Inverse Problems **13** (1997) 621–645. Printed in the UK PII: S0266-5611(97)70352-6

Microwave imaging of time-varying radar targets

J Bertrand† and P Bertrand‡

† CNRS and University Paris VII, F-75251 Paris, Cedex 05, France

‡ ONERA, BP 72, F-92322 Chatillon, France

Received 13 December 1995

Abstract. The study deals with the introduction of coherent imaging ideas in the kinetic analysis of systems. It is developed in the radar context and is devoted to the description of general targets exhibiting motion, scintillation and dispersivity. This description is based on a physical model of small independent reflectors which can be moving and non-permanent. The resulting representations are generalized images which correspond to densities in a position–velocity–time–frequency space. A theoretical approach of the subject is presented in which the imaging problem is expressed as a phase-space representation problem associated with the Weyl–Poincaré group. The physical relevance of the formulation is emphasized and two procedures are proposed to make it practical. The first one is founded on a special wavelet analysis of the target backscattering function and the second one on the introduction of a generalized Wigner function. Connections with previous works on the same subject are discussed.

1. Introduction

The electromagnetic behaviour of a radar target is entirely described by its backscattering function which connects the incident and reflected fields in a scattering experiment. For a given target, this function can either be computed using the electromagnetic theory or acquired experimentally by direct measurement. In the second case, the approach is purely phenomenological and the study eludes the question of the perception of the target as a real object with spatiotemporal extension. Such a form of perception can, however, be very useful and it is the function of radar imaging [1] to reintroduce it by processing the scattering data. Essentially the technique makes use of a model of independent localized reflectors which is assumed to be the cause of the observed phenomena. In the usual applications, the targets are static or in rigid motion and the elementary scatterers are supposed to exist permanently. The object of the present study is to show that scintillating targets can be represented by the same technique provided ephemeral reflectors are considered [2]. This extension of radar imaging is important for the study of the behaviour of active radar targets.

To sketch the method which will be used, it is convenient to recall some results obtained in the particular case of two-dimensional static radar imaging. In that case, it has proved essential to introduce a model of localized scatterers that are able to discriminate between different frequencies and different directions of illumination [3]. The resulting images, called hyperimages, are thus composed of points labelled by frequency and directivity as well as space parameters. Their construction relies mainly on the principle that all imaging techniques have to be equivalently formulated whatever the reference system used. Practically, this requires finding the group of transformations relating the possible reference frames which, in this particular case, is the similarity group of the plane. Once its relevant unitary representation has been found, the essential of the physics has been isolated and there is no further need for a wave equation. The construction of the hyperimage can then be performed using a wavelet analysis associated with the group of invariance [4].

In the more general case of time-varying targets, it is the whole set of inertial reference frames that must be taken into account. The possible transformations between its elements consist of spacetime translations, dilations and Lorentz boosts and constitute a group known as the Weyl–Poincaré group. To proceed further, the unitary representation of the group that arises in the problem must be determined. Then an extension of wavelet analysis to the Weyl–Poincaré group must be performed and leads to a hyperimage obtained as a function of four parameters: position, velocity, instant of appearance and working frequency of the elementary scatterers. The dependence of the procedure on the choice of an initial function, the 'mother' wavelet, is inherent to the use of wavelet analysis. However, in the present situation, a generalization of time–frequency methods [5] makes it possible to set up an intrinsic analysis and to construct a generic hyperimage.

In section 2, the backscattering coefficient of the problem is defined and its transformation in a change of reference system is established. Section 3 is devoted to the introduction of the model of evolutive scatterers and to the description of a hyperimage. The wavelet imaging technique adapted to the Weyl–Poincaré group in one space dimension is then developed in section 4 and the choice of the basic wavelet is discussed in section 5. Going one step further, section 6 gives a generalized Wigner function which provides an intrinsic solution to the imaging problem. Some comparisons with other works are performed in section 7 [6–8]. Finally, to make the paper self-contained, appendices describe briefly the Kirillov construction of phase space, the establishment of uncertainty relations and the affine Wigner functions.

2. The electromagnetic response of a non-inert target

A radar target is said to be non-inert if some of its parts are moving or, more generally, if the electrical properties of its materials vary with time. In all cases, frequency modulations are taking place in the scattering process and their effects can be observed in a radar experiment. These effects are in fact currently exploited in the classical imaging of permanent targets which undergo rigid motions [1, 9]. The treatment of more general situations is also possible provided the notion of backscattering function is introduced in a general way. This will be done by fixing some notations concerning the incident and reflected fields and by expressing their mutual relations.

The radar transmitter and receiver are located at the same spot, far from the target so that the observed waves can be considered as plane waves. Fixing the polarizations at emission and reception allows us to treat the electromagnetic field as a scalar. As a result, a reference frame K is simply determined once the experimentalist has chosen an origin and a scale for space and time. In this frame, the positive-frequency part of an incident field can be written as a superposition of plane waves in the form:

$$\Phi_{\mathrm{in}}(x,t) = \int_0^\infty \hat{\Phi}_{\mathrm{in}}(f) \mathrm{e}^{2\mathrm{i}\pi f(t-x/c)}\,\mathrm{d}f \tag{2.1}$$

where c is the velocity of light and where usual notations are used for space and time.

The echo from the target will have the same form except that it travels in the opposite direction:

$$\Phi_{\mathrm{out}}(x,t) = \int_0^\infty \hat{\Phi}_{\mathrm{out}}(f) \mathrm{e}^{2\mathrm{i}\pi f(t+x/c)}\,\mathrm{d}f. \tag{2.2}$$

The linearity of electromagnetic theory implies that there is a linear operator connecting the fields Φ_{in} and Φ_{out}. We shall write the relation in terms of a kernel H in the following form:

$$\hat{\Phi}_{\mathrm{out}}(f_2) = \int_0^{\infty} H(f_1, f_2)\hat{\Phi}_{\mathrm{in}}(f_1)(f_1/f_2)^{\gamma}\,\mathrm{d}f_1 \tag{2.3}$$

where γ is a constant that will be adjusted later. The kernel H is interpreted as the general complex scattering amplitude or general backscattering coefficient. All target information made available by a radar experiment is contained in $H(f_1, f_2)$ which thus constitutes the data of the imaging problem.

It can be noted that, for static targets at rest, no frequency change occurs and the backscattering function is necessarily of the form:

$$H(f_1, f_2) \equiv h(f_1)\delta(f_1 - f_2) \tag{2.4}$$

where δ represents a Dirac distribution. In that case, relation (2.3) reduces to:

$$\hat{\Phi}_{\mathrm{out}}(f_1) = h(f_1)\hat{\Phi}_{\mathrm{in}}(f_1) \tag{2.5}$$

which shows that the echo signal is the time convolution of the transmitted signal with a function representative of the target.

In a change of reference frame, the coefficient H will undergo a transformation that must be determined before any frame-invariant imaging procedure can be set up. Equivalent frames correspond to different observers moving at constant velocity relatively to each other and using their own spacetime coordinates. Thus a change from system K to system K' is characterized by a dilation $\alpha > 0$, a pure Lorentz transformation of velocity ν and a spacetime translation (ξ, τ). In fact, the family of all these four-parameter transformations constitutes what is called the Weyl–Poincaré group W with one space dimension. This group acts on the spacetime coordinates (x, t) transforming them into (x', t') given by:

$$\begin{aligned} x' &= \alpha\frac{1}{\sqrt{1-(\nu/c)^2}}(x - \nu t) + \xi \\ t' &= \alpha\frac{1}{\sqrt{1-(\nu/c)^2}}(t - x\nu/c^2) + \tau. \end{aligned} \tag{2.6}$$

Introducing the notations:

$$\boldsymbol{x} \equiv \begin{pmatrix} x \\ ct \end{pmatrix} \qquad \boldsymbol{\xi} \equiv \begin{pmatrix} \xi \\ c\tau \end{pmatrix} \tag{2.7}$$

$$\Gamma(\alpha, \nu) \equiv \frac{\alpha}{\sqrt{1-(\nu/c)^2}}\begin{pmatrix} 1 & -\dfrac{\nu}{c} \\ -\dfrac{\nu}{c} & 1 \end{pmatrix} \tag{2.8}$$

we can write the transformation (2.6) in the compact form:

$$\boldsymbol{x}' = \Gamma(\alpha, \nu)\boldsymbol{x} + \boldsymbol{\xi}. \tag{2.9}$$

This transformation can also be symbolically written as

$$\boldsymbol{x}' = g \cdot \boldsymbol{x} \tag{2.10}$$

where g is the group element $(\boldsymbol{\xi}, \alpha, \nu)$.

The composition law of the group W is obtained by performing two successive transformations with the result:

$$(\boldsymbol{\xi}, \alpha, \nu)(\boldsymbol{\xi}', \alpha', \nu') = \left(\boldsymbol{\xi} + \Gamma(\alpha, \nu)\boldsymbol{\xi}', \alpha\alpha', \frac{\nu + \nu'}{1 + \nu\nu'/c^2}\right). \tag{2.11}$$

In a change of reference system, the transformation law of the scattering coefficient H is deduced from that of the fields (2.1) and (2.2). The new fields Φ^g_{in} and Φ^g_{out} are given by the relations:

$$\Phi^g_{\mathrm{in}}(\boldsymbol{x}) = \Phi_{\mathrm{in}}(g^{-1}\cdot\boldsymbol{x})$$
$$\Phi^g_{\mathrm{out}}(\boldsymbol{x}) = \Phi_{\mathrm{out}}(g^{-1}\cdot\boldsymbol{x}).$$

This is written explicitly using (2.1) and inverting (2.6) as

$$\Phi^g_{\mathrm{in}}(x,t) = \int_0^\infty \hat{\Phi}_{\mathrm{in}}(f)\exp\left[2\mathrm{i}\pi\frac{f}{\alpha}\frac{1-\nu/c}{\sqrt{1-(\nu/c)^2}}\left[t-\tau-\frac{1}{c}(x-\xi)\right]\right]\mathrm{d}f. \tag{2.12}$$

Since the expression (2.12) is again of the form (2.1), the transformed field in the frequency domain can be obtained directly

$$\hat{\Phi}^g_{\mathrm{in}}(f) = \alpha\frac{1+\nu/c}{\sqrt{1-(\nu/c)^2}}\mathrm{e}^{-2\mathrm{i}\pi f(\tau-\xi/c)}\hat{\Phi}_{\mathrm{in}}\left(\alpha\frac{1+\nu/c}{\sqrt{1-(\nu/c)^2}}f\right). \tag{2.13}$$

The transformation law of the echo can be obtained in the same way and reads:

$$\hat{\Phi}^g_{\mathrm{out}}(f) = \alpha\frac{1-\nu/c}{\sqrt{1-(\nu/c)^2}}\mathrm{e}^{-2\mathrm{i}\pi f(\tau+\xi/c)}\hat{\Phi}_{\mathrm{out}}\left(\alpha\frac{1-\nu/c}{\sqrt{1-(\nu/c)^2}}f\right). \tag{2.14}$$

Using formulae (2.13), (2.14) and (2.3), it is now possible to compute the transformation law of the backscattering coefficient H in a change of reference frame defined by the parameters (ξ,τ,α,ν). The result is:

$$\begin{aligned} H \xrightarrow{g} H_g(f_1,f_2) &= \alpha\left(\frac{1+\nu/c}{1-\nu/c}\right)^{\gamma-1/2}\exp(-2\mathrm{i}\pi[(f_2-f_1)\tau+(f_2+f_1)\xi/c]) \\ &\quad\times H\left(\alpha\frac{1+\nu/c}{\sqrt{1-(\nu/c)^2}}f_1,\alpha\frac{1-\nu/c}{\sqrt{1-(\nu/c)^2}}f_2\right). \end{aligned} \tag{2.15}$$

According to definition (2.3), the coefficient H must have the dimension of an inverse frequency in order to ensure the same physical dimension to Φ_{in} and Φ_{out}. The occurrence of the factor α in front of (2.15) is the direct consequence of this point. The second factor of the expression can be eliminated by giving to the arbitrary coefficient γ the value $\gamma=\frac{1}{2}$ so that formula (2.3) can be rewritten as

$$\hat{\Phi}_{\mathrm{out}}(f_2) = \int_0^\infty H(f_1,f_2)\hat{\Phi}_{\mathrm{in}}(f_1)\sqrt{f_1/f_2}\,\mathrm{d}f_1. \tag{2.16}$$

In fact, changing γ corresponds to multiplying H by some power of f_1/f_2 which is a dimensionless quantity. With the choice $\gamma=\frac{1}{2}$, the natural scalar product of two backscattering coefficients, i.e.

$$(H,H') = \int_{\mathbb{R}^2_+} H(f_1,f_2)H^*(f_1,f_2)\,\mathrm{d}f_1\,\mathrm{d}f_2 \tag{2.17}$$

does not change in a change of reference system. It can be seen that the transformation defined by (2.15) is a unitary irreducible representation of the Weyl–Poincaré group in the corresponding Hilbert space.

For future computations, it is convenient to introduce the alternative parametrization of group W defined by:

$$\begin{aligned} a_1 &= \alpha \frac{1+v/c}{\sqrt{1-(v/c)^2}} \qquad & b_1 &= -\tau + \xi/c \\ a_2 &= \alpha \frac{1-v/c}{\sqrt{1-(v/c)^2}} \qquad & b_2 &= \tau + \xi/c \end{aligned} \tag{2.18}$$

and to replace spacetime coordinates by light-cone coordinates according to:

$$x_1 = -t + x/c \qquad x_2 = t + x/c. \tag{2.19}$$

The action of group W on coordinates (x_1, x_2) obtained from relations (2.6) can thus be written as

$$g : x_1 \longrightarrow x'_1 = a_1 x_1 + b_1 \qquad x_2 \longrightarrow x'_2 = a_2 x_2 + b_2. \tag{2.20}$$

On this form, one recognizes the action on each coordinate x_i of the affine group A consisting of elements (a_i, b_i), $a_i > 0$ and b_i real, with the composition law:

$$(a_i, b_i)(a'_i, b'_i) = (a_i a'_i, b_i + a_i b'_i).$$

The isomorphism of the Weyl group, W, with the direct product $A \times A$ of two affine groups is thus clearly exhibited.

With these new parameters, the representation (2.15) becomes:

$$H \xrightarrow{g} H_g(f_1, f_2) = (a_1 a_2)^{1/2} \mathrm{e}^{-2\mathrm{i}\pi(b_1 f_1 + b_2 f_2)} H(a_1 f_1, a_2 f_2) \tag{2.21}$$

where $g = (a_1, a_2, b_1, b_2)$.

The two types of parametrization of the Weyl–Poincaré group will be used in the following. Form (2.21) of the representation is more adapted to computations while form (2.15) is essential for physical interpretation.

3. Physical representation of evolutive scatterers

In conformity with the usual practice in radar imaging, we will interpret the reflective properties of the target as due to the independent contributions of elementary localized scatterers. The special point is that we will not assume, as is generally done, that the constitutive reflectors of the model are permanently present during the scattering process and that their behaviour is independent of the radar frequency. In fact, the introduction of such simplifying hypotheses is not a prerequisite for solving the inverse problem as we will now show by recalling some results previously obtained in the broad-band description of static targets [5, 3].

The electromagnetic response of static radar targets can be described by introducing images which are densities of elementary scatterers characterized by their positions, x, and their working frequencies, f. In this approach, the basic model is of a mathematical nature (localized frequency-selective mirrors are not realizable) and the physical reality is expressed by the images which always display a spreading in the (x, f) space. This spreading is representative of the trade-off between the position and frequency resolutions. Two techniques for the computation of these images have been proposed in the past, using either a Wigner function [5] or a wavelet analysis [4]. In both cases, a constructive role can be attributed to the group of changes of coordinates in the target referential. This group is in fact the affine group A of transformations consisting of space translations by a real number b (change of origin) and dilations by $a > 0$ (change of units) [10]. In such transformations, the scatterer coordinates (x, f) go to $(ax+b, a^{-1}f)$ and it can be observed that the volume element $\mathrm{d}x\,\mathrm{d}f$ remains unchanged. Moreover, the action of the group is such that any two points (x, f) and (x', f') in the space of the image can be related by a unique transformation (a, b), i.e. the action is transitive and free. An image is a function $I(x, f)$ which represents a repartition of scatterers and transforms pointwise in a change of reference frame. The expression of this transformation is:

$$I(x, f) \longrightarrow I'(x, f) = a^{-1} I(a^{-1}(x-b), af) \tag{3.1}$$

where the factor a^{-1} ensures that the x integral of I transforms as the square modulus of the backscattering coefficient introduced by (2.5). To emphasize the difference between traditional images, which depend on x alone, and the broad-band images in space (x, f), the latter are referred to as hyperimages.

A representation of evolutive targets can be set up by extending the above method. Because the target exhibits motion and scintillation, the elementary scatterers are now characterized by a velocity v, and an instant of appearance t, in addition to the position x and the frequency f. The space (x, t, v, f) spanned by these parameters will be denoted by $\mathcal{M}$.

Variables (v, f) are connected to the transmitted and reflected radar frequencies (f_1, f_2) by the classical formulae:

$$f = \sqrt{f_1 f_2} \qquad v/c = \frac{f_1 - f_2}{f_1 + f_2} \tag{3.2}$$

which imply $f > 0$ and $|v| < c$. In fact, f represents the frequency (incident and reflected) which is observed in the scatterer referential. The variable v is simply the Doppler velocity associated with the two radar frequencies f_1 and f_2. Relations (3.2) can be inverted under the form:

$$f_1 = f \frac{1 + v/c}{\sqrt{1 - v^2/c^2}} \qquad f_2 = f \frac{1 - v/c}{\sqrt{1 - v^2/c^2}}. \tag{3.3}$$

The set of relations (2.19), (3.2) and (3.3) allows us to operate the change of variables $(x, t, v, f) \longrightarrow (x_1, x_2, f_1, f_2)$ in the space $\mathcal{M}$ of elementary scatterers. This change of variables will be found useful for computations.

In non-static situations, the group of changes of reference systems to consider is the Weyl–Poincaré group whose elements g, characterized by the parameters (ξ, τ, α, ν), act upon variables (x, t) according to (2.9). Under this action, a frequency change occurs which can be inferred from (2.15) and reads:

$$g : f_1 \to f_1' = f_1 \alpha^{-1} \frac{1 - \nu/c}{\sqrt{1 - (\nu/c)^2}} \qquad f_2 \to f_2' = f_2 \alpha^{-1} \frac{1 + \nu/c}{\sqrt{1 - (\nu/c)^2}}. \tag{3.4}$$

The equivalent action on the variables (v, f) introduced by (3.2) is:

$$g : v \to v' = \frac{v - \nu}{1 - v\nu/c^2} \qquad f \to f' = \alpha^{-1} f. \tag{3.5}$$

We note that the transformation on v is just the familiar law of relativistic composition of velocities.

A direct study of transformations (2.9) and (3.5) could show that they preserve a volume element in space (x, t, v, f) and that two arbitrary points of $\mathcal{M}$ are connected in a unique way by an element of the group W (transitive and free action). However, it is worthwhile to imbed these results in a more general framework that will set the imaging technique on firm foundations. In fact, it can be shown, using Kirillov's theory [11], that space $\mathcal{M}$ is closely related to group W. More precisely, $\mathcal{M}$ can be constructed as the co-adjoint orbit of W associated with representation (2.15) of W on the backscattering coefficient. As such, it is canonically endowed with an invariant symplectic form which yields the invariant volume element $\mathrm{d}\mu$ on $\mathcal{M}$ given by

$$\mathrm{d}\mu(x,t,v,f) = \frac{f}{c^2(1-v^2/c^2)}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}v\,\mathrm{d}f. \tag{3.6}$$

Moreover, it is the unique space with all those properties (see appendix A for some of the details). We will refer to $\mathcal{M}$ as the phase space to conform with a usage initiated in mechanics.

Once the space of parameters characterizing the individual reflectors has been set up, the description of the whole target is realized by introducing a function on that space, the *hyperimage* that represents a density of scatterers and transforms pointwise in a change of reference system. The hyperimage will be denoted by $\tilde{I}(x_i, f_i) \equiv \tilde{I}(x_1, x_2, f_1, f_2)$ or $I(x,t,v,f)$, depending on the parametrization chosen on $\mathcal{M}$. The transformation of $\tilde{I}(x_i, f_i)$ in a change of reference system characterized by an element $g = (a_i, b_i) \in W \sim A \times A$ is obtained from (2.20) and reads:

$$\tilde{I}(x_i, f_i) \xrightarrow{g} \tilde{I}_g(x_i, f_i) \equiv \tilde{I}(a_i^{-1}(x_i - b_i), a_i f_i). \tag{3.7}$$

The equivalent requirement on I is obtained by using formulae (2.9) and (3.5). The result is:

$$I(\boldsymbol{x}, v, f) \xrightarrow{g} I_g(\boldsymbol{x}, v, f) \equiv I\left(\Gamma^{-1}(\alpha,\nu)(\boldsymbol{x} - \boldsymbol{\xi}), \frac{v+\nu}{1+v\nu/c^2}, \alpha f\right) \tag{3.8}$$

where $\Gamma(\alpha,\nu)$ is the matrix defined in (2.8) and where notations (2.7) have been used.

These preliminaries open the way to the mathematical formulation of the imaging problem. In fact, the question is to express the hyperimage I in terms of the observed data, that is to say in terms of the backscattering function introduced in section 2. It is clear that the form of the relation must be independent of the reference system that is used to formulate it. Moreover, the relation has to ensure the consistency of the transformations (2.15) and (3.8) in any change of reference system. These conditions are expressed by the commutativity of the following diagram:

$$\begin{array}{ccc} H(f_1, f_2) & \longrightarrow & H_g(f_1, f_2) \\ \downarrow & & \downarrow \\ I(x,t,v,f) & \longrightarrow & I_g(x,t,v,f) \end{array} \tag{3.9}$$

where the vertical arrows represent the imaging procedure and the horizontal ones refer to a change of reference system represented by some transformation $g = (\xi, \tau, \alpha, \nu)$. An imaging procedure which satisfies (3.9) is said to be *covariant* by the Weyl–Poincaré transformations.

In accordance with the classical treatments, we will suppose that the expression of the hyperimage is given by a Hermitian functional of the backscattering function H. The form of this functional must be compatible with the natural constraint:

$$\int_{\mathcal{M}} I(x,t,v,f)\frac{f}{c^2(1-v^2/c^2)}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}v\,\mathrm{d}f = \|H(f_1, f_2)\|^2 \tag{3.10}$$

relating the two fundamental invariants associated with I and H.

The above conditions are necessary requirements to impose on a meaningful hyperimage but are too general to determine completely the functional I. We shall now turn to the wavelet solution of the problem.

4. Imaging technique based on a continuous wavelet analysis

Continuous wavelet analysis is closely related to the theory of coherent states defined from representations of Lie groups [12–15]. Both theories are based on the introduction of a fundamental family of functions obtained by action of a relevant group representation on a fiducial function also called 'mother wavelet' [16]. By construction, this family as a whole is stable by action of the group and constitutes an overcomplete basis on which any square-integrable function can be decomposed. The elements of this basis are called coherent states or wavelets and the computation of the coefficients of the development constitutes what is called wavelet analysis. It must be stressed that the above properties are specific to the continuous wavelet theory and are lost when restricting to orthogonal wavelets. In this latter case, the bases are no longer invariant by action of the group and the choice of the mother wavelet is *a priori* restricted.

Continuous wavelet analysis is particularly useful in coherent imaging when the constructive Lie group can be identified with the physical group of changes of reference systems. This approach, which has proven fruitful in the static case [3, 4], will now be shown to give images satisfying the covariance constraint (3.9) with respect to the Weyl–Poincaré group. Because this group can be interpreted as the direct product of two affine groups (cf section 2), the related wavelet analysis is very close to the classical one [12, 16].

To obtain a relevant system of wavelets, we choose a numerical function $\phi(f_1, f_2)$ (analogous to a backscattering function) and consider the set of all its transforms by the elements of the Weyl–Poincaré group. In this operation, the representation (2.21) of the group is used so that the resulting family is written as

$$\phi_{a_1a_2b_1b_2}(f_1, f_2) = (a_1a_2)^{1/2}\mathrm{e}^{-2\mathrm{i}\pi(b_1f_1+b_2f_2)}\phi(a_1f_1, a_2f_2) \tag{4.1}$$

with a_1, a_2 positive and b_1, b_2 real. Wavelets are thus labelled by elements (a_1, a_2, b_1, b_2) of the group. We shall now show that they are in fact associated with points in phase-space $\mathcal{M}$.

Suppose that the original function ϕ is assigned to a point $P_0 \in \mathcal{M}$ with coordinates $(x_{i0} = 0, f_{i0} = 1)$ or equivalently $(x_0 = t_0 = v_0 = 0, f_0 = 1)$. This suggests the interpretation of ϕ as the backscattering coefficient of a target localized around the spacetime origin, with velocity approximately equal to zero and a working frequency in the vicinity of unity. If a transformation (a_i, b_i) is performed, point P_0 goes to point P defined by the following coordinates:

$$x_i = a_ix_{i0} + b_i \qquad f_i = a_i^{-1}f_{i0} \tag{4.2}$$

or, substituting the values of coordinates (x_{i0}, f_{i0})

$$x_i = b_i \qquad f_i = a_i^{-1}. \tag{4.3}$$

After the change of variables $(x_i, f_i) \longrightarrow (x, t, v, f)$, we finally obtain:

$$x = \frac{c}{2}(b_1 + b_2) \qquad t = \tfrac{1}{2}(b_2 - b_1) \tag{4.4}$$

$$v = c\frac{a_2 - a_1}{a_2 + a_1} \qquad f = \frac{1}{\sqrt{a_1a_2}}. \tag{4.5}$$

This is a one-to-one correspondence between the elements of the group and the points in phase space†. As a result, any wavelet in (4.1) can be characterized equivalently

† This property expresses the fact that group W acts transitively and freely on phase-space $\mathcal{M}$.

by parameters (a_1, a_2, b_1, b_2) or by coordinates (x, t, v, f) and the whole family can be rewritten:

$$\phi_{x,t,v,f}(f_1, f_2) = f^{-1}\mathrm{e}^{-2\mathrm{i}\pi[(f_1+f_2)x/c-(f_1-f_2)t]}\phi\left(\frac{f_1}{f}\frac{1-v/c}{\sqrt{1-v^2/c^2}}, \frac{f_2}{f}\frac{1+v/c}{\sqrt{1-v^2/c^2}}\right). \tag{4.6}$$

The collection of wavelets $\{\phi_{xtvf}\}$ forms an overcomplete system which is stable by action of the covariance group and depends only on the choice of one numerical function ϕ. Indeed, a change of reference frame will only lead to a relabelling of the family. Any backscattering coefficient H can be developed on the system $\{\phi_{xtvf}\}$. The wavelet coefficient of H is defined by the scalar product:

$$C(x, t, v, f) = (H, \phi_{x,t,v,f}). \tag{4.7}$$

Using definition (2.17), we obtain:

$$C(x, t, v, f) = f^{-1}\int_{\mathbb{R}^2} H(f_1, f_2)\mathrm{e}^{2\mathrm{i}\pi[(f_1+f_2)x/c-(f_1-f_2)t]} \times\phi^*\left(\frac{f_1}{f}\frac{1-v/c}{\sqrt{1-v^2/c^2}}, \frac{f_2}{f}\frac{1+v/c}{\sqrt{1-v^2/c^2}}\right)\mathrm{d}f_1\,\mathrm{d}f_2. \tag{4.8}$$

If the analysis is applied to the coefficient H_g seen in another referential and defined in (2.15), the obtained wavelet coefficient is:

$$C'(x, t, v, f) = (H_g, \phi_{x,t,v,f}) \tag{4.9}$$

where g denotes the transformation with parameters (ξ, τ, α, ν). Using the invariance of the scalar product and taking into account the definition (4.6) of $\phi_{x,t,v,f}$, we can rewrite coefficient C' as

$$C'(x, t, v, f) = (H, \phi_{g^{-1}(x,t,v,f)}) \tag{4.10}$$

and, according to formulae (2.9) and (3.5), this is equal to:

$$C'(x, t, v, f) = C\left(\Gamma^{-1}(\alpha, \nu)(\boldsymbol{x}-\boldsymbol{\xi}), \frac{v+\nu}{1+v\nu/c^2}, \alpha f\right). \tag{4.11}$$

Thus, the new wavelet coefficient is deduced from the old one by a formula identical to (3.8).

In the same way as in the pure affine case, the wavelet coefficient is shown to satisfy an isometry property:

$$\int_{\mathcal{M}} |C(x, t, v, f)|^2\,\mathrm{d}\mu\,(x, t, v, f) = \kappa_\phi\|H\|^2. \tag{4.12}$$

where $\mathrm{d}\mu$ is given by (3.6) and the range of integration is the whole space, $\mathcal{M}$, characterized by the intervals $-\infty < x, t < \infty, 0 < f < \infty, -c < v < c$.

The constant κ_ϕ depends only on the basic function ϕ. Its value is given by the integral:

$$\kappa_\phi \equiv \tfrac{1}{4}\int_{\mathbb{R}^2_+} |\phi(f_1, f_2)|^2(f_1 f_2)^{-1}\,\mathrm{d}f_1\,\mathrm{d}f_2. \tag{4.13}$$

The knowledge of the wavelet coefficient allows us to reconstruct the backscattering function by the formula:

$$H(f_1, f_2) = \frac{1}{\kappa_\phi}\int_{\mathcal{M}} C(x, t, v, f)\phi_{x,t,v,f}(f_1, f_2)\,\mathrm{d}\mu\,(x, t, v, f) \tag{4.14}$$

provided the function ϕ is such that the integral (4.13) has a finite value. The above developments allow us to define the kinematical (hyper)image by:

$$I(x,t,v,f) = \frac{1}{\kappa_\phi}|C(x,t,v,f)|^2. \tag{4.15}$$

The result is a positive quantity satisfying the covariance constraint expressed by the diagram (3.9). By virtue of the isometry relation (4.12), the image (4.15) verifies the constraint (3.10). Function I is interpreted as a distribution in x and t of target elements moving at velocity v and reflecting at frequency f. Moreover, integrating I over spacetime yields the expression

$$\int_{\mathbb{R}^2} I(x,t,v,f)\,\mathrm{d}x\,\mathrm{d}t = \frac{c}{2\kappa_\phi f^2}\int_{\mathbb{R}_+^2} |H(f_1,f_2)|^2 \times \left|\phi\left(\frac{f_1}{f}\frac{1-v/c}{\sqrt{1-v^2/c^2}}, \frac{f_2}{f}\frac{1+v/c}{\sqrt{1-v^2/c^2}}\right)\right|^2 \mathrm{d}f_1\,\mathrm{d}f_2. \tag{4.16}$$

The result is a smoothing of $|H|^2$ which commutes with the transformation connecting different observers and is thus endowed with an invariant interpretation. We will find it convenient to perform the change of variables (3.2) and to write the basic function ϕ in terms of variables (f,v) as

$$\phi(f_1,f_2) \equiv \tilde{\phi}(f,v). \tag{4.17}$$

The family (4.6) generated by Weyl–Poincaré transformations is then written in the form:

$$\tilde{\phi}_{x,t,v,f}(f',v') = f^{-1}\exp\left\{-4\mathrm{i}\pi\frac{f'}{c\sqrt{1-(v'^2/c^2)}}(x-v't)\right\}\tilde{\phi}\left(\frac{f'}{f}, \frac{v'-v}{1-v'v/c^2}\right). \tag{4.18}$$

The whole analysis can be performed as above provided the change of variables is taken care of in the scalar product.

In practice, it is essential to use an initial function $\phi(f_1,f_2)$ with good localization properties. The discussion on its choice, which involves a study of the uncertainty relations, will be carried out in the next section.

5. Uncertainty relations for the scatterers coordinates

The presence of uncertainty relations is an inescapable fact which stems from the non-commutativity of the elements of Weyl–Poincaré group W. Its practical manifestation in the image space is the impossibility of having sharp values for all the parameters (x,t,v,f) simultaneously. Actually, some mathematical limits for the resolutions can be exhibited, as we shall see now.

Taking advantage of the direct product structure of group $W \approx A \times A$, we write its infinitesimal generators as $\boldsymbol{f}_i$ and $\boldsymbol{B}_i$ $(i = 1,2)$ defined by (cf appendix B):

$$\boldsymbol{f}_i = f_i\,\cdot \tag{5.1}$$

$$\boldsymbol{B}_i = -\frac{1}{2\mathrm{i}\pi}\left(f_i\frac{\mathrm{d}}{\mathrm{d}f_i} + \frac{1}{2}\right) \tag{5.2}$$

where the dot denotes the usual multiplication.

All commutation relations between the generators are zero except for the following ones:

$$[\boldsymbol{B}_i, \boldsymbol{f}_i] = -\frac{1}{2\mathrm{i}\pi}f_i. \tag{5.3}$$

Thus there are common eigenfunctions of $(\boldsymbol{f}_1, \boldsymbol{f}_2)$ corresponding to well-defined values of the frequencies. In the same way, there are eigenfunctions of $(\boldsymbol{B}_1, \boldsymbol{B}_2)$ that are of the form:

$$f_1^{-2i\pi\beta_1-1/2} f_2^{-2i\pi\beta_2-1/2} \tag{5.4}$$

and correspond to eigenvalues (β_1, β_2). These values are insensitive to dilations and represent the coordinate product $(x_1 f_1, x_2 f_2)$ [17]. By contrast, (5.3) implies that there can be no common eigenfunction for $\boldsymbol{f}_i$ and $\boldsymbol{B}_i$. As a consequence, the product of the standard deviations σ_{f_i} and σ_{β_i}, defined in (B.11) and (B.12), is always bounded from below. More precisely, the following inequalities are established (cf appendix B):

$$\sigma_{f_i}^2 \sigma_{\beta_i}^2 \geqslant \frac{\langle \boldsymbol{f}_i \rangle^2}{16\pi^2} \tag{5.5}$$

where $\langle \boldsymbol{f}_i \rangle$ represents the mean value of the frequencies f_i.

It is possible to find optimal functions for which the minimal values of the left member in (5.5) are attained. They are obtained as products of the analogous minimal functions corresponding to the affine group [18, 17] and are labelled by points (f_{i0}, β_{i0}) of phase space. They can be written as

$$\phi_K(f_1, f_2) = (f_1 f_2)^{-\frac{1}{2}} f_1^{2\pi\lambda_1 f_{10} - 2i\pi\beta_{10}} f_2^{2\pi\lambda_2 f_{20} - 2i\pi\beta_{20}} \mathrm{e}^{-2\pi(\lambda_1 f_1 + \lambda_2 f_2)}. \tag{5.6}$$

In this formula $\lambda_i, i = 1, 2$ are adjustable parameters which characterize the spread of the functions in variables $(f_i, \beta_i = x_i f_i)$. Properties of function (5.6) make it interesting to use as a basic wavelet. In this case, the condition of finiteness of expression (4.13) implies that parameters λ_i are greater than $1/(4\pi f_{i0})$.

When the change of variables (3.2) from (f_1, f_2) to (f, v) is performed on expression (5.6), the result is not factorizable in the new variables. As a consequence, the parameters λ_1 and λ_2 turn out to be inadequate for the controls of the spreads in variables f and v. In fact, when working with those variables, new inequalities have to be considered. To compute their expressions, we first write operators $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ in terms of f, v:

$$\boldsymbol{B}_1 = -\frac{1}{4i\pi}\left(f\frac{\mathrm{d}}{\mathrm{d}f} + 1 + \left(1 - \frac{v^2}{c^2}\right)\frac{\mathrm{d}}{\mathrm{d}v}\right) \tag{5.7}$$

$$\boldsymbol{B}_2 = -\frac{1}{4i\pi}\left(f\frac{\mathrm{d}}{\mathrm{d}f} + 1 - \left(1 - \frac{v^2}{c^2}\right)\frac{\mathrm{d}}{\mathrm{d}v}\right). \tag{5.8}$$

Then the more relevant operators $\boldsymbol{B}^{\pm}$ and $\boldsymbol{f}, \boldsymbol{v}$ are introduced:

$$\boldsymbol{f} = f\,\cdot \qquad \boldsymbol{v} = v\,\cdot \tag{5.9}$$

$$\boldsymbol{B}^+ = \boldsymbol{B}_1 + \boldsymbol{B}_2 = -\frac{1}{2i\pi}\left(f\frac{\mathrm{d}}{\mathrm{d}f} + 1\right) \tag{5.10}$$

$$\boldsymbol{B}^- = \boldsymbol{B}_1 - \boldsymbol{B}_2 = -\frac{1}{2i\pi}\left(1 - \frac{v^2}{c^2}\right)\frac{\mathrm{d}}{\mathrm{d}v} \tag{5.11}$$

The operators $\boldsymbol{B}^+$ and $\boldsymbol{B}^-$ are actually the infinitesimal generators of dilations and boosts in the Weyl–Poincaré group. The operator $\boldsymbol{B}^+$ (respectively $\boldsymbol{B}^-$) commutes with $\boldsymbol{v}$ (respectively $\boldsymbol{f}$) and the only non-zero commutation relations are:

$$[\boldsymbol{B}^+, \boldsymbol{f}] = -\left(\frac{1}{2i\pi} f\right)\cdot \tag{5.12}$$

$$[\boldsymbol{B}^-, \boldsymbol{v}] = -\frac{c}{2i\pi}(1 - v^2/c^2). \tag{5.13}$$

The corresponding uncertainty relations are obtained as in appendix B and read:

$$\sigma_{\beta^+}^2 \sigma_f^2 \geqslant \frac{1}{16\pi^2} \langle f \rangle^2 \tag{5.14}$$

$$\sigma_{\beta^-}^2 \sigma_v^2 \geqslant \frac{c^2}{16\pi^2} \left(1 - \frac{\langle v^2 \rangle}{c^2}\right)^2 . \tag{5.15}$$

Minimal functions $\tilde{\phi}(f, v)$ corresponding to equality in (5.14) and (5.15) can be found by the usual methods which amount to solving the following system of differential equations:

$$\left[\boldsymbol{B}^+ - \beta_0^+ + \mathrm{i}\lambda^+(\boldsymbol{f} - f_0)\right] \tilde{\phi}(f, v) = 0 \tag{5.16}$$

$$\left[\boldsymbol{B}^- - \beta_0^- + \mathrm{i}\lambda^-(\boldsymbol{v} - v_0)\right] \tilde{\phi}(f, v) = 0 \tag{5.17}$$

where $\beta_0^\pm$, f_0 and v_0 are constants fixing the mean values of operators $\boldsymbol{B}^\pm$, $\boldsymbol{f}$ and $\boldsymbol{v}$ respectively and where $\lambda^\pm$ are spread parameters. The solution for the case where $(\beta^\pm = 0, v_0 = 0, f_0 = 1)$ is given, up to a normalization factor, by:

$$\tilde{\phi}(f, v) = f^{-1+2\pi\lambda^+} \mathrm{e}^{-2\pi\lambda^+ f} \left(\frac{1 - v^2/c^2}{4}\right)^{\pi c^2 \lambda^-} . \tag{5.18}$$

The norm of $\tilde{\phi}$ and expression (4.13) will be finite provided the spread parameters λ^+ and λ^- verify:

$$\lambda^- > 0 \qquad \lambda^+ > 1/(2\pi f_0). \tag{5.19}$$

Functions $\tilde{\phi}$ thus obtained are all concentrated around the image point $(f_0 = 1, v_0 = 0, x_0 = t_0 = 0)$ and the two parameters λ^+ and λ^- allow us to control the spreadings in f and v separately. As a consequence, the choice of a $\tilde{\phi}(f, v)$ of type (5.18) as a basic wavelet for family (4.18) is appropriate when discussing hyperimage resolutions in variables (x, t, v, f).

6. Wigner's function as a generic hyperimage

6.1. General form of the phase-space representation

The images delivered by the technique of section 4 depend not only on the backscattering function of the target but also on the basic function which is chosen for the analysis. An illustration of the respective roles of these two functions in the genesis of a hyperimage has been given when writing and discussing relation (4.16). More generally, we will now show that the wavelet hyperimage itself can be considered as an invariant smoothing of a generic hyperimage which depends only on the target backscattering function. This generic hyperimage will still verify (3.9) and (3.10) but, in contrast to the wavelet one, it will not be everywhere positive. In fact, the new representation about to be introduced is a generalization of Wigner's original function [19] that is specially adapted to the Weyl–Poincaré group [20].

The construction of the phase-space representation is performed using results obtained in the case of the affine group and recalled in appendix C. Expression (C.22) with $r = -\frac{1}{2}$, $q = 2r + 1 = 0$ can be readily extended to the present case in the form:

$$P(x_1, x_2, f_1, f_2) = f_1 f_2 \int_{\mathbb{R}^2} \mathrm{e}^{2\mathrm{i}\pi(x_1 f_1 u_1 + x_2 f_2 u_2)} H(f_1\lambda(u_1), f_2\lambda(u_2)) \\ \times H^*(f_1\lambda(-u_1), f_2\lambda(-u_2))\lambda(u_1)\lambda(u_2)\mathrm{e}^{-\frac{1}{2}(u_1+u_2)}\, \mathrm{d}u_1\, \mathrm{d}u_2 \tag{6.1}$$

where

$$\lambda(u) = \frac{u \exp(u/2)}{2 \sinh(u/2)}. \tag{6.2}$$

Remark that function λ can be characterized by the relations:

$$\frac{\lambda(u)}{\lambda(-u)} = \mathrm{e}^{u} \qquad \lambda(u) - \lambda(-u) = u. \tag{6.3}$$

Property (C.50) immediately yields the marginal:

$$\int_{\mathbb{R}^2} P(x_1, x_2, f_1, f_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 = |H(f_1, f_2)|^2. \tag{6.4}$$

To obtain the physically interesting results, it is necessary to change variables from (x_1, x_2, f_1, f_2) to (x, t, v, f) using (2.19) and (3.2). Setting

$$I_W(x, t, v, f) \equiv P(x_1, x_2, f_1, f_2) \tag{6.5}$$

and

$$\hat{H}(f, v) \equiv H(f_1, f_2) \tag{6.6}$$

allows us to write (6.1) as

$$\begin{aligned} I_W(x, t, v, f) = \int_{\mathbb{R}^2} \exp & \left\{ \left(2\mathrm{i}\pi \frac{f}{c\sqrt{1 - v^2/c^2}} \right) \right. \\ & \left. \times \left(x \left(u_1 + u_2 + \frac{v}{c}(u_1 - u_2) \right) - ct \left(u_1 - u_2 + \frac{v}{c}(u_1 + u_2) \right) \right) \right\} \\ & \times \hat{H} \left(f\sqrt{\lambda(u_1)\lambda(u_2)}, \frac{\lambda(u_1) - \gamma(v)\lambda(u_2)}{\lambda(u_1) + \gamma(v)\lambda(u_2)} \right) \\ & \times \hat{H}^* \left(f\sqrt{\lambda(-u_1)\lambda(-u_2)}, \frac{\lambda(-u_1) - \gamma(v)\lambda(-u_2)}{\lambda(-u_1) + \gamma(v)\lambda(-u_2)} \right) \\ & \times \lambda(u_1)\lambda(u_2)\mathrm{e}^{(-1/2)(u_1+u_2)}\, \mathrm{d}u_1\, \mathrm{d}u_2 \end{aligned} \tag{6.7}$$

where

$$\gamma(v) \equiv \frac{1 - v/c}{1 + v/c}. \tag{6.8}$$

The marginal property then becomes:

$$(2/c) \int_{\mathbb{R}^2} I_W(x, t, v, f)\, \mathrm{d}x\, \mathrm{d}t = |\hat{H}(f, v)|^2. \tag{6.9}$$

The Weyl–Poincaré covariance of the correspondence between $\hat{H}$ and I_W is a direct consequence of the above construction. In a transformation labelled by $g = (\xi, \tau, \alpha, \nu)$, $\hat{H}$ can be seen from (2.15) to transform as

$$\hat{H} \xrightarrow{g} \hat{H}'(f, v) = \alpha \exp \left(4\mathrm{i}\pi \frac{f}{c\sqrt{1 - v^2/c^2}} (\xi - v\tau) \right) \hat{H} \left(\alpha f, \frac{v + \nu}{1 + v\nu/c^2} \right). \tag{6.10}$$

The concomitant transformation on I_W is found to be:

$$I_W(\boldsymbol{x}, v, f) \xrightarrow{g} I'_W(\boldsymbol{x}, v, f) \equiv I_W \left(\Gamma^{-1}(\alpha, \nu)(\boldsymbol{x} - \boldsymbol{\xi}), \frac{v + \nu}{1 + v\nu/c^2}, \alpha f \right) \tag{6.11}$$

where Γ is defined in (2.8). Remark that the choice $q = 0$ that has been made in (C.22) ensures the transformation of I_W as a dimensionless quantity.

6.2. Localization properties

In the one-dimensional case, the requirement of localization played a central role in the determination of satisfactory phase-space distributions. Its extension to the present case is possible after a careful discussion on the concept of localized bright points and their backscattering functions.

Here the counterpart of a signal localized in frequency is a backscattering function of the form:

$$H(f_1, f_2) = (f_1^0 f_2^0)^{1/2}\delta(f_1 - f_1^0)\delta(f_2 - f_2^0) \tag{6.12}$$

or

$$\hat{H}(f, v) = \frac{c}{2}(1 - v^2/c^2)\delta(f - f_0)\delta(v - v_0). \tag{6.13}$$

It is interpreted as the response of a target which reflects with frequency f_2^0 when receiving frequency f_1^0 or, equivalently (cf (3.2)), active at frequency $f_0 = \sqrt{f_1^0 f_2^0}$ and moving at velocity $v_0 = c(f_1^0 - f_2^0)(f_1^0 + f_2^0)^{-1}$. Substituting expression (6.13) into (6.7), we find that the result is a phase-space representation, I_W, localized at frequency f_0 and velocity v_0 given explicitly by:

$$I_W = f_0 \frac{c}{2}\left(1 - \frac{v^2}{c^2}\right)\delta(f - f_0)\delta(v - v_0). \tag{6.14}$$

It can be observed that the position and the time of existence of the reflector are undetermined.

Next, consider the case of a point target localized in $x = x_0$ and present at time $t = t_0$, i.e. localized in $\boldsymbol{x} = \boldsymbol{x}_0$ in the notations (2.7). The backscattering coefficient $H(\boldsymbol{x}_0; f_1, f_2)$ of such a point can be deduced from its transformation properties in a change of observer. Namely, after a change of reference system characterized by $g = (\xi, \tau, \alpha, \nu)$, the transformed coefficient H_g, which has the expression (2.15), must correspond to a point localized in the transformed coordinates $\boldsymbol{x}_0' = \Gamma(\alpha, \nu)\boldsymbol{x}_0 + \boldsymbol{\xi}$. This yields a constraint on H which can be written as

$$\begin{aligned} H(\Gamma(\alpha, \nu)\boldsymbol{x}_0 + \boldsymbol{\xi}; f_1, f_2) &= \alpha \exp(2\mathrm{i}\pi[(f_2 - f_1)\tau + (f_2 + f_1)\xi/c]) \\ &\quad \times H\left(\boldsymbol{x}_0; \alpha\frac{1 + \nu/c}{\sqrt{1 - (\nu/c)^2}} f_1, \alpha\frac{1 - \nu/c}{\sqrt{1 - (\nu/c)^2}} f_2\right). \end{aligned} \tag{6.15}$$

To solve this equation for H, take the derivatives on both sides with respect to the group parameters (ξ, τ, ν, α) and set $\xi = \tau = \nu = 0, \alpha = 1$. The following system of partial differential equations is thus obtained:

$$\begin{aligned} &\frac{2\mathrm{i}\pi}{c}(f_2 + f_1)H = \frac{\partial H}{\partial x_0} \\ &2\mathrm{i}\pi(f_2 - f_1)H = \frac{\partial H}{\partial t_0} \\ &H + f_1\frac{\partial H}{\partial f_1} + f_2\frac{\partial H}{\partial f_2} = x_0\frac{\partial H}{\partial x_0} + t_0\frac{\partial H}{\partial t_0} \\ &f_1\frac{\partial H}{\partial f_1} - f_2\frac{\partial H}{\partial f_2} = -ct_0\frac{\partial H}{\partial x_0} - \frac{x_0}{c}\frac{\partial H}{\partial t_0}. \end{aligned}$$

The solution up to a constant factor is found to be:

$$H(\boldsymbol{x}_0; f_1, f_2) = (f_1 f_2)^{-1/2}\exp\left[\frac{2\mathrm{i}\pi}{c}(f_1(x_0 - ct_0) + f_2(x_0 + ct_0))\right] \tag{6.16}$$

or, in (f, v) variables:

$$\hat{H}(\boldsymbol{x}_0; f, v) = \frac{1}{f} \exp\left[\frac{4\mathrm{i}\pi f}{c\sqrt{1 - v^2/c^2}}(x_0 - vt_0)\right]. \tag{6.17}$$

The computation of the corresponding phase-space distribution, $I_W(x, t, v, f)$, shows that it correctly describes the above target as localized in x_0 at time t_0. Namely:

$$I_W(x, t, v, f) = \frac{c}{2} f^{-2} \delta(x - x_0)\delta(t - t_0). \tag{6.18}$$

The velocity and frequency of this reflector are indeterminate.

The above localization properties of I_W are direct consequences of similar properties for the localized affine distribution (C.22) we started from. We now turn to a result that is specific of the two-dimensional case. Consider a point of the target reflecting equally well all frequencies and moving at a constant velocity v_0. Its backscattering coefficient is computed by identifying such a point to a perfect mirror in uniform motion. Suppose a wave $\Phi_I(f) = \delta(f - f_1)$ is incident on the mirror situated at $x = x_0 + v_0 t$. The outgoing wave is of the form $\Phi_S = F\delta(f - f_2)$ where the frequency f_2 and the reflection coefficient F, determined by the boundary condition at the mirror, are given by:

$$f_2 = f_1 \frac{1 - v_0/c}{1 + v_0/c} \qquad F = \exp\left(-4\mathrm{i}\pi \frac{x_0 f_1}{v_0 + c}\right). \tag{6.19}$$

The definition (2.16) of the backscattering coefficient then yields:

$$H(x_0, v_0; f_1, f_2) = \sqrt{\frac{1 - v_0/c}{1 + v_0/c}}\, \delta\left(f_2 - f_1 \frac{1 - v_0/c}{1 + v_0/c}\right) \exp\left(-4\mathrm{i}\pi \frac{x_0 f_1/c}{1 + v_0/c}\right) \tag{6.20}$$

or, in variables (v, f):

$$\hat{H}(x_0, v_0; f, v) = \frac{c}{2f}(1 - v_0/c)\delta(v - v_0) \exp\left(-4\mathrm{i}\pi \frac{x_0 f}{c\sqrt{1 - (v_0/c)^2}}\right). \tag{6.21}$$

Substituting expression (6.20) into definition (6.1) gives:

$$\begin{aligned} P(x_1, x_2, f_1, f_2) = & f_1 f_2 \frac{1 - v_0/c}{1 + v_0/c} \int_{\mathbb{R}^2} \mathrm{e}^{2\mathrm{i}\pi(x_1 f_1 u_1 + x_2 f_2 u_2)} \mathrm{e}^{-4\mathrm{i}\pi(1+v_0/c)^{-1}(x_0/c) f_1 u_1} \\ & \times \delta\left(f_2 \lambda(u_2) - f_1 \lambda(u_1) \frac{1 - v_0/c}{1 + v_0/c}\right) \delta\left(f_2 \lambda(-u_2) - f_1 \lambda(-u_1) \frac{1 - v_0/c}{1 + v_0/c}\right) \\ & \times \lambda(u_1)\lambda(u_2) \mathrm{e}^{-(1/2)(u_1+u_2)}\, \mathrm{d}u_1\, \mathrm{d}u_2. \end{aligned} \tag{6.22}$$

The delta distributions imply:

$$\begin{aligned} \frac{\lambda(u_1)}{\lambda(-u_1)} &= \frac{\lambda(u_2)}{\lambda(-u_2)} \\ f_2(\lambda(u_2) - \lambda(-u_2)) &= f_1(\lambda(u_1) - \lambda(-u_1)) \frac{1 - v_0/c}{1 + v_0/c} \end{aligned} \tag{6.23}$$

and hence, because of the form (6.2) of λ:

$$u_1 = u_2 \tag{6.24}$$

$$f_2 = f_1 \frac{1 - v_0/c}{1 + v_0/c}. \tag{6.25}$$

The result is:

$$P(x_1, x_2, f_1, f_2) = f_2 \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{i}\pi(x_1 f_1 + x_2 f_2 - 2(x_0/c) f_1 (1+v_0/c)^{-1})u} \delta\left(f_2 - f_1 \frac{1 - v_0/c}{1 + v_0/c}\right) \mathrm{d}u. \tag{6.26}$$

The u integral yields a Dirac distribution and the change of variables defined by (2.19) and (3.2) finally gives:

$$I_W(x,t,v,f) = \frac{c^2}{4f}(1 - v_0^2/c^2)^{3/2}\delta(v - v_0)\delta(x - x_0 - v_0 t). \tag{6.27}$$

This expression of a distribution concentrated on the surface $v = v_0$ and $x = x_0 + v_0 t$ is quite satisfactory.

6.3. Connection with the wavelet solution

Function (6.7) verifies a unitarity property which has the form:

$$\int I_W(x,t,v,f) I'_W(x,t,v,f)\,\mathrm{d}\mu\,(x,t,v,f) = |(H,H')|^2 \tag{6.28}$$

where I_W and I'_W are the distributions corresponding to the backscattering functions H and H' respectively and where $\mathrm{d}\mu$ is the invariant measure (3.6). This relation is a direct consequence of a similar one for the affine distributions which is recalled in (C.10).

An important consequence of unitarity is the possibility of the covariant smoothing of I_W it provides. Indeed choose for (H', I'_W) in (6.28) a wavelet ϕ and its corresponding distribution Φ. The complete family $\phi_{x,t,v,f}$ is obtained according to (4.6) by action of the Weyl–Poincaré group. Besides, due to the covariance property expressed by relations (6.10) and (6.11), the associated family of hyperimages $\Phi_{x,t,v,f}$ has the form:

$$\Phi_{x,t,v,f}(\boldsymbol{x}', v', f') = \Phi\left(\Gamma(f,v)(\boldsymbol{x}' - \boldsymbol{x}), \frac{v' - v}{1 - (vv'/c^2)}, f^{-1}f'\right). \tag{6.29}$$

The unitarity equality (6.28) allows us to write:

$$\int I_W(x,t,v,f)\Phi^*_{x_0,t_0,v_0,f_0}(x,t,v,f)\,\mathrm{d}\mu\,(x,t,v,f) = |(H, \phi_{x_0,t_0,v_0,f_0})|^2. \tag{6.30}$$

The right-hand side of this relation is exactly the squared modulus of the wavelet coefficient $C(x_0, t_0, v_0, f_0)$ (cf (4.7)) which is thus given a new interpretation as an invariant regularization of the phase-space distribution I_W. This regularization is in fact a convolution on the Weyl–Poincaré group.

The operation defined by (6.30) has the merit of leading to a positive image. However, it involves a spreading of the function I_W which depends on the localization properties of the basic wavelet. From this point of view, the most attractive wavelets will be those leading to sharply localized images Φ_{x_0,t_0,v_0,f_0}. The construction of such wavelets has been discussed in section 5.

7. Relations of the study with some previous works

In [6], Feig and Grünbaum describe an assembly of moving objects by their response $\psi_e(t)$ to a pulse $\psi(t)$. In a narrow-band situation (high carrier frequency f_0, low velocities of observed points), they assume that the response of the target is independent of frequency and write the echo as

$$\psi_e(t) = \int_{R^2} D(r,y)\exp(-2\mathrm{i}\pi yt)\psi(t - 2r/c)\frac{2\,\mathrm{d}r}{c}\,\mathrm{d}y \tag{7.1}$$

where the function $D(r,y)$ is supposed to characterize a distribution of elementary scatterers located at r and having velocity $v \equiv cy/2f_0$. Due to the narrow-band approximation, the Doppler effect has been reduced to a translation y in the frequency.

In the imaging procedure proposed in the present paper, the elementary scatterers were allowed to have additional degrees of freedom, namely the frequencies they were able to reflect and their instants of appearance. It is thus interesting to investigate the relation between the two approaches and to find out under which conditions it is possible to recover the 'range-Doppler' characterization of the target given by the function D.

Noticing that the Fourier transforms of the fields $\psi_e(t)$, $\psi(t)$ are respectively equal to $\hat{\Phi}_{\text{out}}$, $\hat{\Phi}_{\text{in}}$ given in relations (2.1) and (2.2) and using definition (2.16) of the backscattering coefficient H, we obtain the following relation between H and D:

$$H(f_1, f_2) = \sqrt{f_2/f_1} \int_{\mathbb{R}} D(r, f_1 - f_2) e^{-4i\pi r f_1/c} \frac{2\,dr}{c}. \tag{7.2}$$

Since the analysis takes place in the vicinity of a high frequency f_0 and since the velocities v are supposed to be low, the frequencies f_1 and f_2 can be written as

$$f_1 \approx f_0 \left(1 + \frac{v}{c}\right) \qquad f_2 \approx f_0 \left(1 - \frac{v}{c}\right). \tag{7.3}$$

In this approximation, the factor f_2/f_1 in (7.2) can be neglected. Moreover, it will not make any difference in the following whether we extend the frequency integrals over $\mathbb{R}$ or $\mathbb{R}^+$. Under these conditions, relation (7.2) is rewritten in terms of physical variables as:

$$\tilde{H}(f, y) \equiv H(f_1, f_2) = \int_{\mathbb{R}} D(r, y) e^{-4i\pi(f_0 + y/2)r/c} \frac{2\,dr}{c} \tag{7.4}$$

and the following isometry formula holds:

$$(c/2) \int_{\mathbb{R}^2} |\tilde{H}(f, y)|^2 \,df\,dy = \int_{\mathbb{R}^2} |D(r, y)|^2 2\frac{dr}{c}\,dy. \tag{7.5}$$

The wavelet coefficient of H defined in (4.7) now becomes:

$$C(r, t, v, f) = \int_{\mathbb{R}^2} \tilde{H}(f', 2f_0 v'/c) e^{-4i\pi(f'/c)(-r+tv')} \phi(f' - f, v' - v)\,df'\,dv'. \tag{7.6}$$

For the present case, we choose the basic function ϕ of the form

$$\phi(f, v) = C_{\epsilon, f_0}(f)\delta(v) \tag{7.7}$$

where C_{ϵ, f_0} denotes the characteristic function of the interval $[f_0 - \epsilon, f_0 + \epsilon]$. If H is expressed in terms of D according to (7.4), the computation of the wavelet coefficient taking into account all the approximations, leads to the formula:

$$|C(r, t, v, f_0)|^2 = |D(r, 2f_0 v/c)|^2. \tag{7.8}$$

The r and v parameters of the individual scatterers appearing in the wavelet coefficient are the same as those in D. The time variable has disappeared, since the points have been supposed to be permanently present, and the frequency is fixed at f_0.

Consider now the image $I_W(x, t, v, f) \equiv P(x_1, x_2, f_1, f_2)$ constructed in section 6 using a generalized Wigner function. The narrow-band approximation of distribution $P(x_1, x_2, f_1, f_2)$ can be obtained by developing λ about $u = 0$ in (6.1) and changing variables to $u_i' = f_i u_i$, $i = 1, 2$. The result is:

$$P(x_1, x_2, f_1, f_2) = \int_{\mathbb{R}^2} e^{2i\pi(x_1 u_1' + x_2 u_2')} \times H(f_1 + u_1'/2, f_2 + u_2'/2) H^*(f_1 - u_1'/2, f_2 - u_2'/2)\,du_1'\,du_2'. \tag{7.9}$$

This function is easily expressed in terms of variables (x, t, v, f) as defined by (2.19) and (3.3) and yields the narrow-band approximation of $I_W(x, t, v, f)$ itself.

As a first test showing that this function represents correctly the elementary targets described in [6], we consider a point for which the density D is given by:

$$D(r, y) = \delta(r - r_0)\delta(y - y_0). \tag{7.10}$$

Computation of the corresponding backscattering coefficient by formula (7.2) and substitution of the resulting expression in (7.9) leads to:

$$I_W(x, t, v, f) = (1/f_0)\delta(x - r_0)\delta(v - v_0) \qquad y_0 \equiv 2f_0 v_0/c. \tag{7.11}$$

This result gives a phase-space interpretation of the points composing the target that is in accordance with the initial definition in [6, 7]. In the case of an arbitrary target, integrating $I_W(x, t, v, f)$ over $t \equiv (x_2 - x_1)/2$ yields:

$$\int_{\mathbb{R}} I_W(x, t, v, f)\,\mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{-4\mathrm{i}\pi f u/c} D(x + u/2, f_1 - f_2) D^*(x - u/2, f_1 - f_2) \frac{2\,\mathrm{d}u}{c}. \tag{7.12}$$

Recalling the assumption that the velocity is low and that the band is narrow and centred at a high-frequency f_0, we obtain finally:

$$\int_R I_W\,\mathrm{d}t\,\mathrm{d}f = |D(x, 2f_0 v/c)|^2. \tag{7.13}$$

This relation shows another aspect of the consistency of our results with those of [6]. In the approximation of narrow band and low velocity that they consider, we recover their density by integrating the pseudodistribution I_W over time and frequency. But our whole analysis has shown that the latter parameters were essential attributes of the elementary scatterers and that they should not be ignored.

The extension of the analysis in [6] to the wide-band, arbitrary velocity case has been attempted in [8]. In that work, the echo, $e(t)$, is assumed to be given in terms of the incident field, $s(t)$, by a formula generalizing (7.1) as

$$e(t) = \int_{R \times R^+} D(\xi, y)\sqrt{y}\, s(y(t - \xi))\,\mathrm{d}\xi\,\mathrm{d}y \tag{7.14}$$

where D is supposed to represent a 'density' of targets at distance $c\xi$ and velocity v such that $y = (1-v/c)/(1+v/c)$. Since in addition the function D belongs to $L^2(R \times R^+; y^{-2}\,\mathrm{d}\xi\,\mathrm{d}y)$, the dimensions of the incoming and outgoing fields in (7.14) have to be different. So there is some ambiguity in the definition of D and to make a comparison with our work, we will write:

$$H(f_1, f_2) = \int_R D(\xi, f_2/f_1)\mathrm{e}^{-2\mathrm{i}\pi f_2 \xi} f_1^{\gamma_1} f_2^{\gamma_2}\,\mathrm{d}\xi \tag{7.15}$$

and determine the unknown exponents γ_1, γ_2 by requiring the isometry formula in the form:

$$\int_{R^+ \times R^+} |H(f_1, f_2)|^2\,\mathrm{d}f_1\,\mathrm{d}f_2 = \int_{R \times R^+} |D(\xi, a)|^2 \frac{\mathrm{d}\xi}{a}\,\mathrm{d}a. \tag{7.16}$$

The values of the parameters are found to be:

$$\gamma_1 = -\tfrac{1}{2} \qquad \gamma_2 = 0. \tag{7.17}$$

It is possible to compute the phase-space representation of a point target corresponding to $D = \delta(x - \xi_0)\delta(y - y_0)$. Substitution of (7.15) into (6.1) with this form of D yields an expression of $I_W(x, t, v, f)$ that is proportional to Dirac distributions:

$$I_W(x, t, v, f) \propto \delta(v - v_0)\delta\left(x - \frac{\xi_0}{2}(1 - v_0/c) - v_0 t\right). \tag{7.18}$$

The interpretation of this formula is that a point of velocity v_0 is localized on a trajectory $x = v_0 t +$ constant while there is complete indetermination on its reflecting frequency f or its instant of appearance t. Indeed, a point with velocity v_0 cannot be permanently localized in x_0.

In the general case, it may be tempting to perform the integral of I_W with respect to t and f. However, it does not allow us to recover any density in position and velocity. This is related to the existence of an uncertainty relation between these variables which prevents them from having simultaneously definite values.

8. Conclusion

A technique of microwave imaging has been proposed for the study of time-varying radar targets. It has been developed for application to one-dimensional situations where the targets are characterized by their response functions $H(f_1, f_2)$ expressed in terms of the incoming and outgoing frequencies. In practice, the technique allows us to compute generalized images (called hyperimages) which are not only descriptive of the features of the targets but also of their kinematics.

As usual in radar imaging, the construction has been based on the adoption of a model of independent reflectors. The novelty lies essentially in the fact that each reflector has been endowed with a working frequency and an instant of existence in addition to its space position and its velocity. The resulting hyperimage is thus described by a function $I(x, t, v, f)$ of the four reflectors' coordinates. An expression of this function has been obtained by requiring the invariance of the imaging procedure in any change of inertial reference system. This has led us to express the hyperimage in terms of a wavelet transform generated by the Weyl–Poincaré group which actually governs the changes of the reference system. The main advantage of this approach is that all images constructed by different observers using the same mother wavelet Φ are directly related by changes of variables. In particular, this shows that the control of the uncertainty relations for the hyperimage pixels depends only on the choice of the numerical function Φ, whatever the reference system in use.

Going a step further, we have introduced a generalized Wigner function which provides an intrinsic treatment of the data and can be seen as the root of the previous wavelet analysis. Though it is not positive everywhere, this function of the four parameters (x, t, v, f) has strongly appealing features. It does not depend on any *a priori* choice of a basic function and provides the maximal resolutions in some circumstances. In particular it gives a sharp localization on specific curves for some well defined targets. Furthermore, the previous wavelet-based hyperimage can be recovered by performing a smoothing, in fact a convolution on the Weyl–Poincaré group of two such generalized Wigner functions, one corresponding to the analysed scattering function H and the other to the basic function Φ.

The method applies in priority to the analysis of scintillating targets. However, it can also be used to give a new interpretation to the Doppler-distance imaging of bidimensional static targets [1]. In that case, a target rotation is simulated and the target can be analysed as a moving object [21]. Finally, one may wonder why a relativistic group should be necessary to describe objects moving far below the velocity of light. Indeed, a low-velocity approximation would be quite sufficient. But, in the present context where the underlying Maxwell equations are Weyl–Poincaré invariant, the consideration of physically correct transformations has greatly simplified the work.

Appendix A. The Kirillov construction of phase space

In this appendix, the co-adjoint orbits of the Weyl–Poincaré group (i.e. the orbits of the co-adjoint representation) are constructed. The orbit associated with representation (2.15) yields the phase space $\mathcal{M}$ spanned by the parameters of the hyperimage (x, t, v, f).

The element g of group $W = A \times A$, where A is the affine group, can be represented by (a_1, a_2, b_1, b_2), $a_1, a_2 > 0$, b_1, b_2 real, and the group law is given by the multiplication of matrices with:

$$g = \begin{pmatrix} a_1 & 0 & b_1 \\ 0 & a_2 & b_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.1}$$

An element X of the Lie algebra $\mathcal{W}$ of W can then be written as

$$X = \begin{pmatrix} \alpha_1 & 0 & \beta_1 \\ 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.2}$$

The adjoint representation $\mathrm{ad}(g)$ of group W acting on $\mathcal{W}$ is defined by:

$$\mathrm{ad}(g)X \equiv gXg^{-1}. \tag{A.3}$$

Its action on the coordinates (ξ_1, ξ_2) of X defined by $\xi_i \equiv (\alpha_i, \beta_i)$ for $i = 1, 2$ can easily be deduced. In particular, the element g^{-1} is represented by:

$$\mathrm{ad}(g^{-1}) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{A.4}$$

where

$$M_i = \begin{pmatrix} 1 & 0 \\ a_i^{-1} b_i & a_i^{-1} \end{pmatrix}. \tag{A.5}$$

The coadjoint representation of the group W acting in the dual $\mathcal{W}^*$ of the Lie algebra $\mathcal{W}$ is defined by:

$$\mathrm{ad}^*(g) \equiv (\mathrm{ad}(g^{-1}))^{\mathrm{T}} \tag{A.6}$$

where T denotes the transpose. If an element X^* in $\mathcal{W}^*$ is represented by its coordinates $\xi_i^* \equiv (\alpha_i^*, \beta_i^*)$ in the dual basis, the adjoint representation has the following form:

$$\mathrm{ad}^*(g) \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} \equiv \begin{pmatrix} \xi_1^{'*} \\ \xi_2^{'*} \end{pmatrix} = \begin{pmatrix} M_1^{\mathrm{T}} & 0 \\ 0 & M_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix}. \tag{A.7}$$

If a particular point $\xi^* \equiv (\xi_1^*, \xi_2^*)$ has been chosen, the set of all possible transforms $\xi^{'*} \equiv (\xi_1^{'*}, \xi_2^{'*})$ obtained by (A.7) constitutes the orbit of ξ^*. There are orbits of dimension four, two and zero. The maximal ones are labelled by the signs of β_i^* and the orbit associated with representation (2.21) is seen to correspond to $\beta_i^* > 0$: this is the orbit $\mathcal{O}$ we are interested in and it is isomorphic to the space $\mathcal{M}$ used in the text.

A convenient parametrization is obtained by choosing a ξ^* such that

$$\alpha_i^* = 0 \qquad \beta_i^* = 1. \tag{A.8}$$

Then $\xi^{'*}$ computed by (A.7) is given by:

$$\alpha_i^{'*} = a_i^{-1} b_i \qquad \beta_i^* = a_i^{-1} \tag{A.9}$$

and the isomorphism between the orbit and the group $A \times A$ is thus displayed.

The orbit $\mathcal{O}$ is canonically endowed with an invariant symplectic form B defined by:

$$B = \sum_{i=1}^{2} \mathrm{d}\alpha_i^* \wedge \frac{\mathrm{d}\beta_i^*}{\beta_i^*} \tag{A.10}$$

giving rise to an invariant volume element equal to:

$$\prod_{i=1,2} \mathrm{d}\alpha_i^* \, \mathrm{d}\beta_i^* / \beta_i^*. \tag{A.11}$$

The variables x_i, f_i, used in the text, are recovered by setting:

$$x_i \equiv \alpha_i^* / \beta_i^* \qquad f_i = \beta_i^*. \tag{A.12}$$

It can be readily verified that they transform as in (2.20). The invariant volume element can be written in terms of these variables:

$$\mathrm{d}\mu\,(x_i, f_i) = \prod_{i=1,2} \mathrm{d}x_i \, \mathrm{d}f_i \tag{A.13}$$

or in terms of the physical variables $(\boldsymbol{x}, v, f)$ as

$$\mathrm{d}\mu\,(\boldsymbol{x}, v, f) = \frac{f}{c^2(1 - v^2/c^2)} \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}v \, \mathrm{d}f. \tag{A.14}$$

Thus the space of the image parameters is completely characterized as the orbit $\mathcal{M} \subset \mathcal{W}^*$ and supports a definite action of the Weyl–Poincaré group W.

Appendix B. The uncertainty relations

Generally speaking, the so-called uncertainty relations come from the non-commutation of some of the operations that are currently performed on the analysed signal. In the present case, the transformations considered were the changes of reference systems whose action on the backscattering coefficient $H(f_1, f_2)$ is given by (2.15). With this parametrization, the infinitesimal generators are the operators $\boldsymbol{f}_i$ and $\boldsymbol{B}_i$ characterizing respectively translations and dilations of the variables $x_i, i = 1, 2$ defined in (2.19). Their action on $H(f_1, f_2)$ is obtained from the representation (2.21) as follows:

$$(\boldsymbol{B}_i H)(f_1, f_2) \equiv -\frac{1}{2\mathrm{i}\pi} \frac{\mathrm{d}}{\mathrm{d}a_i} H_g(f_1, f_2) \bigg|_{(a_1=a_2=1, b_1=b_2=0)} \tag{B.1}$$

$$= -\frac{1}{2\mathrm{i}\pi} \left(f_i \frac{\mathrm{d}}{\mathrm{d}f_i} + \frac{1}{2} \right) H(f_1, f_2) \tag{B.2}$$

$$(\boldsymbol{f}_i H)(f_1, f_2) = f_i H(f_1, f_2). \tag{B.3}$$

The commutation relations between operators $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are all equal to zero except for the following:

$$[\boldsymbol{B}_i, \boldsymbol{f}_i] = -\frac{1}{2\mathrm{i}\pi} f_i. \tag{B.4}$$

Diagonalization of the operators $\boldsymbol{B}_i$ is performed using an adapted Mellin transform introduced according to:

$$\mathcal{M}[H](\beta_1, \beta_2) = \int_{R_+^2} H(f_1, f_2) f_1^{2\mathrm{i}\pi\beta_1 - \frac{1}{2}} f_2^{2\mathrm{i}\pi\beta_2 - \frac{1}{2}} \, \mathrm{d}f_1 \, \mathrm{d}f_2. \tag{B.5}$$

This transformation is invertible and isometric:

$$\int_{R^2} |\mathcal{M}[H](\beta_1, \beta_2)|^2 \, \mathrm{d}\beta_1 \, \mathrm{d}\beta_2 = \|H\|^2. \tag{B.6}$$

A direct computation using (B.2) and (B.5) yields:

$$\mathcal{M}[B_i H](\beta_1, \beta_2) = \beta_i \mathcal{M}[H](\beta_1, \beta_2). \tag{B.7}$$

This result is of the utmost importance for computations and will be used repeatedly in the following.

Let f_{i0} and β_{i0} denote the mean values of $\boldsymbol{f}_i$ and $\boldsymbol{B}_i$ defined by:

$$f_{i0} \equiv \langle \boldsymbol{f}_i \rangle \equiv \int_{R_+^2} f_i |H(f_1, f_2)|^2 \, \mathrm{d}f_1 \, \mathrm{d}f_2 \tag{B.8}$$

$$\beta_{i0} \equiv \langle \boldsymbol{B}_i \rangle \equiv \int_{R_+^2} [\boldsymbol{B}_i H(f_1, f_2)] H^*(f_1, f_2) \, \mathrm{d}f_1 \, \mathrm{d}f_2 \tag{B.9}$$

$$= \int_{\mathbb{R}^2} \beta_i |\mathcal{M}(\beta_1, \beta_2)|^2 \, \mathrm{d}\beta_1 \, \mathrm{d}\beta_2. \tag{B.10}$$

Let σ_{f_i} and σ_{β_i} represent the standard deviations corresponding to operators $\boldsymbol{f}_i$ and $\boldsymbol{B}_i$ respectively. Their computation gives:

$$\sigma_{f_i}^2 = \int_{R_+^2} (f_i - f_{i0})^2 |H(f_1, f_2)|^2 \, \mathrm{d}f_1 \, \mathrm{d}f_2 \tag{B.11}$$

$$\sigma_{\beta_i}^2 = \int_{R^2} (\beta_i - \beta_{i0})^2 |\mathcal{M}(\beta_1, \beta_2)|^2 \, \mathrm{d}\beta_1 \, \mathrm{d}\beta_2. \tag{B.12}$$

The uncertainty relations between the self-adjoint operators $\boldsymbol{f}_i$ and $\boldsymbol{B}_i$ are given by the general formula:

$$\sigma_{f_i}^2 \sigma_{\beta_i}^2 \geqslant -\tfrac{1}{4} \langle [\boldsymbol{f}_i, B_i] \rangle^2. \tag{B.13}$$

Thus, using (B.4), we obtain finally:

$$\sigma_{f_i}^2 \sigma_{\beta_i}^2 \geqslant \frac{\langle \boldsymbol{f}_i \rangle^2}{16\pi^2}. \tag{B.14}$$

The same procedure allows us to obtain the uncertainty relations between $\boldsymbol{B}^+$ and $\boldsymbol{f}$ or between $\boldsymbol{B}^-$ and $\boldsymbol{v}$.

Appendix C. The affine group and some of its associated Wigner functions

The affine or '$ax+b$'-group is the set of pairs (a, b), a positive, b real, with the composition law inherited from the multiplication of matrices

$$\begin{vmatrix} a & b \\ 0 & 1 \end{vmatrix}. \tag{C.1}$$

Thus

$$(a, b)(a', b') = (aa', b + ab'). \tag{C.2}$$

It is known [10] that the affine group has only two inequivalent unitary irreducible representations $U^{\pm}$. These may be realized in the space $L^2(\mathbb{R}^+, f^{2r+1}\, \mathrm{d}f)$ of functions $S(f)$ according to:

$$U_{a,b}^{\mp} S(f) = a^{r+1} \mathrm{e}^{\pm 2\mathrm{i}\pi bf} S(af). \tag{C.3}$$

Representations corresponding to different values of r are unitarily equivalent. Only U^+ will be used here and it will be denoted simply by U.

The pseudodistributions we are interested in are real sesquilinear forms of S given by:

$$P[S](x,f) \equiv \int_0^\infty \int_0^\infty K(x,f;v,v')S(v)S^*(v')\,\mathrm{d}v\,\mathrm{d}v'. \tag{C.4}$$

Under an affine transformation, they are required to transform pointwise, apart from a scaling factor a^q, q real. Thus

$$(a,b) : P(x,f) \longrightarrow a^q P(a^{-1}(x-b), af). \tag{C.5}$$

The correspondence between S and P is said to be affine-covariant provided the following relation holds:

$$P[U_{a,b}S](x,f) = a^q P[S](a^{-1}(x-b), af). \tag{C.6}$$

The general form of pseudodistributions (C.4) satisfying relation (C.6) is found to be:

$$P(x,f) = f^{2r-q+2}\int_0^\infty \int_0^\infty \mathrm{e}^{2\mathrm{i}\pi x f(v-v')}K(v,v')S(fv)S^*(fv')\,\mathrm{d}v\,\mathrm{d}v' \tag{C.7}$$

with the condition $K^*(v,v') = K(v',v)$ for P to be real.

Many specific distributions of this form have been studied [5, 22, 23]. Here we need only consider the subclass corresponding to *diagonal kernels*, i.e. distributions of the form:

$$P(x,f) = f^{2r-q+2}\int_0^\infty \int_0^\infty \mathrm{e}^{2\mathrm{i}\pi x f(\lambda(u)-\lambda(-u))}S(f\lambda(u))S^*(\lambda(-u))\mu(u)\,\mathrm{d}u \tag{C.8}$$

with λ a positive function and μ arbitrary. This amounts to choosing the kernel K in (C.7) as

$$K(v,v') = \int_{-\infty}^{\infty} \delta(\lambda(u)-v)\delta(\lambda(-u)-v')\mu(u)\,\mathrm{d}u. \tag{C.9}$$

The arbitrariness of functions λ and μ is limited by requiring that the distributions P have useful properties like unitarity and localizability.

The unitarity property, also called Moyal property in the case of the usual Wigner function, is written as

$$\int_{\mathbb{R}\times\mathbb{R}^+} P[S](x,f)P'[S'](x,f)f^{2q}\,\mathrm{d}t\,\mathrm{d}f = |(S,S')|^2 \tag{C.10}$$

where

$$(S,S') \equiv \int_0^\infty S(f)S'^*(f)f^{2r+1}\,\mathrm{d}f. \tag{C.11}$$

This property will be true for the distribution (C.8) provided the following conditions are satisfied:

(i) the functions $(\lambda(u)-\lambda(-u))$ and $\dfrac{\lambda(u)}{\lambda(-u)}$ (C.12)

are monotonic functions of u

(ii) $|\mu(u)|^2 = \left|\dfrac{\mathrm{d}}{\mathrm{d}u}(\lambda(u)-\lambda(-u))\dfrac{\mathrm{d}}{\mathrm{d}u}\ln\left(\dfrac{\lambda(u)}{\lambda(-u)}\right)\right|(\lambda(u)\lambda(-u))^{2r+2}$. (C.13)

There are two basic localization conditions concerning functions localized either in frequency S_{f_0} or in space S_{x_0}. The corresponding pseudodistributions, P_{f_0} and P_{x_0}, are required to have the same properties. This is expressed by the following correspondences

$$S_{f_0}(f) = f^{-r}\delta(f-f_0) \longrightarrow P_{f_0}(x,f) = f^{1-q}\delta(f-f_0) \tag{C.14}$$

$$S_{x_0}(f) = f^{-r-1}\mathrm{e}^{-2\mathrm{i}\pi f x_0} \longrightarrow P_{x_0}(x,f) = f^{-1-q}\delta(x-x_0). \tag{C.15}$$

The factors f arising in these expressions are needed for the covariance: a function S_{f_0}, for example, localized in f_0 must be localized in af_0 after an affine transformation (a, b) has been performed.

The condition for (C.14) to hold is just:

$$\mu(0) = 1. \tag{C.16}$$

To satisfy (C.15), on the other hand, there are two conditions:

$$\text{the mapping: } u \longrightarrow \lambda(u) - \lambda(-u) \text{ is one-to-one} \tag{C.17}$$

$$\mu(u) = \left|\frac{\mathrm{d}}{\mathrm{d}u}(\lambda(u) - \lambda(-u))\right| (\lambda(u)\lambda(-u))^{r+1}. \tag{C.18}$$

Finally, we investigate the possibility of satisfying simultaneously conditions (C.12), (C.13) and (C.16)–(C.18). Conditions (C.13) and (C.18) yield the following equation on $\lambda(u)$:

$$\frac{\mathrm{d}}{\mathrm{d}u}(\lambda(u) - \lambda(-u)) = \frac{\mathrm{d}}{\mathrm{d}u}\ln\left(\frac{\lambda(u)}{\lambda(-u)}\right). \tag{C.19}$$

Integrating this equation with the condition $\lambda(0) = 1$ and setting $V(u) = \lambda(u) - \lambda(-u)$ leads to the following expressions:

$$\lambda(u) = \frac{V(u)\mathrm{e}^{V(u)}}{\mathrm{e}^{V(u)} - 1} \qquad \lambda(-u) = \frac{V(u)}{\mathrm{e}^{V(u)} - 1}. \tag{C.20}$$

Hence

$$\mu(u) = \left|\frac{\mathrm{d}}{\mathrm{d}u}V(u)\right| (\lambda(u)\lambda(-u))^{r+1}. \tag{C.21}$$

Substituting these formulae into (C.8) and taking (C.17) into account to change variables from u to $u' = V(u)$, we find:

$$\begin{aligned} P(x, f) = {} & f^{2r-q+2}\int_0^\infty \int_0^\infty \mathrm{e}^{2\mathrm{i}\pi x f u} S\left(f\frac{u\exp(u/2)}{2\sinh(u/2)}\right) \\ & \times S^*\left(f\frac{u\exp(-u/2)}{2\sinh(u/2)}\right)\left(\frac{u}{2\sinh(u/2)}\right)^{2(r+1)} \mathrm{d}u. \end{aligned} \tag{C.22}$$

Thus there is a unique pseudodistribution of the diagonal class that is unitary and localized. It still has other properties, one of which is easily obtained. Integrating expression (C.22) of P with respect to x, we find:

$$\int_{\mathbb{R}} P(x, f)\,\mathrm{d}x = |S(f)|^2. \tag{C.23}$$

provided $q = 2r + 1$. This function P plays a central role among the affine invariant distributions and can be considered as the true analogue for the affine group of the usual Wigner function.

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