WIDE-BAND AMBIGUITY FUNCTION AND $a \cdot x + b$ GROUP

LOUIS AUSLANDER* AND IZIDOR GERTNER*†

Abstract. In this paper a wide-band ambiguity function is defined. Conditions when the narrow-band approximation is valid are stated. It is shown that the wide-band ambiguity function is a coefficient function of the unitary representation of the $a \cdot x + b$ group. The relation to the "wavelet" expansion is demonstrated. Computational algorithm for the wide-band ambiguity function is presented.

1. Introduction. The classical narrow-band ambiguity function is defined for signals where the Doppler effect is approximated by a frequency shift that is constant across the bandwidth. Therefore the signal spectral shape is not changed. Such signals are called quasiharmonic. Let $s(t)$ be a quasiharmonic signal:

$$s(t) = | s(t) | e^{i\phi(t)} \tag{1}$$

$| s(t) |$ can be directly interpreted as the envelope of the real signal and the quantity

$$\frac{1}{2\pi} \cdot \frac{d\phi(t)}{dt} \tag{2}$$

as its instantaneous frequency.

Let $S(f) = \mathcal{F}\{s(t)\}$ be the Fourier transform of $s(t)$ with f_0 as its central frequency defined as:

$$f_0 = \int_{-\infty}^{\infty} f \mid S(f) \mid^2 df$$

The Doppler shift is then:

$$f_d = 2\pi \frac{v}{c} \cdot f_0 \tag{3}$$

$$\begin{aligned} where: c - &\quad \textit{the signal propagation velocity} \\ v - &\quad \textit{the target velocity} \end{aligned}$$

The spectra of the signal is linearly shifted by the quantity f_d. This approximation is valid when the energy is concentrated near the central frequency f_0 . We call such signals narrow-band. The use of these signals has been widely spread in classical radar and communication.

Unfortunately, in many cases such as signals arising in sonar, seismology, oceanography the narrow-band approximation is not valid. The use of the narrow band ambiguity function may cause an error in the estimation of a velocity and

*Center for Large Scale Computation, The Graduate School and University Center, CUNY, 25 West 43rd street , suite 400, New York , N.Y. 10036

†Supported by DARPA-ACMP 770997 and in part by IMA with funds provided by NSF

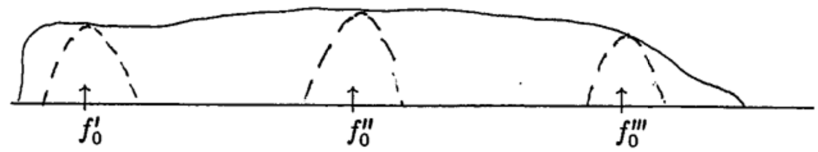

Figure 1: Wide-band signal spectra.

distance. For these signal the deviation from the central frequency is large. Thus when the target is moving there is no uniform Doppler shift for the entire spectra. We can illustrate this heuristically in the following figure 1.

Let f_0' *and* f_0'' be two frequencies in the signal spectra. Consider the neighborhood of the frequencies f_0' and f_0'' According to (3) the Doppler shift will be different in each neighborhood thus causing the signal spectra to strech or to compress. Therefore a more general than narrow-band signal model is required. In [SW66] a signal model that is independent of bandwidth and central frequency was proposed. We will describe this model below.

Let $s_1(t) \in L^2(\Re)$ represent a signal with a constant propogation velocity c . Let $s_2(t)$ be an echo of $s_1(t)$ from a target moving with velocity v . Then

$$s_2(t) \;=\; As_1(a\cdot t - \tau) \tag{4}$$

$$\begin{aligned} where: A & \quad - The\ energetic\ parameter\ of\ the\ signal \\ a &= \frac{c-v}{c+v} \quad - The\ Doppler\ strech - compress\ factor \\ \tau &= \frac{2R_0}{c+v} \quad - The\ signal\ delay\ at\ time\ t=0 \\ R &= R_0 + vt \end{aligned}$$

For our purpose we will ignore signal propogation attenuation and we will normalize to one the signal energy.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \mid s_2(t) \mid^2 dt \\ &= A^2 \int_{-\infty}^{\infty} \mid s_1(a\cdot t - \tau) \mid^2 dt \\ &= \frac{A^2}{a} \int_{-\infty}^{\infty} \mid s_1(t) \mid^2 dt \\ &= \frac{A^2}{a} \end{aligned}$$

Thus the conservation of energy requires that $A = \sqrt{a}$ and the received signal can be represented as:

$$\begin{aligned} s_2(t) &= \sqrt{a}\cdot s_1(a\cdot t - \tau) \\ &= \sqrt{a}\cdot s_1(a(t-\tau')). \end{aligned} \tag{5}$$

$$\tau' = \frac{2R_0}{c-v}. \tag{6}$$

The problem is to estimate the Doppler parameter a and the delay τ. The maximum likelihood estimator of these parameters is the *max* of the crosscorrelation function :

$$\max_{a,\tau} \quad | \sqrt{a} \int_{-\infty}^{\infty} s_1(t) \cdot s_1^*(a(t-\tau))dt|^2 \tag{7}$$

This suggests the definition of the wide-band ambiguity function:

$$WA_{f,g}(a,\tau) = \sqrt{a} \int_{-\infty}^{\infty} f(t) \cdot g^*(a(t-\tau))dt \tag{8}$$

The maximum likelihood estimator requires one to find the maximum of the wide -band ambiguity function surface between the received signal and the reference signals.

In this paper we will present properties of the wide-band ambiguity function. We will show that wide-band ambiguity function is a unitary representation of the $a \cdot x + b$ group. We will present a computational algorithm to compute the wide-band ambiguity function.

2. Wide-band ambiguity function properties and the $a \cdot x + b$ group.

In this section we will show how to connect the wide-band ambiguity function with a representation of the affine or $a \cdot x + b$ group. Recently, the concept of wavelet expansion has been introduced. A good survey has been presented in [ID88]. We will also show how to relate the wide-band ambiguity function with the wavelets. First we will recall the representation of the $a \cdot x + b$ group. Reader who is familiar with the subject may skip the subsection.

2.1. Relation of the narrow-band and wide-band ambiguity functions. In this section we will show when the wide-band ambiguity function can be approximated by the narrow-band ambiguity function. To do this, let $s(t)$ be a signal and $S(f)$ the Fourier transform of $s(t)$. Then

$$\begin{aligned} s(a(t-\tau)) &= \int_{-\infty}^{\infty} S(f) e^{2\pi i f a(t-\tau)} df \\ &= \int_{-\infty}^{\infty} e^{-2\pi i f a \tau} \cdot S(f) e^{2\pi i f a t} df. \end{aligned} \tag{9}$$

Let f_0 be the central frequency of the spectra $S(f)$. Also denote

$$\begin{aligned} \Delta f &= f - f_0 && - \textit{the frequency deviation}, \\ \delta &= \frac{2v}{c+v} && - \textit{the relative velocity}, \\ a &= 1 - \frac{2v}{c+v} = 1 - \delta && - \textit{the Doppler factor}. \end{aligned}$$

Then

$$(10)\qquad s(a(t-\tau)) = e^{-2\pi i f_0 \delta(t-\tau)} \cdot \int_{-\infty}^{\infty} S(f) e^{2\pi i f(t-\tau)} \cdot e^{-2\pi i \Delta f \delta(t-\tau)} df$$

There are two factors affecting the signal behavior in time

$$e^{-2\pi i f_0 \delta(t-\tau)} \ and \ e^{-2\pi i \Delta f \delta(t-\tau)}$$

The first one shifts the signal spectra ; the second one stretches/compresses the signal envelope. If the target-source velocity $v \ll c$ then $\delta \ll 1$ and thus

$$(11)\qquad e^{-2\pi i \Delta f \delta(t-\tau)} \approx 1$$

In this case

$$(12)\qquad s(a(t-\tau)) = e^{2\pi i f_d (t-\tau)} s(t-\tau),$$

where

$$f_d = -f_0 \delta = f_0(a-1)$$

is the classical Doppler shift. From (8) and (5) we get that the wide-band ambiguity function can be written as:

$$\begin{aligned} WA_{h,g}(a,\tau) = &\ \sqrt{a} \int_{-\infty}^{\infty} h(t) e^{-2\pi i f_0 (a-1)(t-\tau)} \cdot g^*(t-\tau) dt \\ = &\ \sqrt{a} e^{2\pi i f_0 (a-1)\tau} \int_{-\infty}^{\infty} h(t) \cdot g^*(t-\tau) e^{-2\pi i f_0 (a-1) t} dt \\ (13)\qquad &\ \sqrt{a} e^{2\pi i f_0 (a-1)\tau} \cdot NA_{h,g}(a,\tau), \end{aligned}$$

where

$$(14)\qquad NA_{h,g}(a,\tau) = \int_{-\infty}^{\infty} h(t) \cdot g^*(t-\tau) e^{-2\pi i f_0 (a-1) t} dt$$

is the narrow-band ambiguity function.

The approximation error is given as:

$$(15)\qquad WA_{h,g}(a,\tau) - NA_{h,g}(a,\tau) = NA_{h,g}(a,\tau) \left\{ \sqrt{a} \cdot e^{2\pi i f_0 (a-1)\tau} \right\}$$

An upper bound on the relative approximation error was obtained by Swick[SW69] in terms of J - the Woodward signal duration and β the Woodward signal bandwidth:

$$(16)\qquad | WA(\tau,a) | - \sqrt{a} \,| NA(\tau,a) | \leq \frac{\sqrt{a}}{2\pi} \cdot | \delta | \cdot J \cdot \beta$$

In any case from the above , when the narrow-band approximation is acceptable, we can determine the limiting strech factor a .

From the above discussion we can conclude that factors affecting the narrow-band approximation are:

(1) δ - the target velocity,
(2) $\Delta f = f - f_0$ the signal frequency deviation,
(3) $t - \tau$ - the observation interval.

For sonar signals we usually have the deviation Δf very large, the observation time is usually large also, since one wants to receive more energy in order to improve signal to noise ratio. In this case the narrow band ambiguity function is not valid and one should use the wide-band ambiguity function.

2.2. The $a \cdot x + b$ group.

The $a \cdot x + b$ group is a set $\Re \setminus \{0\} \times \Re$ equipped with the group multiplication law:

$$(a, b) \circ (a', b') = (a \cdot a', b + a \cdot b') \tag{17}$$

The unity of the group is $(1, 0)$:

$$\begin{aligned} (1,0) \circ (a, b) &= (a, b) \\ (a, b) \circ (1,0) &= (a, b) \end{aligned} \tag{18}$$

The right and left inverse of (a, b) is $(\frac{1}{a}, -\frac{b}{a})$:

$$\begin{aligned} (a, b) \circ \left(\frac{1}{a}, -\frac{b}{a}\right) &= (1,0) \\ \left(\frac{1}{a}, -\frac{b}{a}\right) \circ (a, b) &= (1,0) \end{aligned}$$

The group is non-commutative:

$$\begin{aligned} (a, b) \circ (a', b') &= (aa', b + ab') \\ (a', b') \circ (a, b) &= (aa', b' + a' \cdot b). \end{aligned}$$

2.3. Unitary representation. A representation of the group $\Re \setminus \{0\} \times \Re$ in a space $L^2(\Re)$ is a pair:

$$\{\mathfrak{U}, L^2(\Re)\}$$

where $\mathfrak{U}$ is a mapping which assigns to every element $(a, b) \in \Re \setminus \{0\} \times \Re$ a linear mapping

$$\mathfrak{U}_{(a,b)} : L^2(\Re) \Rightarrow L^2(\Re)$$

such that

$$\mathfrak{U}_{(1,0)} = 1_{L^2(\Re)} \tag{19}$$

$$\mathfrak{U}_{((a,b)\circ(a',b'))} = \mathfrak{U}_{(a,b)} \circ \mathfrak{U}_{(a',b')}, \quad \forall\, (a, b), (a', b') \in \Re \setminus \{0\} \times \Re \tag{20}$$

In particular

$$\mathfrak{U}_{(a,b)} \circ \mathfrak{U}_{(\frac{1}{a}, -\frac{b}{a})} = \mathfrak{U}_{(1,0)}$$

If $L^2(\Re)$ is a Hilbert space, a linear representation $\{\mathfrak{U}, L^2(\Re)\}$ is said to be unitary if the automorphism $\mathfrak{U}_{(a,b)}$ forms a unitary operator of

$$L^2(\Re) \quad \forall (a, b) \in \Re \setminus \{0\} \times \Re$$

$\mathfrak{U}$ defines a morphism $(a,b) \to \mathfrak{U}_{(a,b)}$ of the group $\Re \setminus \{0\} \times \Re$ into a unitary group $\underline{\mathfrak{U}}(L^2(\Re))$ of $L^2(\Re)$ operators such that

$$\mathfrak{U}_{(a,b)^{-1}} = \left(\mathfrak{U}_{(a,b)}\right)^* \quad \forall\, (a,b) \in \Re \setminus \{0\} \times \Re \tag{22}$$

Unitarity is equivalent to:

$$\left\|\mathfrak{U}_{(a,b)} f\right\| = \|f\| \quad \forall\, (a,b), \in \Re \setminus \{0\} \times \Re;\ f \in L^2(\Re) \tag{23}$$

or

$$\langle \mathfrak{U}_{(a,b)} f, \mathfrak{U}_{(a,b)} g\rangle = \langle f, g\rangle \quad \forall\, (a,b), \in \Re \setminus \{0\} \times \Re \quad f, g \in L^2(\Re) \tag{24}$$

Define the translation representation

$$\left(T_{(1,s)}\right)(f) = f(t-s) \qquad (1,s) \in \Re \setminus \{0\} \times \Re. \tag{25}$$

It is easy to see that the representation T is unitary and commutative

$$\begin{aligned} T_{(1,s),(1,s')} &= T_{(1,s+s')} \\ &= T_{(1,s)} \circ T_{1,s')} \end{aligned} \tag{26}$$

The unitarity of the translation representation follows from:

$$\begin{aligned} \langle T_{(1,s)} h, T_{(1,s)} g\rangle &= \int_{-\infty}^{\infty} h(t-s) \cdot g^*(t-s) dt \\ &= \langle h, g\rangle \end{aligned} \tag{27}$$

Define the dilation representation by

$$\left(D_{(\lambda,0)} h\right)(t) = \mid \lambda \mid^{-\frac{1}{2}} \cdot h\left(\frac{t}{\lambda}\right) \qquad \forall (\lambda, 0) \in \Re \setminus \{0\} \times \Re. \tag{28}$$

It is easy to see that representation D is commutative

$$\begin{aligned} D_{(\lambda,0)} \circ D_{(\lambda',0)} &= D_{(\lambda,0)\circ(\lambda',0)} \\ &= D_{(\lambda\cdot\lambda',0)} \end{aligned} \tag{29}$$

and

$$\begin{aligned} D_{(\lambda',0)} \circ D_{(\lambda,0)} &= D_{((\lambda',0)\circ(\lambda,0))} \\ &= D_{(\lambda'\cdot\lambda,0)} \end{aligned} \tag{30}$$

It easy to see that the dilation representation is unitary

$$\begin{aligned}\langle D_{(\lambda,0)}h, D_{(\lambda,0)}g\rangle &= \int_{-\infty}^{\infty} \mid \lambda \mid^{-\frac{1}{2}} \cdot h\left(\frac{t}{\lambda}\right) \cdot \mid \lambda \mid^{-\frac{1}{2}} \cdot g^{*}\left(\frac{t}{\lambda}\right) dt \\ &= \int_{-\infty}^{\infty} h(t) \cdot g^{*}(t) dt \\ &= \langle h, g\rangle \end{aligned} \tag{31}$$

Consider the product of dilation and translation representations in the set of unitary representations.

$$DT_{\lambda,s)} \triangleq D_{(\lambda,0)} \circ T_{(1,s)} \tag{32}$$

$$\begin{aligned} DT_{(\lambda,s)} &= D_{(\lambda,0)} \circ \left(T_{(1,s)}h\right)(t) \\ &= D_{(\lambda,0)}h(t-s) \\ &=\mid \lambda \mid^{-\frac{1}{2}} \cdot h\left(\frac{t-s}{\lambda}\right) \end{aligned} \tag{33}$$

The translation and the dilation representations are not commutative

$$\begin{aligned} \left(T_{(1,s)} \circ D_{(\lambda,0)}\right) h &=\mid \lambda \mid^{-\frac{1}{2}} \cdot h\left(\frac{t-\lambda s}{\lambda}\right) \\ &= D_{(\lambda,0)} \circ T_{(1,\lambda s)} \end{aligned} \tag{34}$$

We will introduce a different notation for this unitary representation and call it affine coherent states. This notion has been borrowed from quantum mechanics and is used in the wavelet theory[AK68],[ID88]

$$h^{(\lambda,s)}(t) =\mid \lambda \mid^{-\frac{1}{2}} \cdot h\left(\frac{t-s}{\lambda}\right) \tag{35}$$

The inner product of the affine coherent states and any $g() \in L^{2}(\Re)$ is the wide -band ambiguity function that we have already defined

$$\begin{aligned} WA_{g,h)}(\lambda,\tau) &= \langle g, h\rangle \\ &=\mid \lambda \mid^{-\frac{1}{2}} \cdot \int_{-\infty}^{\infty} g(t) \cdot h^{*}\left(\frac{t-\tau}{\lambda}\right) dt \end{aligned} \tag{36}$$

In the language of unitary representation theory this is called the matrix coefficient.

2.4. Wide-band ambiguity function properties.

In this subsection we will summarize the wavelet expansion and we will show the relation to the wide-band ambiguity function. We will use the notations as in [ID88]. The wavelet expansion is connected to the representation of the affine group. Define points of the grid from constants λ_0 , τ_0 > 0, $\lambda_0 \neq 1$. The points of the grid are given by:

$$\lambda_m = \lambda_0^m$$
$$\tau_{mn} = n \cdot \tau_0 \cdot \lambda_0^m \; ; m, n \in Z$$

Define a function on a grid by discrete translation and dilution operators

$$h_{mn}(t) = D_{(\lambda_m,0)} \circ T_{(1,\tau_{mn})} h(t)$$

(37)
$$= \lambda_0^{-\frac{m}{2}} h \left(\lambda_0^{-m} t - n\tau_0 \right)$$

We will say $h_{mn}(t)$ is a frame, if there exits constants $0 < A \leq B$ such that for any signal $s(t) \in L^2(\Re)$ the following hold

(38)
$$A \mid\mid s() \mid\mid^2 \leq \sum_m \sum_n \mid \langle h_{mn}, s \rangle \mid^2 \leq B \mid\mid s() \mid\mid^2$$

Estimates for the bounds A, B are given in[ID88]. If h_{mn} such defined is not a frame, then we can construct a frame from h_{mn} by the following procedure [ID88] Define a operator

(39)
$$P = I - \frac{2 \sum_m \sum_n h_m n \langle h_{mn}, \cdot \rangle}{A + B}$$

where

$$I - identity\ operator,$$
$$A, B - are\ constants$$

Then

(40)
$$\tilde{h}_{0n} = \left(\frac{2}{A+B} \sum_k P^k \right) h_{0n}$$

and the frame $\tilde{h}_{mn}$ is obtained from $\tilde{h}_{0n}$ by applying the dilution operator

$$\tilde{h}_{mn} = D_{(\lambda_m,0)} \tilde{h}_{0n}(t)$$

(41)
$$= \lambda_0^{-\frac{m}{2}} \tilde{h}_{0n} \left(\lambda_0^{-m} t \right)$$

Then the signal $s(t)$ can be expanded

(42)
$$s(t) = \sum_m \sum_n \langle h_{mn}, s \rangle \tilde{h}_{mn}$$

The inner product $\langle h_{mn}, s \rangle$ is a wide-band cross-ambiguity function as we have defined previously and sampled at the points of the grid that have been defined.

If $\frac{B}{A}$ close enough to one ,then the expansion can be approximated [ID88]

(43)
$$s(t) = \frac{2}{A+B} \sum_m \sum_n \langle h_{mn}, s \rangle h_{mn}$$

In the next section we will present an algorithm to compute the wide-band ambiguity function

3. Wavelets and the wide-band ambiguity function. In this section we will present properties of the wide-band ambiguity function which follow directly from the affine group representation properties.

[P.1]

$$\begin{aligned} WA_{(g,h)}(1,\tau) &= \int_{-\infty}^{\infty} g(t) \cdot h^*(t-\tau)dt \\ &= \mathcal{R}_{g,h}(\tau) \end{aligned} \tag{44}$$

where $\mathcal{R}_{g,h}(\tau)$ -is the cross correlation of two signals.

[P.2]

$$\begin{aligned} WA_{g,g}(1,0) &= \int_{-\infty}^{\infty} \mid g(t) \mid^2 dt \\ &= 1. \end{aligned} \tag{45}$$

[P.3]

$$\begin{aligned} WA_{g,h}(\frac{1}{\lambda},\tau) &\leq \mid WA_{g,h}(1,0) \mid \\ &= 1. \end{aligned} \tag{46}$$

[P.4]

$$\begin{aligned} WA_{g,h}(\frac{1}{\lambda},\tau) &= \sqrt{\frac{1}{\lambda}} \cdot \int_{-\infty}^{\infty} g(t) \cdot h^*(\frac{t-\tau}{\lambda})dt \\ &= \sqrt{\frac{1}{\lambda}} \cdot \int_{-\infty}^{\infty} G(\frac{f}{\lambda}) \cdot H^*(f)e^{2\pi i f\tau}df \\ &= \sqrt{\frac{1}{\lambda}} \cdot \int_{-\infty}^{\infty} G(f) \cdot H^*(f\lambda)e^{2\pi i f \frac{1}{\lambda}\tau}df \end{aligned} \tag{47}$$

where $G(f), H(f)$ are the Fourier transform of $g(t), h(t)$.

[P.5] The unitary representation corresponding to the inverse $(\lambda,\tau)^{-1} \in \Re \setminus \{0\} \times \Re$ of the element (λ,τ) is

$$\begin{aligned} h^{((\lambda,\tau)^{-1})}(t) &= h^{(\frac{1}{\lambda},-\frac{\tau}{\lambda})}(t) \\ &= \sqrt{\lambda}h(\lambda t+\tau) \end{aligned} \tag{48}$$

The matrix coefficient is then

$$\begin{aligned} WA_{g,h}(\frac{1}{\lambda},-\frac{\tau}{\lambda}) &= \langle g, h^{(\frac{1}{\lambda},-\frac{\tau}{\lambda})}\rangle \\ &= \langle h^{(\lambda,\tau)}, g\rangle \\ &= WA^*_{g,h}(\lambda,\tau) \end{aligned} \tag{49}$$

or

$$| WA_{g,h}(\frac{1}{\lambda}, -\frac{\tau}{\lambda}) |^2 = | WA_{g,h}(\lambda, \tau) |^2 \tag{50}$$

[P.6]

$$\begin{aligned}\int_{-\infty}^{\infty} WA_{g,h}(\frac{1}{\lambda}, \tau) e^{-2\pi\nu\tau} d\tau &= \sqrt{\frac{1}{\lambda}} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\frac{f}{\lambda}) \cdot H^*(\bar{f}) d\tau df \\ &= \sqrt{\frac{1}{\lambda}} \cdot G(\frac{\nu}{\lambda}) \cdot H^*(\nu) \\ &= \sqrt{\frac{1}{\lambda}} \cdot G(\nu) \cdot H^*(\nu\lambda) \qquad (51)\end{aligned}$$

4. Computation of the wide-band ambiguity function.

We have seen in previous paragraphs the use of wide-band ambiguity function in wavelet expansion and in maximum likelihood estimator of strech/compress parameter a and delay parameter τ in signals arising in sonar. The maximum likelihood estimator is

$$\max_{a,\tau} | WA_{h,g}(a, \tau) |^2$$

where $h()$ is a received signal, $g()$ is a reference signal. In passive sonar case $h(t), g(t)$ may be the received signals at different locations and the locations and the parameters a, τ are relative compress/strech and relative delay factors. From the computation point of view it is the same problem. In the wavelet case the parameters are specified by the discrete grid. We will demonstrate the algorithm for the sonar signal case.

Let $[\tau_{min}, \tau_{max}]$ be reasonable delay interval and $[a_{min}, a_{max}]$ strech/compress factor interval. From the desired accuracy choose the quantization level:

$$\tau_i = \frac{\tau_{max} - \tau_{min}}{N} \cdot i \qquad i = 1, 2 \cdots N$$

and

$$a_j = \frac{a_{max} - a_{min}}{M} \cdot j \qquad j = 1, 2 \cdots M$$

Thus we want to compute for each a_j

$$\max_{\tau} | WA_{h,g}(a_j, \tau) |^2,$$

The above computation assumes that there is only one target in the estimation range and that the noise level is such that after matched correlation processing it is possible, with high certainty, to determine the existence of the signal and locate its maxima with prespecified accuracy. Property P.4 of the wide-band ambiguity

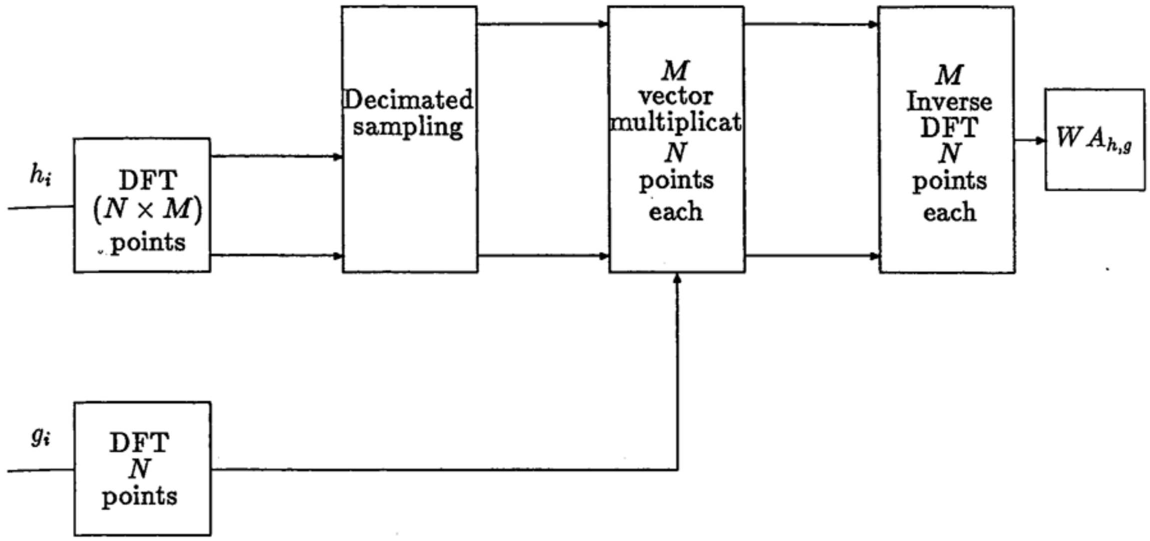

Figure 2: The computational flowchart for wide-band ambiguity function

function will be used for computation. The computational flowchart is presented in the figure 2 above.

The most computationally intensive is the second stage. It includes N-point DFT and another $N \times M$ DFT. From these points M sequencies of length N generated corresponding to different strech/compress factors. The algorithm to compute the wide-band ambiguity function is summarized below:

(1) Precompute for the reference signal its scaled DFT -$N \times M$ points.

(2) Compute for the received signal N point DFT.

(3) Perform M decimated sampling and M vector multiplication of length N each.

(4) Compute M inverse DFT on N points each.

(5) Find maximum on the array of $N \times M$ numbers.

As it is seen from the flowchart the algorithm can be implemented in parallel /pipelined manner. If the narrow-band approximation is valid then more efficient algorithm can be use[AT88], [AG88].

5. Conclusion. The computation of the wide-band ambiguity function is computationally intensive process. It includes the computation of a very long discrete Fourier transform of the reference signal. Then the transformed signal is sampled with the decimated rate determined by the strech/compress Doppler factor. The discrete Fourier transform of the received signal is then vector multiplied by the decimated reference signal. Ultimately the inverse discrete Fourier transform is taken of the product. The computational algorithm is being mapped onto the tree architecture of signal processors .

In the "wavelet case" the computation result is used as a matrix coefficient in the likelihood estimator of the delay and Doppler factors.

REFERENCES

[SW66] DAVID A. SWICK, *An Ambiguity function independent of assumption about bandwidth and carrier frequency*, NRL Report 6471 , Dec. 15 1966..

[SW69] DAVID A. SWICK, *An Review of Wide-band ambiguity functions*, NRL Report 6994 ,Washington , D.C. Dec. 1969.

[ID88] INGRID DAUBECHIES, *The wavelet transform time-frequency localization and signal analysis*, submitted for publication.

[AK68] ASLAKSEN E.W. AND KLAUDER J.R., *Unitary representation of the affine group*, J.Math.Physics 9 (1968), pp. 208–211.

[AT88] L. AUSLANDER, R. TOLIMIERI, *Computing decimated finite cross-ambiguity functions*, IEEE Transactions on Acoustics, Speech, and Signal Processing, vol. 36, No (March 1988), pp. 359–363.

[AG88] L. AUSLANDER, I.GERTNER, *Computing the ambiguity function on various domains*, submitted to publication.