The Concept of Hyperimage in Wide-Band Radar Imaging

J. Bertrand and P. Bertrand

Abstract—**A method is proposed to construct radar (or sonar) hyperimages from the knowledge of a backscattering function. Compared to usual imaging, the new feature is to treat on equal footing all parameters characterizing the image: frequency, directivity and position of the bright points in the two-dimensional static case; frequency, velocity, position, and date of existence in the one-dimensional imaging of nonstationary targets. The main interest of such a formulation is to allow the control of the reciprocal relations that are always present in microwave imaging. This possibility can be useful for processing data obtained by today's wide-band radars.**

I. Introduction

USUAL radar imaging associates distributions of bright points with the electromagnetic response of a target and the result generally takes the form of a bidimensional image with optical features. Due to the wavelengths that are used, the spatial resolution of the scatterers is often lower than in optics, but it can be argued that this negative result can always be compensated by processing a broader band and a larger aperture. However, it must be noticed that such an operation can only be carried out at the expense of information concerning the behavior of the scatterers in frequency and directivity. In fact, there is a tradeoff between the resolutions in the two types of variables, and this suggests the placement of all variables on an equal footing when constructing an image in order to properly describe the reflecting behavior of the target. This step is essential when dealing with data obtained by wide-band and wide-aperture radars. Taking into account the whole set of variables leads, in fact, to giving the concept of hyperimage a central role in the inverse problem under consideration. In the static case, for example, such a formulation consists in describing the target in terms of points labeled by frequency and directivity as well as space variables. The resulting representation can effectively be considered as a generalization since the classical images can be recovered as special sections of the hyperimage. An illustration of the practical interest of this approach has previously been given in the case of monostatic bidimensional radar imaging [1]. The object of the present study is to apply the method in other situations and, thus, illustrate its value in general coherent imaging.

The analytical expression of the hyperimages is partly determined by the physical principle of invariance under changes of the reference system: all imaging techniques have to be equivalently formulated whatever the system used. In particular, two images of the same target, obtained with two distinct reference systems, must be related by a well-defined geometric transformation. To take advantage of this fact, the first task in any situation is to identify the relevant group of transformations associated with the changes of reference systems and to write down their effects on the microwave response of the target. This allows one to establish a natural constraint on the imaging procedure by requiring compatibility between the transformation of the observed data and the transformation of the constructed hyperimages. However, this condition is not sufficient to determine the exact form of the hyperimages and specifications relative to their interpretation have to be added. In the following, a general procedure is given which is founded mainly on the use of a continuous wavelet analysis, and for which hyperimages are always positive [2].

In Section II, the notion of hyperimage is exemplified in the special case of one-dimensional (1-D) radar imaging. The adapted wavelet analysis is described and shown to lead to an image given as a distance-frequency representation of the target. In other sections, more complex situations are handled from the same point of view. In Section III, two-dimensional (2-D) static radar targets are considered and hyperimages are constructed from bistatic data. The case of nonstatic targets is taken up in Section IV where the technique is applied with elementary scatterers which may be in motion and ephemeral.

II. Distance-Frequency Diagram as Example of Hyperimage

The essential aspects of the technique are now described by applying it to the particular case of 1-D static target imaging. We suppose that a transmitter sends out an acoustic or electromagnetic wave on a target and that a receiver, located at the same spot, records the scattered wave. The observation is supposed to take place in the far-field region and possible polarizations are fixed at emission and reception so that the field can be considered as scalar. The numerical outcome of the experiment consists of the values of a complex backscattering function $H(f)$ for frequencies in a band that may be arbitrarily large. Under the hypothesis that the target is made of independent point reflectors, the problem of its description will be solved by associating with H, a function $I(x, f)$ that represents a repartition of bright points located at x and reflecting at frequency f.

The form of the correspondence between the response $H(f)$ and the hyperimage $I(x, f)$ has to be independent of the choice of reference system in which it is set up. To exploit

Manuscript received October 27, 1995; revised April 21, 1996.
J. Bertrand is with LPTM, University Paris VII, 75251 Paris, France.
P. Bertrand is with ONERA/DES, 92322 Chatillon, France.
Publisher Item Identifier S 0196-2892(96)06824-6.

this constraint, the first task is to identify the transformations connecting the possible reference frames. These transformations consist of a change of origin and a change of units of length and time. Their effect on the coordinates is expressed by the translation b and the dilation $a > 0$ on x. The same dilation will also be applied to the time in order to preserve the constancy of the light velocity c. The corresponding transformation law of the coefficient $H(f)$ is found to be

$$(a,b): \quad H(f) \longrightarrow H_{(a,b)}(f) \equiv a\, e^{-4i\pi b(f/c)}\, H(af). \tag{1}$$

The factor a ensures that H transforms as a length in a dilation. A further change of frame defined by (a', b') will transform $H_{(a,b)}$ into the function given by

$$a\, e^{-4i\pi b'(f/c)}\, H_{(a,b)}(a'f) = aa'\, e^{-4i\pi(b'+a'b)(f/c)}\, H(a'af). \tag{2}$$

Thus, two successive transformations (a, b), (a', b') act as a single one defined by $(a'a, b' + a'b)$. Their composition law, written symbolically as

$$(a', b')(a, b) = (a'a, b' + a'b) \tag{3}$$

is in fact a group law and the set of transformations (a, b), $a > 0, b$ real, is called the affine group. The group structure ensures that no reference frame is privileged and that the transformations connecting any two of them depend only on those frames characteristics.

It results from (1) that the scalar product of scattering functions defined by

$$(H, H') \equiv \int_0^{\infty} H(f)\, H'^{*}(f)\, f\; df \tag{4}$$

is invariant under the group action. Namely, we have

$$\begin{aligned}(H_{(a,b)}, H'_{(a,b)}) &= a^2 \int_0^{\infty} H(af)\, H'^{*}(af)\, f\; df \\ &= (H, H').\end{aligned}$$

The associated norm of a backscattering function given by

$$\|H\|^2 \equiv (H, H) \tag{5}$$

is thus also invariant and transformation (1) constitutes a unitary representation of the affine group with respect to that norm.

The action of a change of reference frame defined by (a, b) on the image coordinates (x, f) is

$$(a,b): \quad x \longrightarrow ax + b, \qquad f \longrightarrow a^{-1} f. \tag{6}$$

To obtain a description of the target that could be considered as universal, we require the image function to transform as

$$I(x, f) \longrightarrow a\, I(a^{-1}(x - b), af) \tag{7}$$

in the same change of reference frame. The factor a in this expression ensures that the quantity I behaves as a length. This is necessary for the subsequent interpretation of I as a space distribution of radar cross section. In that way, the integral $\int I(x, f)\; dx$ on any interval yields a quantity which transforms as a surface, exactly like $|H(f)|^2$ [3].

The compatibility of transformations (1) and (7) is expressed by the *covariance requirement* of the theory. This constraint is analytically represented by the commutative diagram

$$\begin{array}{ccc} H(f) & \longrightarrow & a\, e^{-4i\pi b(f/c)}\, H(af) \\ \downarrow & & \downarrow \\ I(x,f) & \longrightarrow & a\, I(a^{-1}(x-b), af) \end{array} \tag{8}$$

where the horizontal arrows represent a change of reference frame and the vertical arrows symbolize the imaging procedure. To go a step further in the definition of $I(x, f)$, we now turn to wavelet analysis [4].

Ideally, one would like to decompose $H(f)$ into elementary coefficients each pertaining to a reflector that is perfectly localized in space (x, f). But, this is impossible because of the reciprocity relations that prevent the variables x and f from being simultaneously defined with arbitrary precision. Instead, it will be shown how to construct a reference family of backscattering functions which correspond to elementary reflectors represented by "blobs" in image space. In practice, a *basic function* $\phi(f)$ is chosen and associated with a reflector located *about* the point $x_0 = 0$ and reflecting *mainly* frequency $f_0 = 1$. Point(x_0, f_0) can be moved to any arbitrary point (x, f) by a transformation (a, b) acting as follows:

$$x = ax_0 + b, \qquad f = a^{-1} f_0 \tag{9}$$

i.e., since $(x_0, f_0) = (0, 1)$

$$x = b, \qquad f = a^{-1}. \tag{10}$$

In this manner, the image parameters are in one-to-one correspondence with the elements of the affine group. Applying transformation (1) to function ϕ with the values $a = f^{-1}$ and $b = x$, we obtain a family $\{\phi_{xf}\}$ of so-called *wavelets* defined by

$$\phi_{xf}(f') = \frac{1}{f}\, e^{-4i\pi f'(x/c)}\, \phi\!\left(\frac{f'}{f}\right). \tag{11}$$

Each wavelet ϕ_{xf} is uniquely associated with a point (x, f) of the image space. It can be interpreted as the backscattering function of an elementary reflector having a small extension around x and reflecting mainly in the vicinity of frequency f. Remark that this *system of wavelets* is common to all observers in that a change of reference frame performs only a relabeling of the elements. A development of the backscattering function $H(f)$ on this set of functions is possible. The coefficients of the decomposition are defined by

$$C(x, f) = (H, \phi_{xf}) \tag{12}$$

or, explicitly (cf (4)) as

$$C(x, f) = \int_0^{\infty} H(f')\, \frac{1}{f}\, e^{4i\pi f'(x/c)}\, \phi^{*}\!\left(\frac{f'}{f}\right) f'\; df'. \tag{13}$$

This coefficient is a function of the variables x and f and a straightforward computation yields the following *isometry relation*

$$\int_{\mathbb{R}\times\mathbb{R}^{+}} |C(x, f)|^2\; dx df = (c/2)\kappa_{\phi}\, \|H\|^2 \tag{14}$$

where κ_ϕ is a constant depending only on the basic function ϕ and given by

$$\kappa_\phi \equiv \int_0^\infty |\phi(y)|^2 \, dy. \tag{15}$$

In practice, H can always be normalized so that the norm $\|H\|$ defined by (5) is equal to one. If the function ϕ is such that $\kappa_\phi < \infty$, a reconstruction formula does exist and expresses the function H in terms of the coefficients C as follows:

$$H(f) = \frac{2}{c\kappa_\phi} \int_{-\infty}^{\infty} \int_0^\infty C(x', f') \, \frac{1}{f'} \, e^{-4i\pi f(x'/c)} \, \phi(f/f') \, dx' df'. \tag{16}$$

Property (14) is essential as it allows to regard $(2/c\kappa_\phi)|C(x,f)|^2$ as a probability density on space (x, f). This probability gives a dimensionless representation of the target. To obtain an expression which transforms in accordance with (7) and (8), we are led to attribute a physical dimension to the pixels and define the *hyperimage* by

$$I(x, f) = \frac{2}{c\kappa_\phi} |C(x, f)|^2 \, f^{-1}. \tag{17}$$

In this way, the integral of the function $I(x, f)$ over the space variable x transforms as a surface and can be interpreted as a smoothed form of the cross section $|H(f)|^2$. The nature of the smoothing depends on the choice of the function ϕ. It is given explicitly by

$$\int_{-\infty}^{\infty} I(x, f) \, dx = \frac{1}{\kappa_\phi f^3} \int_0^\infty |H(f')|^2 \, |\phi(f'/f)|^2 \, f'^2 \, df'. \tag{18}$$

This introduces the problem of the choice of ϕ. In fact, there is a tradeoff between sharp smoothing and precise space localization.

Studies relative to the time-frequency representation of signals [5] have underscored the interest of basic functions of the form [6]

$$\phi_\lambda(f) = K(\lambda) \, f^{2\pi\lambda - 1} \, e^{-2\pi\lambda f} \tag{19}$$

where $K(\lambda)$ is a normalization constant. The function is best localized either in x or in f according to the values of the parameter λ. Fig. 1 shows the graph of ϕ_λ in x-space for different values of λ. Notice the increasing number of oscillations of the function as the parameter λ becomes larger. Any function ϕ_λ can be used to construct a wavelet set provided expression (15) is finite, i.e., $\lambda > (1/2\pi)$. Actually, since the backscattering functions are always band-limited, it is possible to let this limitation aside by introducing an appropriate cutoff for small values of the variable f.

III. Extension to Bidimensional Static Situations

The above method of adapted wavelet analysis is, in fact, general and allows one to tackle different imaging situations. It has been applied in particular to the construction of four-dimensional (4-D) radar hyperimages from a monostatic scattering coefficient $H(f, \theta)$ obtained in the laboratory [1]. It will now be used in the reformulation of bistatic radar imaging [7] [8].

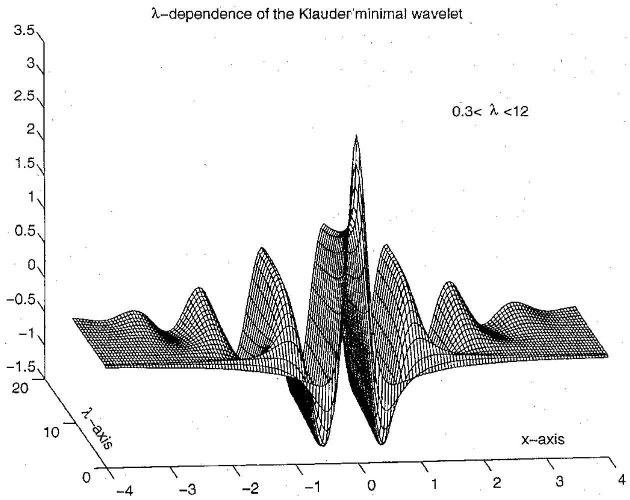


Fig. 1. Graph displaying the λ-dependence of the Klauder minimal wavelet in x-space.

The measures are performed in the far-field region and the polarizations are fixed at emission and reception so that the electromagnetic field can be considered a scalar. The data will consist of the scattering coefficient $H(\boldsymbol{k}_i, \boldsymbol{k}_r)$ obtained as a function of incident and reflected wave vectors $\boldsymbol{k}_i \equiv (k, \theta_i)$ and $\boldsymbol{k}_r \equiv (k, \theta_r)$. The frequency $f \equiv ck/2$ is not changed since the target is at rest. The angles (θ_i, θ_r) are measured from the x-axis as shown on Fig. 2. It is convenient to define the bistatic angle β as

$$\beta \equiv \theta_i - \theta_r \tag{20}$$

and the mid-angle θ

$$\theta \equiv (1/2)(\theta_i + \theta_r). \tag{21}$$

We will consider separately the two situations where either the bistatic or the incident angle is fixed [9]. In the case of constant bistatic angle, the treatment parallels that of the monostatic case [1].

A. Constant Bistatic Angle

In that case, the changes of reference systems consist of changes of origin, units, and orientation of the axes. On space coordinates $\boldsymbol{x}$, they induce translations by $\boldsymbol{b}$, dilations by $a > 0$ and rotations by an angle ϕ. The coordinates $\boldsymbol{x}'$ in the new reference frame are given in terms of $\boldsymbol{x}$ by

$$\boldsymbol{x}' = aR_\phi \boldsymbol{x} + \boldsymbol{b} \tag{22}$$

where R_ϕ denotes the action of the ϕ-rotation explicitly written as

$$R_\phi \boldsymbol{x} = (x_1 \cos\phi - x_2 \sin\phi, x_1 \sin\phi + x_2 \cos\phi). \tag{23}$$

When two successive transformations of type (22) are performed on the coordinates, the result is equivalent to a single transformation written symbolically as

$$(a, \phi, \boldsymbol{b})(a', \phi', \boldsymbol{b}') = (aa', \phi + \phi', \boldsymbol{b} + aR_\phi \boldsymbol{b}'). \tag{24}$$

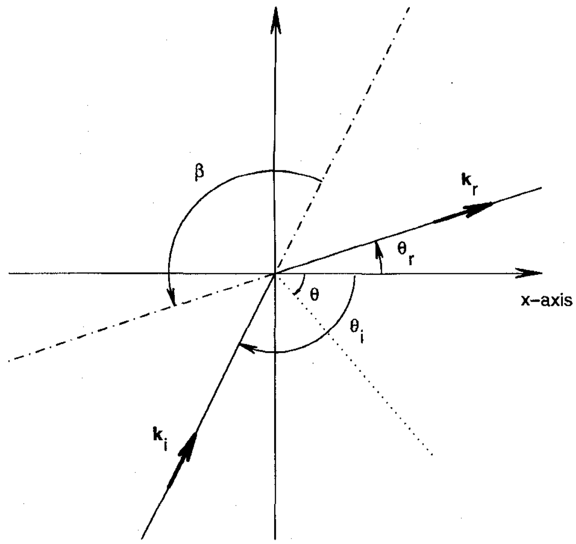


Fig. 2. Definition of parameters in a bistatic setting.

In the transformation defined by $(a, \phi, \boldsymbol{b})$, $(\boldsymbol{k}_i, \boldsymbol{k}_r)$ will go to $(\boldsymbol{k}'_i, \boldsymbol{k}'_r)$ defined by

$$\boldsymbol{k}'_i = (a^{-1}k, \theta_i + \phi), \quad \boldsymbol{k}'_r = (a^{-1}k, \theta_r + \phi). \tag{25}$$

The set of all those transformations labeled by $(a, \phi, \boldsymbol{b})$ forms the similarity group, which is identical to the group occurring in the monostatic case. The difference with that case will show when considering the mode of action of the group on the data.

The scattering coefficient will be written in the form $H(k, \theta; \beta_0)$ where the bistatic angle β_0 is a constant. A change of reference frame characterized by $(a, \phi, \boldsymbol{b})$ induces a transformation on H which is found to be

$$\begin{aligned}(a, \phi, \boldsymbol{b}) :\ H(k, \theta; \beta_0) \quad &\longrightarrow \\ H'(k, \theta; \beta_0) &= a\, e^{i\pi(\boldsymbol{k}_i - \boldsymbol{k}_r)\cdot \boldsymbol{b}}\, H(ak, \theta - \phi; \beta_0) \\ &= a e^{2i\pi k|\cos(\beta_0/2)|\,\boldsymbol{u}\cdot\boldsymbol{b}} H(ak, \theta - \phi; \beta_0)\end{aligned} \tag{26}$$

where $\boldsymbol{u} \equiv (\cos\theta, \sin\theta)$ is the unit vector in the θ direction. When $\beta_0 = 0$, this transformation is the same as in the monostatic case. The factor a in front of (26) ensures that the radar cross section given by $|H|^2$ transforms as a surface. The invariant norm of H is given by

$$\| H \|^2_{\beta_0} \equiv \int_0^\infty \int_0^{2\pi} |H(k, \theta; \beta_0)|^2\, k\, dk d\theta. \tag{27}$$

The description of the target will be performed in terms of independent scatterers that are able to have a directivity and a frequency dependence in addition to their position $\boldsymbol{x}$. This is necessary to handle at once the bulk of data coming from the values of H on a wide band and angle. Thus, the hyperimage will be a function I of parameters $(\boldsymbol{x}, k, \theta)$ at the fixed biscattering angle β_0. The action of a change of reference system on these parameters is obtained from (22), (25) and the image must therefore transform as

$$\begin{aligned}&(a, \phi, \boldsymbol{b}) :\ I(\boldsymbol{x}, k, \theta; \beta_0) \longrightarrow \\ &I'(\boldsymbol{x}, k, \theta; \beta_0) = I(a^{-1}R_{-\phi}(\boldsymbol{x} - \boldsymbol{b}), ak, \theta - \phi; \beta_0).\end{aligned} \tag{28}$$

To be insensitive to the choice of frame, the construction of the hyperimage I in terms of H must satisfy a consistency requirement expressed by the commutativity of the following diagram

$$\begin{array}{ccc} H(k, \theta; \beta_0) & \longrightarrow & H'(k, \theta; \beta_0) \\ \downarrow & & \downarrow \\ I(\boldsymbol{x}, k, \theta; \beta_0) & \longrightarrow & I'(\boldsymbol{x}, k, \theta; \beta_0) \end{array} \tag{29}$$

where H' and I' are given by (26) and (28), respectively, and the vertical arrows represent the imaging construction. In addition, the norm of H must be recovered from the image according to

$$\int I(\boldsymbol{x}, k, \theta; \beta_0) d\boldsymbol{x}\, k dk d\theta = \| H \|^2_{\beta_0}. \tag{30}$$

The hyperimage I is constructed explicitly using a wavelet analysis adapted to the representation (26) of the similarity group transformations. First, we choose a basic function $\Phi(k, \theta; \beta_0)$ that may be associated with the scattering coefficient of a target located around $\boldsymbol{x} = 0$, reflecting mainly at a frequency corresponding to $k = 2f/c = 1$ and in the direction $\theta_r = \beta_0/2$ when illuminated at the same frequency in direction $\theta_i = -\beta_0/2$. Function Φ is naturally assigned to the image point $(\boldsymbol{x}, k, \theta) = (0, 1, 0)$. In a change of reference system labeled by $(a, \phi, \boldsymbol{b})$, point $(\boldsymbol{x}, k, \theta) = (0, 1, 0)$ goes over to $(\boldsymbol{x}, k, \theta) = (\boldsymbol{b}, a^{-1}, \phi)$. The transformed function Φ' obtained from (26) is a wavelet that is associated with the transformed image point and will be denoted by $\Phi_{\boldsymbol{x},\boldsymbol{k}}$. It is given explicitly by

$$\Phi_{\boldsymbol{x},\boldsymbol{k}}(k', \theta'; \beta_0) = (1/k) e^{-2i\pi|\cos(\beta_0/2)|\boldsymbol{k}'\cdot\boldsymbol{x}}\, \Phi(k'/k, \theta' - \theta; \beta_0) \tag{31}$$

where $\boldsymbol{k} = (k, \theta)$ and $\boldsymbol{k}' = (k', \theta')$. Thus a basis of scattering coefficients has been constructed, each of which is associated with an image point. It is the same for all observers and a change of reference system will only lead to a relabeling of its elements. The wavelet coefficient of a scattering function $H(k, \theta; \beta_0)$ on this basis is defined by

$$\begin{aligned}C(\boldsymbol{x}, \boldsymbol{k}; \beta_0) = &\int_0^\infty \int_0^{2\pi} H(k', \theta'; \beta_0) \\ &e^{2i\pi|\cos(\beta_0/2)|\boldsymbol{k}'\cdot\boldsymbol{x}}\Phi^*(k'/k, \theta' - \theta; \beta_0)\frac{k'}{k}\, dk' d\theta'.\end{aligned} \tag{32}$$

This coefficient satisfies the isometry relation given by

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} |C(\boldsymbol{x}, \boldsymbol{k}; \beta_0)|^2\, d\boldsymbol{x} d\boldsymbol{k} = \frac{\kappa_\Phi}{\cos^2(\beta_0/2)} \| H \|^2_{\beta_0} \tag{33}$$

where the constant κ_Φ is defined by

$$\kappa_\Phi \equiv \int_0^\infty \int_0^{2\pi} |\Phi(k, \theta; \beta_0)|^2\, \frac{dk}{k} d\theta. \tag{34}$$

If the constant κ_Φ is finite, the basic function Φ is said to be admissible and a reconstruction formula for the function H can be established. In that case, the hyperimage is defined by

$$I(\boldsymbol{x}, \boldsymbol{k}; \beta_0) = \kappa_\Phi^{-1} \cos^2(\beta_0/2) |C(\boldsymbol{x}, \boldsymbol{k}; \beta_0)|^2. \tag{35}$$

According to (33), this expression satisfies (30). Moreover, integrated over $\boldsymbol{x}$, the hyperimage $I(\boldsymbol{x}, \boldsymbol{k}; \beta_0)$ yields a smoothing of $|H|^2$ given explicitly by

$$\int_{\mathbb{R}} I(\boldsymbol{x}, \boldsymbol{k}; \beta_0)\, d\boldsymbol{x} = \kappa_\Phi^{-1} \int_0^\infty \int_0^{2\pi} |H(k', \theta'; \beta_0)|^2 |\Phi(k'/k, \theta' - \theta)|^2 \frac{k'}{k^2} dk' d\theta' \quad (36)$$

B. Constant Incident Angle

In that case, a fixed direction of reference does exist and the only admissible changes of reference systems consist in translations by the vector $\boldsymbol{b}$ and dilations by the positive number a. This is a subset, in fact, a subgroup, of the transformations considered in Section III-A.

The scattering coefficient is now written in terms of the constant incidence angle θ_i and of the bistatic angle β (cf. Fig. 2) and denoted by $H(k, \beta; \theta_i)$. Its transformation law under a change of frame characterized by $(a, \boldsymbol{b})$ is obtained from (26), setting $\phi = 0$ and performing the adequate change of variables. The result is

$$(a, \boldsymbol{b}) : \ H(k, \beta; \theta_i) \longrightarrow H'(k, \beta; \theta_i) = a\, e^{-2i\pi k \cos(\beta/2) \boldsymbol{v} \cdot \boldsymbol{b}} H(ak, \beta; \theta_i) \quad (37)$$

where $\boldsymbol{v}$ is the unit vector defined by

$$\boldsymbol{v} = (\cos(\beta/2 + \theta_i), \sin(\beta/2 + \theta_i)). \quad (38)$$

The invariant norm is

$$\| H \|_{\theta_i}^2 = \int_0^\infty \int_0^{2\pi} |H(k, \beta; \theta_i)|^2\, k\, dk d\beta. \quad (39)$$

The parameters describing the elementary scatterers must be the position $\boldsymbol{x}$, wavenumber k and directivity β. However, not all of them can be attained by a wavelet analysis associated with transformation (37) and we have to proceed in two steps. First, we set up the wavelet analysis for a given bistatic angle $\beta = \beta_0$. Choose a function $\Phi_{\beta_0}(k, \beta; \theta_i)$ and associate it with the image point $(\boldsymbol{x} = 0, k = 1, \beta_0)$. A transformation labeled by $(a, \boldsymbol{b})$ moves the point to $(\boldsymbol{x} = \boldsymbol{b}, k = a^{-1}, \beta_0)$ and the function Φ_{β_0} to $\Phi_{\boldsymbol{x},k,\beta_0}$ defined by

$$\Phi_{\boldsymbol{x},k,\beta_0}(k', \beta'; \theta_i) = (1/k) e^{-2i\pi |\cos(\beta'/2)| k' \boldsymbol{v}' \cdot \boldsymbol{x}}\, \Phi_{\beta_0}(k'/k, \beta'; \theta_i). \quad (40)$$

A wavelet coefficient could be defined as usual but it would fail to satisfy an isometry relation. So, we proceed with the second step which sets a way of linking together the wavelets $\Phi_{\boldsymbol{x},k,\beta_0}$ for all possible values of β_0. A natural choice is to define

$$\Phi_{\beta_0}(k, \beta; \theta_i) \equiv \Phi(k, \beta - \beta_0; \theta_i) \quad (41)$$

and to associate function Φ with the image point $(\boldsymbol{x} = 0, k = 1, \beta_0 = 0)$. A generalized wavelet coefficient is then defined by

$$C(\boldsymbol{x}, k, \beta_0; \theta_i) = \int_0^\infty \int_0^{2\pi} H(k', \beta'; \theta_i) e^{2i\pi |\cos(\beta'/2)| k' \boldsymbol{v}' \cdot \boldsymbol{x}} \times \Phi(k'/k, \beta' - \beta_0, \theta_i) |\cos(\beta'/2)| (k'/k)\, dk' d\beta' \quad (42)$$

where $\boldsymbol{v}' = (\cos(\beta'/2 + \theta_i), \sin(\beta'/2 + \theta_i))$. It has been tailored to lead to an isometry relation that reads

$$\int_{\mathbb{R}^2} \int_0^\infty \int_0^{2\pi} |C(\boldsymbol{x}, k, \beta_0; \theta_i)|^2\, d\boldsymbol{x} k dk d\beta_0 = \kappa_\Phi \, \| H \|_{\theta_i}^2 \quad (43)$$

where

$$\kappa_\Phi \equiv \int_0^\infty \int_0^{2\pi} |\Phi(k, \beta; \theta_i)|^2 \frac{dk}{k} d\beta. \quad (44)$$

With this definition, a reconstruction formula for H can be written down in terms of the coefficients C provided $\kappa_\Phi < \infty$ and the hyperimage is defined by

$$I(\boldsymbol{x}, k, \beta; \theta_i) = \kappa_\Phi^{-1} |C(\boldsymbol{x}, k, \beta; \theta_i)|^2. \quad (45)$$

IV. Imaging of Nonstationary Targets

This section concerns composite targets whose parts may be moving relative to each other and scintillating. In the approximation of high radar frequency and low velocity of scatterers, a range-Doppler analysis of such a target has been proposed in [10] and [11]. But a complete description, valid for any frequency band and any velocity, requires the introduction of hyperimages. Their construction by the wavelet method will now be given [12].

In a 1-D situation, suppose a radar experiment has been performed and the backscattering coefficient $H(f_1, f_2)$ has been obtained as a function of incident (f_1) and returning (f_2) frequencies. Applying the above ideas, we find that the group $\mathcal{W}$ of transformations linking two observers using coordinates (x, t) and (x', t') is made of space-time translations (ξ, τ), dilations α and Lorentz boosts characterized by a velocity ν. Group $\mathcal{W}$ is usually referred to as the Weyl–Poincaré group. Explicitly, the effect on coordinates (x, t) of a change of reference system characterized by $g \equiv (\xi, \tau, \alpha, \nu)$ may be written as

$$\boldsymbol{x}' = \Gamma(\alpha, \nu)\, \boldsymbol{x} + \boldsymbol{\xi} \quad (46)$$

where

$$\boldsymbol{x} \equiv \begin{pmatrix} x \\ ct \end{pmatrix}, \qquad \boldsymbol{\xi} \equiv \begin{pmatrix} \xi \\ c\tau \end{pmatrix} \quad (47)$$

$$\Gamma(\alpha, \nu) \equiv \frac{\alpha}{\sqrt{1 - (\nu/c)^2}} \begin{pmatrix} 1 & -\nu/c \\ -\nu/c & 1 \end{pmatrix} \quad (48)$$

with c the velocity of light.

Computation of the transformation law of the function $H(f_1, f_2)$ in such a change of reference system gives

$$H(f_1, f_2) \longrightarrow H'(f_1, f_2) = \alpha e^{-2i\pi[(f_2 - f_1)\tau + (f_2 + f_1)(\xi/c)]} \times H\left(\alpha \frac{1 + (\nu/c)}{\sqrt{1 - (\nu/c)^2}} f_1, \alpha \frac{1 - (\nu/c)}{\sqrt{1 - (\nu/c)^2}} f_2\right). \quad (49)$$

This transformation preserves the natural scalar product

$$(H, H') = \int H(f_1, f_2) H'^*(f_1, f_2)\, df_1 df_2. \quad (50)$$

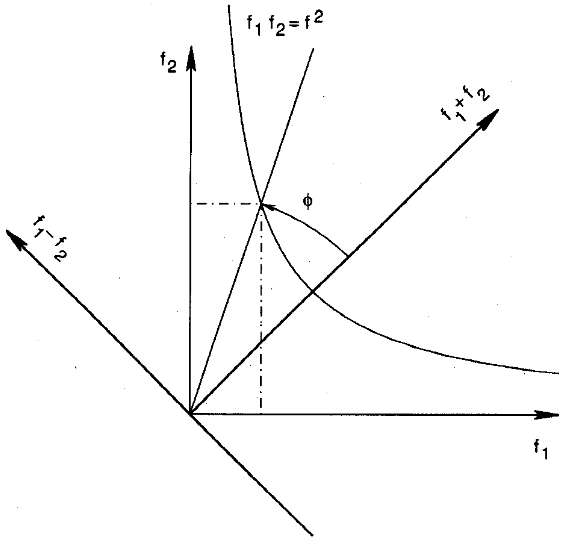


Fig. 3. Geometrical interpretation of the change of variables (51). Angle ϕ is such that $\tan\phi \equiv \frac{v}{c}$.

The target is described in terms of elementary scatterers having an instant of existence t and a velocity v in addition to their position x and frequency f. Variables (v, f) are connected to the transmitted and reflected radar frequencies (f_1, f_2) by the classical formulas

$$f = \sqrt{f_1 f_2}, \quad v/c = \frac{f_1 - f_2}{f_1 + f_2} \tag{51}$$

which imply $f > 0, |v| < c$. The relation between the two types of variables is illustrated in Fig. 3. In fact, f represents the frequency observed in the scatterer referential and v is the radar Doppler velocity associated with frequencies f_1, f_2. The transformation law of variable v in a boost characterized by a velocity ν is given by the relativistic composition of velocities while the frequency f is changed only by a dilation α. Thus, a general change of reference system characterized by (ξ, τ, ν, α) acts upon (v, f) as follows:

$$v \longrightarrow v' = \frac{v - \nu}{1 - (v\nu/c^2)} \qquad f \longrightarrow f' = \alpha^{-1} f. \tag{52}$$

The hyperimage is a function I of coordinates (x, t, v, f) transforming as

$$I(\boldsymbol{x}, v, f) \xrightarrow{g} I_g(\boldsymbol{x}, v, f) \equiv I\Big(\Gamma^{-1}(\alpha, \nu)(\boldsymbol{x} - \boldsymbol{\xi}), \frac{v + \nu}{1 + v\nu/c^2}, \alpha f\Big). \tag{53}$$

In fact, all subsequent computations are greatly simplified by noticing that the group $\mathcal{W}$ is isomorphic to the direct product of two affine groups $A = \{(a, b)\}$ of the type described in Section II.

To perform the wavelet construction of the hyperimage I, choose a function $\phi(f_1, f_2)$ corresponding to the response of a target labeled by the point $(x_0 = t_0 = v_0 = 0, f_0 = 1)$. In a change of reference system labeled by (ξ, τ, ν, α), point $(x_0, t_0, v_0, f_0) = (0, 0, 0, 1)$ goes to $(x, t, v, f) = (\xi, \tau, -\nu, \alpha^{-1})$. The transformed function ϕ' obtained from ϕ according to (49) will be associated with point (x, t, v, f) and denoted by ϕ_{xtvf}. It is given by

$$\phi_{x,t,v,f}(f_1, f_2) = f^{-1} e^{-2i\pi[(f_1+f_2)x/c - (f_1 - f_2)t]} \times \phi\Big(\frac{f_1}{f}\frac{1 - v/c}{\sqrt{1 - v^2/c^2}}, \frac{f_2}{f}\frac{1 + v/c}{\sqrt{1 - v^2/c^2}}\Big). \tag{54}$$

The wavelet coefficient of $H(f_1, f_2)$ on this wavelet basis is defined by

$$C(x, t, v, f) = f^{-1} \int_{\mathbb{R}^2} H(f_1, f_2) e^{2i\pi[(f_1+f_2)x/c - (f_1 - f_2)t]} \times \phi^*\Big(\frac{f_1}{f}\frac{1 - v/c}{\sqrt{1 - v^2/c^2}}, \frac{f_2}{f}\frac{1 + v/c}{\sqrt{1 - v^2/c^2}}\Big)\, df_1 df_2. \tag{55}$$

The isometry property is

$$\int_{\mathcal{D}} |\, C(x, t, v, f)\, |^2 \, d\mu(x, t, v, f) = \omega_\phi \|H\|^2 \tag{56}$$

where

$$d\mu(x, t, v, f) \equiv c^{-2}(1 - v^2/c^2)^{-1} f\, dx dt dv df \tag{57}$$

and the range of integration $\mathcal{D}$ is given by the intervals $(-\infty < x, t < \infty, 0 < f < \infty, -c < v < c)$.

The constant ω_ϕ is defined by

$$\omega_\phi \equiv (1/4) \int_0^\infty \int_0^\infty |\phi(f_1, f_2)|^2 (f_1 f_2)^{-1}\, df_1 df_2. \tag{58}$$

As usual, a reconstruction formula giving $H(f_1, f_2)$ in terms of the wavelet coefficients can be obtained provided $\omega_\phi < \infty$. In that case, the hyperimage for this nonstatic situation is defined by

$$I(x, t, v, f) \equiv \omega_\phi |C(x, t, v, f)|^2. \tag{59}$$

Practically, only 2-D sections of the hyperimage I will be displayed, and it is essential to use a function ϕ adapted to relevant variables in order to keep control of the uncertainties. Theoretical considerations lead to choose

$$\phi(f, v) = \Big(\frac{1 - v^2/c^2}{4}\Big)^{\pi c \lambda^-} f^{-1+2\lambda^+} e^{-2\pi\lambda^+ f} \tag{60}$$

where $\lambda^\pm$ are parameters controlling independently the spreadings of ϕ in v and f.

Among other applications, it is interesting to notice that the present treatment is able to give an interpretation of range-Doppler analysis based on a simulated rotation for the imaging of static 2-D targets [13].

V. Conclusion

The great variety of data that can be obtained in microwave scattering experiments require one to enlarge the concept of image and to introduce new target representations called hyperimages. We have shown that these hyperimages can be constructed using a continuous wavelet analysis adapted to the situation at hand. Actually, the group of transformations generating the basis of wavelets proceeds directly from the transformations connecting the possible reference frames. In that way, the construction of the hyperimage is well defined independently of the reference system, and depends only on a numerical function monitoring the tradeoff between resolutions.

REFERENCES

[1] J. Bertrand, P. Bertrand, and J. P. Ovarlez, "Frequency-directivity scanning in laboratory radar imaging," *Int. J. Imag. Syst. Technol.*, vol. 5, pp. 39-51, 1994.

[2] J. Bertrand and P. Bertrand, "Theoretical radar imaging: From images to hyperimages," in *Proc. IGARSS*, Italy, 1995, pp. 474–476.

[3] J. Bertrand, P. Bertrand, and J. P. Ovarlez, "Dimensionalized wavelet transform with application to radar imaging," in *Proc. IEEE-ICASSP*, Toronto, ON, Canada, 1991, pp. 2909–2912.

[4] G. Kaiser, *A Friendly Guide to Wavelets*. Boston, MA: Birkhäuser, 1994, ch. 3.

[5] J. Bertrand and P. Bertrand, "Affine time-frequency distributions" in *Time-Frequency Signal Analysis—Methods and Applications*, B. Boashash, Ed. Melbourne, Australia: Longman-Cheshire, 1992, pp. 118–140.

[6] J. R. Klauder, " Path integrals for affine variables," in *Functional Integration: Theory and Applications*, J. P. Antoine and E. Tirapegui, Eds. New York: Plenum, 1980, pp. 101–118.

[7] B. D. Jersak, B. D. Krenek, and A. J. Blanchard, "Generation of holographic synthetic aperture radar images from bistatic waterline measurements of a complex metallic object," in *Proc. IGARSS*, Italy, 1995, pp. 2255–2257.

[8] H.-J Li, F.-L. Lin, Y. Shen, and N. H. Farhat, "A generalized interpretation and prediction in microwave imaging involving frequency and angular diversity," *J. Electromagnetic Waves Appl.*, vol. 4, pp. 415-430, 1990.

[9] J. Bertrand, P. Bertrand, and M. Roisin, "Bistatic radar imaging in laboratory," in *Image Processing: Theory and Applications*, G.Vernazza *et al.*, Eds. New York: Elsevier Sci., 1993.

[10] E. Feig and F. A. Grünbaum, "Tomographic methods in range-Doppler radar," *Inverse Problems*, vol. 2, pp. 185-195, 1986

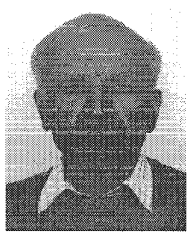
[11] E. Feig, "Range-Doppler Imaging," *Int. J. Imag. Syst. Technol.*, vol. 1, p. 125, 1989.

[12] J. Bertrand and P. Bertrand, "Reflectivity study of time-varying radar targets," in *Proc. EUSIPCO*, Brussels, Belgium, 1992, pp. 1813–1816.

[13] J. Bertrand, P. Bertrand, and L. Vignaud, "The concept of kinematical image and its practical use in radar target studies," in *Proc. Fourth Int. Workshop Time-Varying Image Processing Moving Object Recognition*, Florence, Italy, 1993.

J. Bertrand received the Bachelors degree from the University of Washington, Seattle, and the Doctorate degree in mathematical physics from the University of Paris, France, in 1969.

She is presently a Senior Researcher with the National Center for Scientific Research (CNRS), University Denis Diderot, Paris, France. Her main areas of work have been quantum mechanics, group theory with applications to physics and signal processing, and stochastic processes in quantum field theory. Her present research interests are in quantization problems, theoretical signal processing, and radar imaging.



P. Bertrand followed the whole curriculum of the University of Paris.

He was with the French Atomic Energy Agency, Saclay, France, and with Euratom, Brussels, Belgium. He is now a Senior Research Engineer with the Office National d'Etudes et de Recherches Aerospatiales (ONERA), Palaiseau, France. His main areas of research have been theoretical plasma physics, wave propagation in random media, and aeroacoustics. His present research interests are in electromagnetism, signal analysis, and radar imaging.