Chapter 5

Affine Time-Frequency Distributions

Jacqueline Bertrand and Pierre Bertrand

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1 Introduction

In this chapter, we depart from the usual family of time-frequency distributions singled out by L. Cohen in the context of quantum mechanics [1] and recalled in this book [2]. In fact, although most available representations have been shown to belong to Cohen's class [3], the existence of outer solutions in signal analysis cannot be ruled out by theoretical arguments as we shall see below. Affine time-frequency distributions occur in the framework of this extension.

Formally, the Cohen class [4] can be written as:

$$\rho_S(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi t(v'-v)} g(v'-v,\tau) \, e^{-j2\pi\tau(f-\frac{v+v'}{2})} S(v) S^*(v') d\tau dv dv' \tag{1}$$

where $S(f)$ is a general complex signal. Essential properties of ρ_S include its transformation as:

$$\rho_S(t,f) \to \rho_{S'}(t,f) = \rho_S(t-t_0, f-f_0) \tag{2}$$

when the signal is shifted in time and frequency according to:

$$S(f) \to S'(f) = e^{-j2\pi t_0 f} S(f-f_0) \tag{3}$$

and its invariance under a constant phase change ϕ of the form:

$$S(f) \to e^{i\phi}S(f) \tag{4}$$

Conversely, the above properties are sufficient to construct form (1) which can be proved to be the most general sesquilinear functional satisfying (2)-(3). A specific feature of signal theory lies in the formulation of phase invariance which is not based on the quantum like relation (4) but on the transformation:

$$S(f) \to e^{-i\phi}Z_-(f) + e^{i\phi}Z_+(f) \tag{5}$$

where $Z_-(f)$ and $Z_+(f)$ stand for the negative and positive frequency parts of S. This notion of phase invariance is needed to preserve the real character of any physical signal. Expression (1) is not invariant under transformations (5) of the signal unless it contains only positive (or negative) frequencies. This remark leads to characterize the Cohen class of signal theory by formula (1) in which the signal of interest is represented by its analytic form $Z(f)$ [5][6].

An important effect of the restriction of (1) to analytic signals is to suppress its justification by covariance arguments since frequency translations occuring in (2)-(3) then fail to preserve the space of signals. Such a lack of theoretical foundation would not be significant if the obtained distributions were above reproach. The reality is somewhat different and it is well known for example that the interest of Wigner-Ville representations is manifest only in narrow-band situations. This latter point is illustrated on figure 5.1 which gives the Wigner-Ville representation of a time-localized real signal $\delta(t - t_0)$ whose analytic form is given by:

$$Y(f)e^{-j2\pi f t_0}$$

with $Y(f)$ equal to the Heaviside unit step function. Clearly the localized character of the signal is not correctly displayed by the representation, especially at low frequencies.

As a matter of fact, motivations exist for improving the theoretical formulation of the subject. This can be done by introducing from the start the group A of affine transformations:

$$t \to at + b \tag{6}$$

which represents the effects of clock changes. This group is basic in signal analysis and has already received much attention in the context of wide-band ambiguity functions [7][8], as recalled in the chapter by J.Speiser, H.Whitehouse and J. Allen of this book [9]. Transformations (6) commute with the phase changes of type (5) and thus the two constraints of affine covariance and phase invariance can work together. Their exploitation allows to exhibit a class of

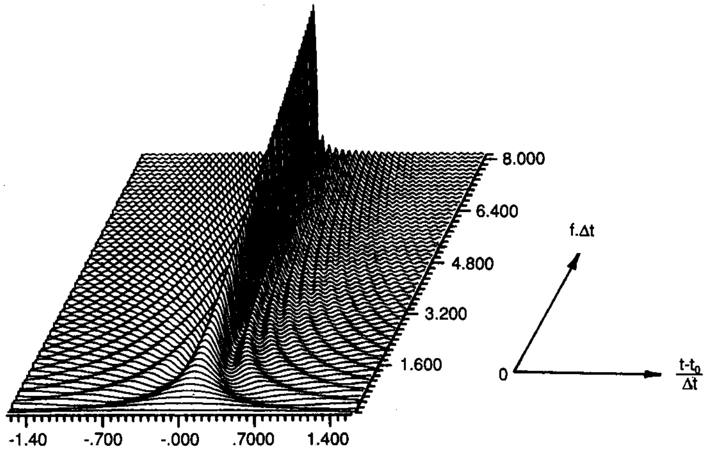


Figure 5.1. Wigner-Ville function of a sharp pulse signal.

representations which is the analogue of Cohen's class relative to the affine group (section 2). In the following, two different methods relying heavily on group theory will be used to derive explicit expressions of joint time-frequency distributions adapted to the analysis of wide-band signals (section 3 and 4). Affine smoothing is introduced in section 5 as a means to get positive distributions. Finally some hints for the implementation of the affine distribution are given in section 6.

2 The general affine covariant class

The affine group A consists of translations and dilations in time given by:

$$t \to at + b \qquad a > 0, b \in R \tag{7}$$

and leads to frequency dilations of the form:

$$f \to a^{-1} f \tag{8}$$

The composition law of elements (a, b) belonging to A is:

$$(a, b)(a', b') = (aa', b + ab') \tag{9}$$

Analytic signals are elements of the space $L^2(R^+)$ of square integrable functions on the positive axis. When a transformation (a, b) is performed, signal Z becomes Z_{ab}, defined by:

$$Z(f) \to Z_{ab}(f) = a^{r+1} e^{-j2\pi fb} Z(af) \tag{10}$$

This transformation law on signals constitutes a representation of group A since the following relation is satisfied:

$$Z_{aa',b+ab'} = (Z_{a'b'})_{ab}$$

In (10), the real scaling index r is left free as its choice depends on the physical context in which the signal arises. The inner product of signals Z and Z' is then defined by:

$$(Z, Z') = \int_0^\infty Z(f) Z'^*(f) f^{2r+1} df \tag{11}$$

It is invariant by transformation (10). Namely:

$$(Z, Z') = (Z_{ab}, Z'_{ab})$$

A joint time-frequency distribution will be defined here as a real sesquilinear functional of the signal which can be written as:

$$P(t, f) = \int \mathcal{K}(t, f; v, v') Z(v) Z^*(v') dv dv' \tag{12}$$

where the kernel $\mathcal{K}$ is such that:

$$\mathcal{K}^*(t, f; v, v') = \mathcal{K}(t, f; v', v) \tag{13}$$

The first task is to find out the most general form of the kernel that is compatible with affine covariance. The transformation law of P is chosen in accordance with (7), (8) to be:

$$P(t, f) \rightarrow a^q P(a^{-1}(t - b), af) \tag{14}$$

Here again, we allow a real free scaling factor q to be able to adapt the dimension to the physical meaning intended for P. The requirement of affine covariance can then be written in terms of P_Z and $P_{Z_{ab}}$ corresponding respectively to Z and Z_{ab}:

$$P_{Z_{ab}}(t, f) = a^q P_Z(a^{-1}(t - b), af) \tag{15}$$

The most general form satisfying this constraint is easily seen to be:

$$\begin{aligned} P_Z(t, f) \quad = \quad & f^{2r-q+2} \int_0^\infty \int_0^\infty e^{j2\pi t f(v - v')} K(v, v') \\ & Z(fv) Z^*(fv') dv dv' \end{aligned} \tag{16}$$

where the kernel K is such that

$$K^*(v, v') = K(v', v)$$

This expression has the same position wih respect to the affine group as Cohen's formula to the group of time and frequency translations.

Formula (16) can be extended to the whole time-frequency plane and will then represent arbitrary complex signals S according to :

$$P_S(t,f) = | f |^{2r-q+2} \int_0^{\infty} \int_0^{\infty} e^{j2\pi t f(v-v')} K(v,v')S(fv)S^*(fv')dvdv' \tag{17}$$

provided $K(v,v') = K(v',v)$ which implies K real. In this way, real signals which correspond to S such that:

$$S(f) = S^*(-f)$$

lead to $P(t,f)$ satisfying:

$$P(t,f) = P(t,-f)$$

The construction founded on the affine group ensures that no spurious interference occurs between positive and negative frequency parts. In the following, we will work with the analytic signal but the results will always be extendable in the form (17).

An interesting subclass of (16) is obtained when the kernel K is chosen diagonal, thus leading to the expression [10][11]:

$$P(t,f) = f^{2r-q+2} \int_{-\infty}^{\infty} e^{j2\pi t f(\lambda(u)-\lambda(-u))} Z(f\lambda(u))Z^*(f\lambda(-u))\mu(u)du \tag{18}$$

where μ is such that $\mu^*(-u) = \mu(u)$ and λ is any positive function verifying the following conditions:

(i) $\lambda(0) = 1$

(ii) The correspondence $u \rightarrow \frac{\lambda(u)}{\lambda(-u)}$ is a one-to-one mapping from R to R^+.

If in addition μ is real, positive and even, P satisfies the time reversal covariance; namely $P(-t,f)$ corresponds to the time reversed analytic signal $Z^*(f)$.

The affine time-frequency representation (18) has many properties; we mention the following, assuming that $\mu(0) = 2 \mid \lambda'(0) \mid$.

Instantaneous spectrum interpretation

$$\int_{-\infty}^{\infty} P_Z(t,f)dt = f^{2r-q+1} \mid Z(f) \mid^2 \tag{19}$$

Probabilistic expression of the time delay
Computation of the conditional mean:

$$\tau_g(f) \equiv \frac{\int_{-\infty}^{\infty} tP(t,f)dt}{\int_{-\infty}^{\infty} P(t,f)dt}$$

using (18) leads to the formula:

$$\tau_g(f) = -\frac{1}{2\pi}\frac{d\phi}{dt} \tag{20}$$

where ϕ is the argument of the analytic signal $Z(f)$. Obtaining (20) requires derivability of μ at $u = 0$ and this implies $\mu'(0) = 0$ since μ is even.

Usual narrow-band interpretation
If Z is different from zero only in a neighborhood of frequency f, its contribution to (18) will occur around $u = 0$. Developing λ about $u = 0$, we can write (18) as:

$$P(t,f) \simeq f^{2r-q+1} \int_{-\infty}^{\infty} e^{j2\pi t f u} Z(f(1+\frac{u}{2}))Z^*(f(1-\frac{u}{2}))f du \tag{21}$$

This coincides with the Wigner-Ville function provided $q = 2r + 1$.

In the following, two types of constraints are introduced directly on the kernel K (equation 16); they both lead to form (18) with specific expressions of λ and the above properties will always be satisfied.

3 The tomographic construction

In this approach, the emphasis is laid on the probabilistic aspect of the description. To give a feeling for the method, we first recall the situation in the case of the Wigner-Ville function W_S corresponding to an arbitrary complex signal $S(f)$ [5].

Function W_S can have negative values and hence is not a true probability. However it retains some probabilistic features. In particular, mean values of quantities $Q(t,f)$ defined on the time-frequency plane are computed using W_S as follows:

$$< Q >= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_S(t,f)Q(t,f)dtdf \tag{22}$$

In addition, W_S satisfies the marginal condition:

$$\int_{-\infty}^{\infty} W_S(t,f)dt = \mid S(f) \mid^2 \tag{23}$$

The r.h.s. of this equation is a positive density and this allows to interpret mean values of functions Q invariant by time translations (i.e. independent of t) as probabilistic averages:

$$< Q >= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_S(t, f) Q(f) dt df = \int_{-\infty}^{\infty} \mid S(f) \mid^2 Q(f) df$$

In fact the probability density occuring in this expression could as well be directly introduced as the square modulus of the signal components in a basis of functions invariant, up to a phase, by time translations. This latter approach permits to generalize the marginal condition (23) as:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_S(t, f) \delta(\gamma - t \cos\varphi + f \sin\varphi) dt df = \mid S(\gamma, \varphi) \mid^2 \quad (24)$$

where φ characterizes a given direction in the time-frequency plane. The quantity $S(\gamma, \varphi)$ is the coefficient of S on a chirp-like basis and is given, modulo a phase, by:

$$S(\gamma, \varphi) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\sin\varphi}} e^{j\pi\left(\frac{2\gamma t}{\sin\varphi} + \frac{t^2}{\tan\varphi}\right)} S(t) dt \quad (25)$$

This basis is derived from the condition of invariance (up to a phase) by time-frequency translations in the φ-direction. From an analytic point of view relation (24) has the form of a Radon transform [12] and its inversion gives a way of constructing the Wigner function [13].

Now, we will apply the same procedure to the affine group. We refer to [14] for more details. Because of this affine covariance, the instantaneous spectrum property becomes:

$$\int_{-\infty}^{\infty} P_Z(t, f) dt = f^{2r-q+1} \mid Z(f) \mid^2 \quad (26)$$

Again, it is interpreted as a time marginal and yields a density for functions Q on the time-frequency half-plane Γ^+ which are invariant by time translations. The generalization of (26) is obtained by considering arbitrary subgroups G_ξ of the affine group A (equation 7) labelled by a real number ξ and defined by the following relation connecting the parameters a, b of A:

$$b = \xi(1 - a)$$

The trajectories of G_ξ in Γ^+ through the point (t_0, f_0) are hyperbolas with equation:

$$t = \frac{\beta}{f} + \xi \quad (27)$$

where $\beta = f_0(t_0 - \xi)$. For each ξ, the associated basis is obtained by requiring its invariance up to a phase by action of G_ξ. The generalized marginal condition can then be written as:

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} P_Z(t,f)\delta(t - \frac{\beta}{f} - \xi) f^{q-1} dt df = | \mathcal{M}_\xi(\beta) |^2 \qquad (28)$$

where $\mathcal{M}_\xi$ is the Mellin transform of Z defined by:

$$\mathcal{M}_\xi(\beta) = \int_0^{\infty} Z(f) f^{j2\pi\beta + r} e^{j2\pi\xi f} df \qquad (29)$$

Considered for all values of β and ξ, the tomographic condition (28) defines P_Z by its Radon transform with respect to hyperbolas. An analogous transformation has also been used by Maas [15] in a different context. The inversion is easily performed using a Fourier transformation and yields [14]:

$$\begin{aligned} P_0(t,f) &= f^{2r-q+2} \int_{-\infty}^{\infty} e^{j2\pi t f u} Z(f\lambda_0(u)) Z^*(f\lambda_0(-u)) \\ &\quad \left(\frac{u}{2\sinh\frac{u}{2}}\right)^{2r+2} du \end{aligned} \qquad (30)$$

where $\lambda_0(u) = \dfrac{u e^{u/2}}{2\sinh\frac{u}{2}}$

Thus, P_0 belongs to the family of diagonal forms (18) with:

$$\mu(0) = 2\lambda'(0) = 1$$

As such, it satisfies relations (19) – (21). Moreover, the following important properties of this distribution together with its tomographic construction lead to consider P_0 as the true analogue of Wigner's function for the affine group.

Unitarity or 'Moyal' property

Unitarity of P_0 is expressed by the relation:

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} P_0(t,f) P_0'(t,f) f^{2q} dt df = | (Z, Z') |^2 \qquad (31)$$

where P_0, P_0' are the distributions corresponding to Z and Z' respectively.

In general, P_0 is not everywhere positive and this formula gives a way of constructing a smoothed positive version of (30). This will be developed in section 5.

Localization
Signals Z_{t_0} localized at time t_0 are defined so that their transformation by the affine group yields a signal localized at time at_0+b. Hence they must satisfy the relation:

$$Z_{at_0+b}(f) = a^{r+1}e^{-j2\pi fb}Z_{t_0}(af)$$

Up to a constant factor, the solution of this equation is:

$$Z_{t_0}(f) = f^{-r-1}e^{-j2\pi t_0 f} \tag{32}$$

The P_0-representation of such a signal is given by:

$$P_{t_0}(t,f) = f^{-q-1}\delta(t-t_0) \tag{33}$$

This result is in agreement with both constraints of localization in t_0 and of affine covariance defined by:

$$a^q P_{t_0}(a^{-1}(t-b), af) = P_{at_0+b}(t,f)$$

More generally, it can be verified that all signals of the form:

$$Z(f) = f^{-r-1}e^{-j2\pi\beta\ln f}e^{-j2\pi t_0 f} \tag{34}$$

are represented by localized distributions:

$$P_0(t,f) = f^{-q-1}\delta(t-t_0-\frac{\beta}{f}) \tag{35}$$

These states concentrated on hyperbolas in the time-frequency half-plane can be viewed as explicit realizations of "Doppler invariant" signals [16].

Extended covariance
It turns out that the joint distribution P_0 given by (30) is in fact covariant by a larger group G_0 of transformations $g \equiv (a,b,c)$ where (a,b) is an element of A and c is real. This group acts on the signal Z as:

$$Z(f) \rightarrow Z^g(f) \equiv a^{r+1}e^{-j2\pi(bf+c\ln f)}Z(af) \tag{36}$$

The resulting change on P_0 is:

$$P_Z \rightarrow P_{Z^g}(t,f) = a^q P_Z(a^{-1}(t-b-\frac{c}{f}), af) \tag{37}$$

Conversely the requirement of covariance by this three-parameter group G_0 would allow the construction of (30) up to an arbitrary function. This hints to a new method for constructing affine time-frequency distributions that will be presented in the next section.

4 Two special families of affine distributions

4.1 Extension of the affine covariance

Affine distributions are now constructed by requiring an extended covariance with respect to three-parameter groups which, like G_0 above, contain the affine group [17]. All such groups are well known from mathematical studies [18]. The most familiar one is probably $SL(2, R)$ which has been studied from a phase space viewpoint by A. and J. Unterberger [19]. It leads to a Wigner function which is not of the diagonal form (18). All other groups form a family G_k labelled by a real number k and defined as follows. For $k \neq 1, G_k$ is the group of elements $g = (a, b, c)$ with multiplication law:

$$gg' = (aa', b + ab', c + a^k c')$$

Group G_1 has the same elements (a, b, c) but its composition law is:

$$gg' = (aa', b + ab' + a(\ln a)c', c + ac')$$

Consider now the action of these groups G_k within the set of analytic signals Z belonging to $L^2(R^+)$ and denote by Z^g the signal transformed by g. Three different formulas are obtained according to the value of k:

$$Z^g(f) = a^{r+1} e^{-j2\pi(bf + cf^k)} Z(af), k \neq 0, 1 \tag{38}$$

$$Z^g(f) = a^{r+1} e^{-j2\pi(bf + c\ln f)} Z(af), \ k = 0 \tag{39}$$

$$Z^g(f) = a^{r+1} e^{-j2\pi(bf + cf\ln f)} Z(af), k = 1 \tag{40}$$

These can be shown to be the only admissible transformations. In particular, other representations of G_{-1} involving frequency translations are not allowed because they would drive Z out of the class of analytic signals. The action of G_k in the time-frequency half-plane can be rigorously derived by identifying phase space with the Kirillov orbit corresponding to each of the above representations [20]. This action, denoted as follows:

$$(t, f) \rightarrow g.(t, f)$$

induces a transformation law on P according to:

$$P(t,f) \rightarrow P^g(t,f) \equiv a^q P(g^{-1}.(t,f)) \tag{41}$$

The computation of P^g for each value of k yields:

$$P^g(t,f) = a^q P(a^{-1}(t-b-kcf^{k-1}), af), k \neq 0, 1 \tag{42}$$

$$P^g(t,f) = a^q P\Big(a^{-1}\Big(t-b-\frac{c}{f}\Big), af\Big),\ k=0 \tag{43}$$

$$P^g(t,f) = a^q P(a^{-1}(t-b-c-c\ln f), af), k=1 \tag{44}$$

The constraint of covariance by G_k is:

$$P_{Z^g}(t,f) = a^q P_Z(g^{-1}.(t,f)) \tag{45}$$

and can be written down using (38)–(40) and (42)–(44). The time-frequency distributions satisfying (45) and time reversal invariance are found to be of the general form (18) and are given explicitely by [21]:

$$P_k(t,f) = f^{2r-q+2} \int_{-\infty}^{\infty} e^{j2\pi t f(\lambda_k(u)-\lambda_k(-u))} Z(f\lambda_k(u))Z^*(f\lambda_k(-u))\mu_k(u)du \tag{46}$$

where

$$\lambda_k(u) = \Big(k\frac{e^{-u}-1}{e^{-ku}-1}\Big)^{\frac{1}{k-1}} \tag{47}$$

and μ_k is a real, positive and even function. Graphs of $\lambda_k(u)$ are given in figure 5.2. The cases of $G_k, k \neq 0, 1, G_0$ and G_1 had to be considered individually but the final result (46) is valid for any real k provided function (47) is defined by continuity for $k=0$ and $k=1$.

For any k the functions $\lambda_k(u)$ verify the properties:

$$\lambda_k(u) = e^u \lambda_k(-u)$$

and

$$\lambda_k(0) = 1 \qquad \lambda_k'(0) = \frac{1}{2}$$

As a result, whenever μ is such that $\mu(0) = 1$, the distribution P_k satisfies the general relations (19)–(21).

Remarks

- The action of G_k on P given in formulas (42)–(44) becomes natural when related to the time delay property (20). Indeed the derivative of the phase in (4.1) gives the time translation in (42)–(44).

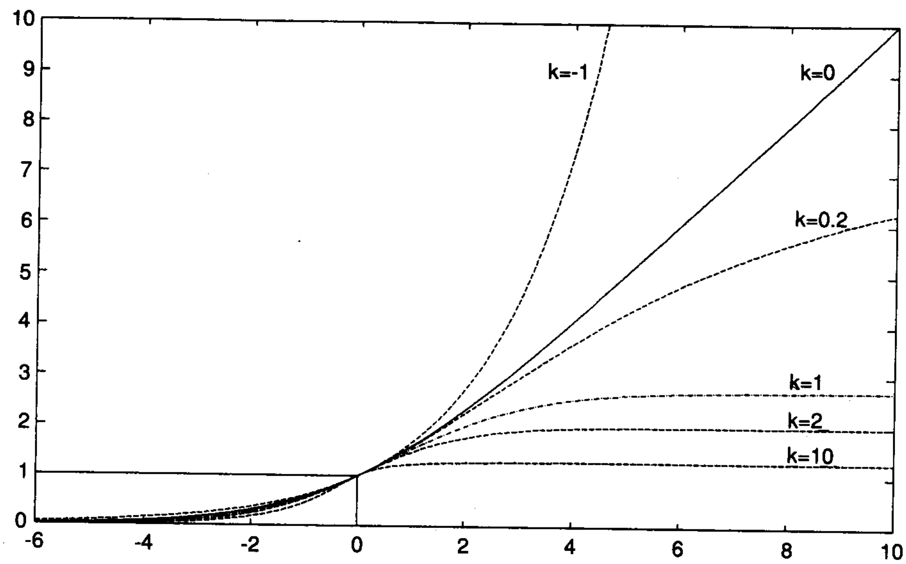


Figure 5.2. Graph of the λ_k-functions (4.5).

- Another interpretation of (46) is obtained by performing a Fourier transformation on time:

$$\begin{aligned}\hat{P}_k(\nu, f) &= \int_{-\infty}^{\infty} e^{-j2\pi t\nu} P_k(t, f)dt \\ &= f^{2r-q+2} \int_{-\infty}^{\infty} \delta(\nu - f\lambda_k(u) + f\lambda_k(-u)) \\ &\quad Z(f\lambda_k(u))Z^*(f\lambda_k(-u))\mu_k(u)du\end{aligned}$$

This operation shows that $\hat{P}_k(\nu, f)$ is related to $Z(f_1)Z^*(f_2)$ by a mere reparametrization of the first quadrant $(f_1, f_2) \rightarrow (\nu, f)$ defined by:

$$f_1 = f\lambda_k(u) \qquad f_2 = f\lambda_k(-u) \qquad \nu = f_1 - f_2$$

and illustrated in figure 5.3. For comparison, one may recall that the change of variables:

$$f_1 = f\lambda_{-1}(u) \qquad f_2 = f\lambda_{-1}(-u)$$

followed by a Fourier transformation on f leads to the symmetrized wide-band ambiguity function [22].

We will now investigate special cases of μ_k leading to distributions satisfying unitarity and/or localization properties.

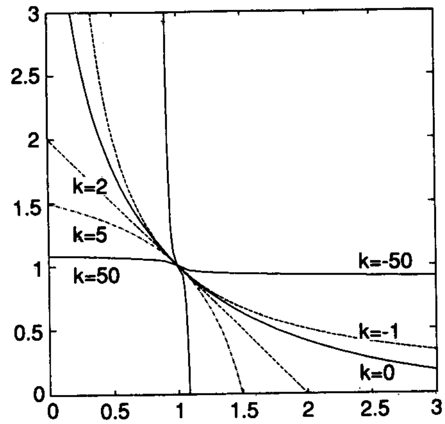


Figure 5.3. f_2/f as a function of f_1/f for various k.

4.2 A family of unitary distributions

The unitary or Moyal property (31) will be satisfied by distributions P_k provided μ_k is given by:

$$\mu_k^U = (\lambda_k(u)\lambda_k(-u))^{r+1}\Big(\frac{d}{du}(\lambda_k(u) - \lambda_k(-u))\Big)^{\frac{1}{2}} \tag{48}$$

From the properties of λ_k, it results that:

$$\mu_k^U(0) \equiv 1$$

Some special cases of unitary distributions will now be exhibited.

– When $k = 2$, formulas (46) and (48) lead to:

$$P_2^U(t,f) = f^{-q}\int_{-2f}^{2f} e^{-j2\pi tv} Z(f-\frac{v}{2})Z^*(f+\frac{v}{2})(f^2-\frac{v^2}{4})^{r+\frac{1}{2}}dv \tag{49}$$

Taking into account the fact that $Z(f) = 0$ when $f < 0$, we recognize the Wigner-Ville function restricted to the analytic signal provided $q = 0$ and $r = -\frac{1}{2}$. Here it must be stressed that the extension of expression (49) to the whole time-frequency plane performed according to formula (17) would not yield the usual Wigner-Ville function.

– When $k = 0$, the unitary distribution P_0^U coincides with the tomographic solution (30).

– When k tends to $\pm\infty$, the limits $\lambda_\pm$ of λ_k and the corresponding distributions $P_\pm^U$ can be written down. Taking the arithmetic mean of P_+^U and P_-^U, we get:

$$\frac{1}{2}(P_+^U + P_-^U) = f^{r-q+\frac{1}{2}}\Re\Big\{e^{-j2\pi tf}Z^*(f)\int_0^\infty e^{j2\pi tv}v^{r+\frac{1}{2}}Z(v)dv\Big\} \tag{50}$$

For $r = -\frac{1}{2}$, $q = 0$, this is exactly the Margenau-Hill-Rihaczek distribution [23].

4.3 A family of distributions with localization properties

We now require that localized signals Z_{t_0} given by (32) be represented by distributions P_{t_0} confined to the line $t = t_0$ (equation 33). This is possible provided the following two conditions are satisfied:

(i) the correspondence $u \rightarrow (\lambda_k(u) - \lambda_k(-u))$ is a one-to-one mapping from R to R.

(ii) μ is equal to:

$$\mu_k^L(u) = \Big(\frac{d}{du}(\lambda_k(u) - \lambda_k(-u))\Big)(\lambda_k(u)\lambda_k(-u))^{r+1} \tag{51}$$

Condition (i) implies that k is necessarily negative or zero for localizable distributions. This excludes, in particular, the case $k = 2$ which has been shown to correspond to functions of the Wigner type. Condition (ii) ensures that:

$$\mu_k^L(0) \equiv 1$$

The case k=0 stands out since the corresponding distribution can cumulate both properties of unitarity and localization. In fact, the only localized distribution P_k^L such that:

$$P_k^L \equiv P_k^U$$

is given by (30).

Another known example of localized distribution P_k^L is obtained for $k = -1$. This case is characterized by the functions:

$$\lambda_{-1}(u) = e^{\frac{u}{2}} \qquad \mu_{-1}(u) = \cosh\frac{u}{2}$$

and the corresponding time-frequency distribution is the "active Wigner function" of A. Unterberger [24]. Its main interest lies in the fact that it allows some analytic computations. Distribution P_{-1}^L can also be seen as the only translation covariant member in a class introduced by R. Altes [25].

For general negative k, the extended covariance (45) can be applied to localized signals Z_{t_0} and their representations P_{t_0}. The operation shows that the family of signals:

$$Z(f) = f^{-r-1}e^{-j2\pi cf^k}e^{-j2\pi t_0 f} \tag{52}$$

is represented by:

$$P_k^L(t,f) = f^{-q-1}\delta(t - t_0 - kcf^{k-1}) \tag{53}$$

This result is reminiscent of an analogous property of the Wigner-Ville function with respect to chirps. However it must be stressed that all signals arising here are analytic.

Distributions P_k^L are non-unitary for $k \neq 0$. Yet they can be paired with distributions P_k^M defined by the duality relation:

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} P_k^L(t,f)P_k'^M(t,f)f^{2q}dtdf = \mid (Z,Z') \mid^2$$

where P_k^L and $P_k'^M$ correspond to Z and Z' respectively. The expressions of P_k^M are given by (46) with

$$\mu_k^M = (\lambda_k(u)\lambda_k(-u))^{r+1}$$

The dual function P_k^M, $k \leq 0$, has a tomographic property, i.e. its integrals along curves $t = t_0 + kcf^{k-1}$ yield a positive density:

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} P_k^M(t,f)\delta(t - t_0 - kcf^{k-1})f^{q-1}dtdf = \mid \mathcal{N}(c,t_0) \mid^2 \qquad (54)$$

where

$$\mathcal{N}(c,t_0) = \int_0^{\infty} Z(f)e^{j2\pi cf^k}e^{j2\pi t_0 f}f^r df$$

It can be shown that the Radon transform in (54) is invertible and this gives a means of constructing P_k^M directly.

5 Affine smoothing

The construction of positive time-frequency representations can be performed through an affine smoothing. One way to proceed is to use Moyal's formula (31) with a special choice of P' as will now be discussed. If $\mathcal{Z}_{(0,f_0)}$ is a reference signal attached to the point $t = 0, f = f_0$, the signal $\mathcal{Z}_{(t,f)}$ is defined by affine transport as follows:

$$\mathcal{Z}_{(t,f)}(f') \equiv \Big(\frac{f}{f_0}\Big)^{-r-1} e^{-j2\pi f' t}\mathcal{Z}_{(0,f_0)}\Big(\frac{f_0}{f}f'\Big) \qquad (55)$$

Consider now distributions $\mathcal{P}_{(0,f_0)}$ and $\mathcal{P}_{(t,f)}$ corresponding respectively to signals $\mathcal{Z}_{(0,f_0)}$ and $\mathcal{Z}_{(t,f)}$ by formula (30). The expression of $\mathcal{P}_{(t,f)}$ is constrained by affine covariance to be equal to:

$$\mathcal{P}_{(t,f)}(t',f') = \Big(\frac{f_0}{f}\Big)^q \mathcal{P}_{(0,f_0)}\Big(\frac{f}{f_0}(t'-t), \frac{f_0}{f}f'\Big) \qquad (56)$$

A smoothed affine distribution $\tilde{P}(t,f)$ of $P(t,f)$ with respect to signal $\mathcal{Z}_{(0,f_0)}$ is then defined by:

$$\tilde{P}(t,f) = f^{-q} \int_{-\infty}^{\infty} \int_{0}^{\infty} P(t',f')\mathcal{P}_{(t,f)}(t',f') f'^{2q} dt' df' \qquad (57)$$

The factor f^{-q} ensures that $\tilde{P}$ transforms under scaling like P. Moreover, $\tilde{P}$ is positive by Moyal's formula (31) and equal to:

$$\tilde{P}(t,f) = \mid (Z, \mathcal{Z}_{(t,f)}) \mid^2 f^{-q} \qquad (58)$$

Using a recent terminology, we can say that the smoothed affine function $\tilde{P}(t,f)$ is, up to factor f^{-q}, the square modulus of the wavelet coefficient relative to the wavelet $\mathcal{Z}_{(0,f_0)}$ [26].

An important property of an adequate smoothing is to perturb as little as possible the original distribution. This will be realized if $\mathcal{P}_{(t,f)}$ is highly localized as was the Gaussian in the case of Wigner's function. To find the analogue for the affine case, we have to consider the specific uncertainty relations. They now occur between the spreads in the f and β variables where β is introduced by (29). Let σ_f denote the standard deviation of f defined by:

$$\sigma_f = \left[\int_0^{\infty} (f - f_0)^2 \mid Z(f) \mid^2 f^{2r+1} df\right]^{\frac{1}{2}} \qquad (59)$$

where $f_0 =< f >$ is the mean value of f. In the same way the standard deviation of β is defined by:

$$\sigma_\beta = \left[\int_{-\infty}^{\infty} (\beta - \beta_0)^2 \mid \mathcal{M}_\xi(\beta) \mid^2 d\beta\right]^{\frac{1}{2}} \qquad (60)$$

where $\beta_0 = \int_{-\infty}^{\infty} \beta \mid \mathcal{M}_\xi(\beta) \mid^2 d\beta$

The operator B corresponding to multiplication by β in the Mellin space is defined by:

$$\mathcal{M}_\xi[BZ] = \beta \mathcal{M}_\xi[Z]$$

and found equal to:

$$B = -\frac{1}{j2\pi}\left(f\frac{d}{df} + r + 1 + j2\pi\xi f\right)$$

Using this result and the isometry property of Mellin's transform which reads:

$$\int_0^{\infty} \mid Z(f) \mid^2 f^{2r+1} df = \int_{-\infty}^{\infty} \mid \mathcal{M}_\xi(\beta) \mid^2 d\beta \qquad (61)$$

we can write σ_β as:

$$\sigma_\beta = \left[\int_0^\infty \mid (B-\beta_0)Z \mid^2 f^{2r+1} df\right]^{\frac{1}{2}}$$

An efficient way of obtaining the uncertainty relations is to consider the following quadratic expression in the real variable λ:

$$h(\lambda) \equiv \| [B - \beta_0 + j2\pi\lambda(f - f_0)]Z \|^2 \geq 0$$

where $\| Z \|^2 \equiv (Z, Z)$ has been defined in (11). Minimizing h with respect to λ yields the inequality:

$$\sigma_\beta^2 \sigma_f^2 \geq \frac{f_0^2}{16\pi^2} \tag{62}$$

The equality holds only if $[B - \beta_0 + j2\pi\lambda(f - f_0)]Z = 0$ and this leads to the analytic minimal states [27] defined by:

$$Z(f) = N f^{2\pi\lambda f_0 - r - 1 - j2\pi\beta_0} e^{-2\pi(j\xi+\lambda)f} \tag{63}$$

Here, N is a normalization factor and $\lambda > 0$ is an adjustable parameter which characterizes the trade-off between the two spreadings along curves $f = f_0$ and $tf = \beta_0$ in the time-frequency plane. For $\beta_0 = \xi = 0$, states (63) are localized in a neighborhood of the curves $t = 0$ and $f = f_0$; they are good candidates for states $\mathcal{Z}_{(0,f_0)}$ occuring in (55). The localized character of these states is illustrated in figure 5.4 which gives an example of their P_0-representation.

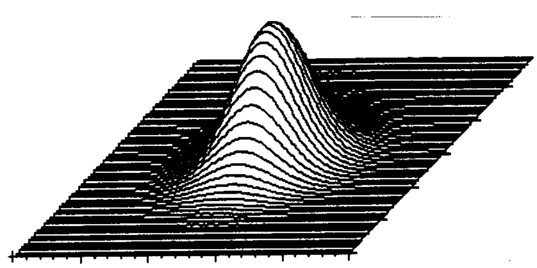


Figure 5.4. P_0-representation of the minimal signal (63).

6 Implementation

The task of computing expressions like (46) may look forbidding. In fact, the implementation is greatly simplified by using the Mellin transform (29) which converts dilation operations into multiplications by a phase factor. Computation can then be carried out with the help of a fast Mellin transform technique. As we shall see, the efficiency of the method is related to the time-frequency intepretation of the Mellin variable given by P_0.

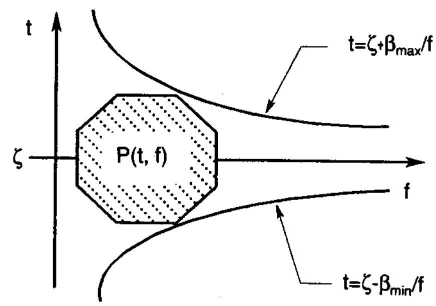


Figure 5.5. Hyperbolic boundaries of a signal.

Technically, the main step is a consistent discretization of the Mellin transform along the same lines as the discrete Fourier case [28]. Start from a signal $Z(f)$ whose P_0-representation can be considered as limited to a bounded domain of the time-frequency half-plane (figure 5.5). It results from the general property (19) that the support of $Z(f)$ is itself bounded. More generally, the integrals of $P_0(t, f)$ along hyperbolas $t = \beta/f + \xi$, ξ given, must vanish outside some β-interval and it results from (28) that the Mellin transform $\mathcal{M}_\xi[Z]$ must also vanish outside the same interval. This observation is at the starting point of the discretization of the transform. It will be developed for $\xi = 0$ but the general case can be easily obtained. Let f_1, f_2 and β_1, β_2 be the extreme points of the supports of Z and $\mathcal{M}_0[Z]$ respectively. The following operations are carried out successively (see table 5.1):

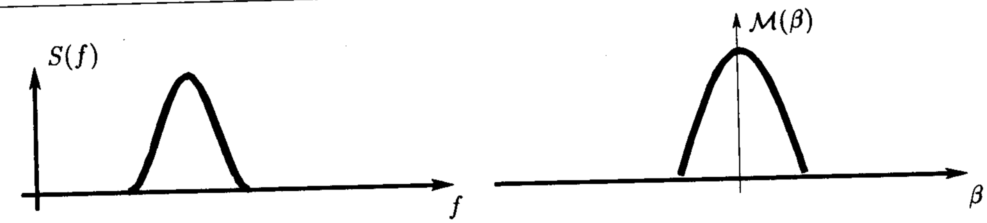
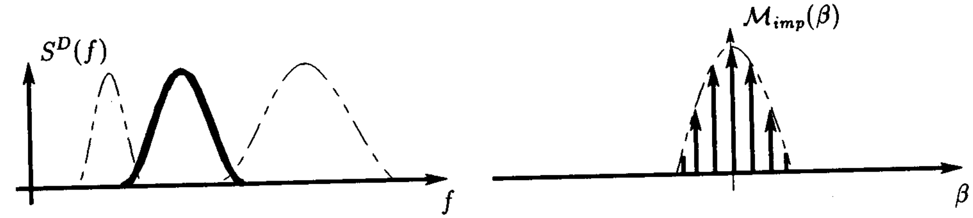
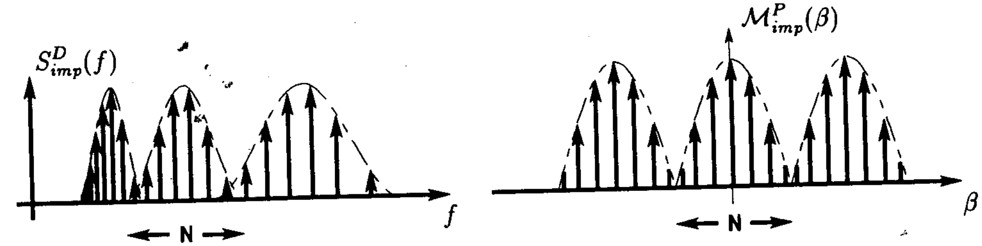
– Dilato-cycling of Z with ratio $Q \geq f_2/f_1$ defines a signal Z^D given by:

$$Z^D(f) = \sum_{n=-\infty}^{\infty} Q^{n(r+1)} Z(Q^n f) \tag{64}$$

– Mellin transform of Z^D. The result $\mathcal{M}_{imp}(\beta)$ is, up to a constant factor, the sampled form of $\mathcal{M}[Z]$:

$$\mathcal{M}_{imp}(\beta) = \frac{1}{\ln Q} \sum_{p=-\infty}^{\infty} \delta(\beta - \frac{p}{\ln Q}) \mathcal{M}(\beta) \tag{65}$$

Table 5.1. Graphical development of the discrete Mellin transform.

$S(f) = \int_{-\infty}^{\infty} \mathcal{M}(\beta) f^{-2i\pi\beta-r-1} d\beta$ $\mathcal{M}(\beta) = \int_{0}^{\infty} S(f) f^{2i\pi\beta+r} df$	
Dilatocycling in f-space ⇕ *Arithmetic sampling in β-space*	
Periodizing in β-space ⇕ *Geometric sampling in f-space*	

– Periodizing $\mathcal{M}_{imp}(\beta)$ with period $\frac{1}{\ln(q)} \geq | \beta_1 - \beta_2 |$ leads to a function denoted by $\mathcal{M}^P_{imp}(\beta)$.

– Inverse Mellin transform of $\mathcal{M}^P_{imp}$. This yields the geometric sampling of Z^D denoted by Z^D_{imp}.

The last two transformations will be consistent with the previous ones provided $Q = q^N$ with N a positive integer. In that case, $\mathcal{M}^P_{imp}$ will be a regularly sampled periodic function and Z^D_{imp} will be both dilato-cyclic and geometrically sampled.

Computation then yields the direct and inverse formulas for the discrete Mellin transform

$$\mathcal{M}^P(\frac{p}{\ln Q}) = \frac{\ln Q}{N} \sum_{k=M}^{M+N-1} q^{k(r+1)} e^{j2\pi k \frac{p}{N}} Z^D(q^k) \tag{66}$$

$$Z^D(q^k) = \frac{q^{-k(r+1)}}{\ln Q} \sum_{p=K}^{K+N-1} e^{-j2\pi p \frac{k}{N}} \mathcal{M}^P(\frac{p}{\ln Q}) \tag{67}$$

The integers K and M are determined by the supports of $\mathcal{M}(\beta)$ and $Z(f)$ respectively. The practical exploitation of these discretized formulas can be carried out using any FFT algorithm.

The application of formulas (66), (67) to the computation of functionals such as (46) makes use of general results concerning the correspondence between multiplication and convolution in the two dual spaces [28]. For an illustration we give a scheme for the implementation of (30) in the case $q = 2r + 1$. The value of the parameter ξ in the Mellin transform has to be chosen in accordance with the time interval of the computation. However, in practice, a time translation of the signal is always possible in order to be able to work with the special form $\xi = 0$. In a first step, the Mellin transform $\mathcal{M}_0[P_0]$ of (30) with respect to f is expressed in terms of the transform $\tilde{\mathcal{M}}_0(\beta)$ of the function $\tilde{Z}(f) \equiv f^{-\frac{r}{2}} Z(f)$. Setting $tf = \eta$ and using properties of $\mathcal{M}$, we obtain:

$$\mathcal{M}[P_0](\beta) = \int_{-\infty}^{\infty} e^{j2\pi\eta u} (\lambda(u)\lambda(-u))^{\frac{r}{2}}$$
$$\left[\int_{-\infty}^{\infty} (\lambda(u))^{-j2\pi\beta'} \tilde{\mathcal{M}}(\beta') \tilde{\mathcal{M}}^*(\beta' - \beta) \right.$$
$$\left. (\lambda(-u))^{-j2\pi(\beta-\beta')} d\beta' \right] du$$

This latter form clearly shows that the final result can be obtained by a succession of FFT operations.

Analogous techniques have also been used for the computation of wide-band ambiguity functions [29] and wavelet coefficients. Whatever the application, it is interesting to note that the exact expression of the Mellin transform and the interpretation of the dual β variable have emerged from the construction of the distribution P_0. This result is one more example of the interest of time-frequency methods in signal analysis.

7 Summary

The constraint of affine covariance is shown to be relevant for the derivation of time-frequency representations. It works on the space of analytic signals and leads to a class of "affine" distributions which is the counterpart of the Cohen class associated with time and frequency translations. A subclass is exhibited with the property, among others, of approaching Wigner-Ville's function when bandwidth goes to zero. In this subclass two families of distributions are selected by a principle of extended covariance. The first one consists of distributions which verify unitarity also called Moyal's property. The second family is composed of distributions ensuring accurate representation for time localized signals. Only one distribution, labelled P_0, belongs simultaneously to the two families. This distribution, which is also obtained by the tomographic approach, plays the same role in the affine class as the Wigner-Ville distribution in Cohen's class. In particular, it associates delta functions on hyperbolas with analytic Doppler-invariant signals, thus exhibiting a property analogous to Wigner-Ville's with respect to chirps.

Extended forms of the various affine distributions are also introduced to obtain representations of complex signals on the whole time-frequency plane. The use of the real signal in these forms has just the effect of producing a symmetrization of the result obtained with the analytic signal. In any case, the construction based on the affine group guarantees that no spurious interference will ever occur between positive and negative frequencies.

A practical result of the study concerns the use of the Mellin transform in signal theory. Indeed distribution P_0 gives a time-frequency interpretation of the dual Mellin variable and thus allows to develop discretization of the transform with correct use of sampling theorems. This leads to a fast procedure for the computation of broad-band functionals containing stretched forms of the signal such as broad-band ambiguity functions, wavelet coefficients and affine distributions.

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