor based comparison of new polyphase sequences," *IEEE Commun. Lett.*, vol. 2, no. 10, pp. 269–270, Oct. 1998.

[41] T. Felhauer, "New class of polyphase pulse compression code with unique characteristics," *Electron. Lett.*, vol. 28, no. 8, pp. 769–771, Apr. 1992.

[42] ——, "Design and analysis of new $P(n, k)$ polyphase pulse compression codes," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-30, no. 3, pp. 865–874, Jul. 1994.

[43] R. Sivaswami, "Self-clutter cancellation and ambiguity properties of subcomplementary sequences," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-18, no. 2, pp. 163–181, Mar. 1982.

[44] J. Guey and M. R. Bell, "Diversity waveform sets for delay-Doppler imaging," *IEEE Trans. Inform. Theory*, vol. 44, no. 4, pp. 1504–1522, Jul. 1998.

[45] E. Prouhet, "Mèmoire sur quelques relations entre les puissances des nombres," *C. R. Acad. Sci. Paris Sèr.*, vol. I 33, p. 225, 1851.

[46] M. Morse, "Recurrent geodesics on a surface of negative curvature," *Trans. Amer. Math Soc.*, vol. 22, pp. 84–100, 1921.

[47] J. P. Allouche and J. Shallit, *Automatic Sequences: Theory, Applications, Generalizations*. Cambridge University Press, 2003.

[48] ——, "The ubiquitous Prouhet-Thue-Morse sequence," in *Sequences and their applications, Proc. SETA'98*, T. H. C. Ding and H. Niederreiter, Eds. Springer Verlag, 1999, pp. 1–16.

[49] D. H. Lahmer, "The Tarry-Escott problem," *Scripta Math*, vol. 13, pp. 37–41, 1947.

[50] S. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. Select. Areas Commun.*, vol. 16, no. 8, pp. 1451–1458, Oct. 1998.