Some invariance properties of the wide-band ambiguity function

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Certain changes in pulse shape do not affect the resolution capability of a sonar system. These changes can be expressed as mathematical transformations that leave invariant important properties of the wide-band ambiguity function. Such transformations provide a large number of signal and filter functions, all of which satisfy a given range-Doppler resolution requirement. The sonar designer can then choose the signal and filter that best satisfy other system constraints (e.g., ease of signal generation and filter synthesis, clutter suppression capability, and peak power).

Subject Classification: 15.2, 15.3.

INTRODUCTION

When echoes are detected by means of a correlation process, the capabilities of a sonar system can be described in terms of the wide-band cross-ambiguity function. Although this function depends upon the sonar signal–filter pair, there are transformations of signal and filter that will not affect certain properties of the ambiguity function.

Such transformations suggest the existence of alternative waveforms that will satisfy a particular range–Doppler resolution requirement. Having found alternatives, the designer can then choose the signal and filter that best satisfy other system constraints.[1]

Invariance transformations are also important to the study of animal echolocation. Dolphins, for example, are capable of using a bewildering variety of echolocation waveforms. Suppose, however, that all these waveforms are related by signal transformations that leave invariant a certain system capability. There would then be strong evidence that the invariant capability is of primary interest to the animal.

I. DEFINITIONS

When a signal $u(t)$ is reflected from a planar target which has a constant velocity βc, where c is the speed of sound, the energy normalized echo has the form $\tilde{s}^{\frac{1}{2}}u[\tilde{s}(t+\tilde{\tau})]$, where $\tilde{s}=(1+\beta)/(1-\beta)$ and $\tilde{\tau}$ is the time delay or range variable.[2] Since $|\beta|<1$, $s>0$. A correlation processor forms the function

$$|\chi_{vu}(\tilde{\tau},\tilde{s})|^2=\left|\tilde{s}^{\frac{1}{2}}\int_{-\infty}^{\infty} v(t)u^*[\tilde{s}(t+\tilde{\tau})]dt\right|^2, \qquad (1)$$

where $v(t)$ is called the filter function. The asterisk symbolizes complex conjugation.

In the presence of additive white noise, maximum output signal-to-noise ratio is realized when $v(t)=\tilde{s}^{\frac{1}{2}}u[\tilde{s}(t+\tilde{\tau})]$. The target parameters $(\tilde{s},\tilde{\tau})$ are generally unknown *a priori*. The receiver must therefore form a series of hypotheses $(\hat{s},\hat{\tau})$, which can be tested by examining the relative magnitude of

$$|\chi_{uu}(\tilde{\tau},\hat{\tau};\tilde{s},\hat{s})|^2=\left|(\tilde{s}\hat{s})^{\frac{1}{2}}\int_{-\infty}^{\infty} u[\hat{s}(t+\hat{\tau})]u^*[\tilde{s}(t+\tilde{\tau})]dt\right|^2$$

$$=\left|s^{\frac{1}{2}}\int_{-\infty}^{\infty} u(t)u^*[s(t+\tau)]dt\right|^2, \qquad (2)$$

where $s=\tilde{s}/\hat{s}$, $\tau=\hat{s}(\tilde{\tau}-\hat{\tau})$. Again, $s>0$. If $s=1$ and $\tau=0$, i.e., if $\hat{s}=\tilde{s}$ and $\hat{\tau}=\tilde{\tau}$, $|\chi_{uu}|^2$ assumes its maximum possible value for a given signal energy. The function $|\chi_{uu}(\tau,s)|^2$ is called the wide-band auto-ambiguity function; $|\chi_{vu}(\tau,s)|^2$ is the wide-band cross-ambiguity function.

The auto-ambiguity function is an important special case of the cross-ambiguity function and corresponds to a system that optimizes output signal-to-noise ratio. It is possible, however, that interference can take forms other than white noise, and in such cases a cross-correlation operation may be advisable.[3]

A signal–filter pair is *Doppler tolerant*[4–6] if

$$\max_{\tau} |\chi_{vu}(\tau,s)|^2\approx|\chi_{vu}(0,1)|^2 \qquad (3)$$

for a large range of s values.

A signal–filter pair is *Doppler resolvent*[7] if

$$\max_{\tau} |\chi_{vu}(\tau,s)|^2\ll|\chi_{vu}(0,1)|^2 \qquad (4)$$

even if s is only slightly different from unity.

Two signal–filter pairs (u_1,v_1), (u_2,v_2) are *equally Doppler resolvent* if

$$\max_{\tau} |\chi_{v_1u_1}(\tau,s)|^2=\max_{\tau} |\chi_{v_2u_2}(\tau,s)|^2 \qquad (5)$$

for any given value of s.

In the above definitions,

$$\max_{\tau} |\chi_{vu}(\tau,s)|^2$$

is the maximum output power of a filter with a specific error on the velocity hypothesis. When passive filters implement the correlation process, the maximum filter response is used both to detect a target and to estimate its range and velocity. Delay hypotheses are automatically implemented by the passage of time. Targets are said to be present when filter output exceeds a particular threshold. Velocity is estimated by comparing the maximum responses for a number of different filters,

$$\max_{\tau} |\chi_{uu}(\tau,s_i)|^2, \quad i=1,\cdots,N,$$

and choosing the largest.

If the real and imaginary parts of the complex signal $u(t)$ are a Hilbert transform pair, the signal is said to be Analytic.[4,8] The Fourier transform $U(\omega)$ of an Analytic signal is one sided, i.e., $U(\omega)$ is zero for $\omega<0$. For an Analytic signal, Parseval's theorem gives

$$|\chi_{uu}(\tau,s)|^2=\left|\frac{1}{2\pi s^{\frac{1}{2}}}\int_0^{\infty} U(\omega)U^*\!\left(\frac{\omega}{s}\right)e^{-j\omega\tau}d\omega\right|^2. \quad (6)$$

II. INVARIANCE PROPERTIES

We are interested in signal transformations that do not affect certain properties of the wide-band ambiguity function. Cross-ambiguity functions will be used whenever possible, since auto-ambiguity results immediately follow as a special case.

I. If $f(t)=u(-t)$ and $g(t)=v(-t)$, then $|\chi_{gf}(\tau,s)|^2=|\chi_{vu}(-\tau,s)|^2$. Time reversal of both signal and filter functions leaves invariant the Doppler resolution properties of the wide-band ambiguity function. If positive and negative range errors are equally probable, the expected range resolution capability is also unaffected.

II. If $F(\omega)=U(\omega)\exp(jk\log\omega)$ and $G(\omega)=V(\omega)\times\exp(jk\log\omega)$, where $F(\omega)$, $U(\omega)$, $G(\omega)$, $V(\omega)$ are the Fourier transforms of the Analytic signals $f(t)$, $u(t)$, $g(t)$, $v(t)$, respectively, then $|\chi_{gf}(\tau,s)|^2=|\chi_{vu}(\tau,s)|^2$. The ambiguity function is unaffected when signal and filter spectra are multiplied by $\exp(jk\log\omega)$.

III. If $f(t)=k^{\frac{1}{2}}u(kt)$ and $g(t)=k^{\frac{1}{2}}v(kt)$, then

$$|\chi_{gf}(\tau,s)|^2=|\chi_{vu}(k\tau,s)|^2$$

and

$$\max_{\tau} |\chi_{gf}(\tau,s)|^2=\max_{\tau} |\chi_{vu}(\tau,s)|^2.$$

An energy invariant scaling of the signal and filter functions does not affect Doppler resolution capability.

IV. $\mathrm{Max}_{\tau}|\chi_{vu}(\tau,s)|^2=\max_{\tau}|\chi_{uv}(\tau,1/s)|^2$. If echoes induced by the signal $u(t)$ are cross correlated with a filter function $v(t)$, then interchanging signal and filter functions has the effect shown above. If s corresponds to a velocity mismatch βc, $1/s$ corresponds to a velocity mismatch of $-\beta c$. The sensitivity of $|\chi_{vu}(\tau,s)|^2$ to a positive velocity mismatch therefore equals the sensitivity of $|\chi_{vu}(\tau,s)|^2$ to a negative velocity mismatch of the same magnitude. If positive and negative velocity mismatches are equally probable, the interchange of signal and filter has no effect upon the expected doppler resolution capability of the system.

All the properties listed in this paper can be proven by direct substitution into one of the expressions for $\chi_{vu}(\tau,s)$, i.e.,

$$\chi_{vu}(\tau,s)=s^{\frac{1}{2}}\int_{-\infty}^{\infty} v(t)u^*[s(t+\tau)]dt$$

$$=\frac{1}{2\pi s^{\frac{1}{2}}}\int_0^{\infty} V(\omega)U^*\!\left(\frac{\omega}{s}\right)e^{-j\omega\tau}d\omega. \quad (7)$$

To prove the fourth property, for example, we change variables in Eq. 7, giving

$$\chi_{vu}(\tau,s)=\frac{s^{\frac{1}{2}}}{2\pi}\int_0^{\infty} U^*(\omega)V(\omega s)e^{-j\omega s\tau}d\omega$$

$$=\chi_{uv}^*(-s\tau,\,1/s) \quad (8)$$

so that, for a given value of s,

$$\max_{\tau} |\chi_{vu}(\tau,s)|^2=\max_{\tau} |\chi_{uv}(\tau,1/s)|^2. \quad (9)$$

V. If $f(t)=u(t+T)$ and $g(t)=v(t+T)$, then

$$\max_{\tau} |\chi_{gf}(\tau,s)|^2=\max_{\tau} |\chi_{vu}(\tau,s)|^2.$$

If $F(\omega)=U(\omega)\exp(j\omega T)$ and $G(\omega)=V(\omega)\exp(j\omega T)$ are substituted into the last expression in Eq. 7, it is easily shown that $\chi_{gf}(\tau,s)=\chi_{vu}(\tau-T+T/s,\,s)$, so that time translation of signal and filter does not affect Doppler resolution properties.

VI. Suppose that $u(t)$ and $v(t)$ are either even or odd; either $u(t)=u(-t)$, $v(t)=v(-t)$ or $u(t)=-u(-t)$, $v(t)=-v(-t)$. If $f(t)=2^{\frac{1}{2}}u(t)[\frac{1}{2}+\frac{1}{2}\,\mathrm{Sgn}t]$ and $g(t)=2^{\frac{1}{2}}v(t)[\frac{1}{2}+\frac{1}{2}\,\mathrm{Sgn}t]$, where

$$\begin{aligned}\mathrm{Sgn}t&=+1,\ t\geq 0,\\ &=-1,\ t<0,\end{aligned} \quad (10)$$

then

$$\max_{\tau} |\chi_{gf}(\tau,s)|^2\geq\max_{\tau} |\chi_{vu}(\tau,s)|^2.$$

Part of an even or odd Doppler tolerant waveform can be discarded, and the waveform will still retain its Doppler tolerance. It also follows that if a given causal waveform $f(t)$ is Doppler resolvent, then $u(t)=2^{-\frac{1}{2}}[f(t)\pm f(-t)]$ is at least as Doppler resolvent as $f(t)$.[9]

The above statement follows from the fact that, if $v(t)$ and $u(t)$ are even or odd,

$$\begin{aligned}\chi_{gf}(-\tau, s) &= 2\int_0^\infty v(t)u^*[s(t-\tau)]dt \\ &= -2\int_0^{-\infty} v(-t)u^*[-s(t+\tau)]dt \\ &= 2\int_{-\infty}^0 v(t)u^*[s(t+\tau)]dt,\end{aligned} \tag{11}$$

so that

$$\begin{aligned}&\tfrac{1}{2}[\chi_{gf}(\tau,s)+\chi_{gf}(-\tau, s)] \\ &= \int_0^\infty v(t)u^*[s(t+\tau)]dt + \int_{-\infty}^0 v(t)u^*[s(t+\tau)]dt \\ &= \chi_{vu}(\tau,s).\end{aligned} \tag{12}$$

Therefore,

$$\begin{aligned}&\max_\tau |\chi_{vu}(\tau,s)| \\ &= \tfrac{1}{2}\max_\tau |\chi_{gf}(\tau,s)+\chi_{gf}(-\tau, s)| \\ &\le \tfrac{1}{2}\max_\tau |\chi_{gf}(\tau,s)| + \tfrac{1}{2}\max_\tau |\chi_{gf}(-\tau, s)| \\ &= \max_\tau |\chi_{gf}(\tau,s)|.\end{aligned} \tag{13}$$

VII. If $F(\omega)=\omega^{-1}U(1/\omega)$ and $G(\omega)=\omega^{-1}V(1/\omega)$, then $|\chi_{gf}(0,s)|^2=|\chi_{vu}(0,1/s)|^2=|\chi_{uv}(0,s)|^2$. The energy invarient transformations $\omega^{-1}U(1/\omega)$, $\omega^{-1}V(1/\omega)$ do not affect ambiguity function behavior along the $\tau=0$ profile if signal and filter are interchanged. In general, the transformations $F(\omega)=(n\omega^{n-1})^{\frac{1}{2}}U(\omega^n)$ have the effect that $|\chi_{ff}(0,s)|^2=|\chi_{uu}(0,s^n)|^2$.

VIII. If $f(t)=t^{-1}u(1/t)$ and $g(t)=t^{-1}v(1/t)$, then $|\chi_{gf}(0,s)|^2=|\chi_{vu}(0,1/s)|^2=|\chi_{uv}(0,s)|^2$. The energy invariant transformations $t^{-1}u(1/t)$, $t^{-1}v(1/t)$ do not affect s-axis ambiguity function behavior if signal and filter are interchanged.

IX. If $f(t)=u(t)\exp(\pm jk\log|t|)$ and $g(t)=v(t)\times\exp(\pm jk\log|t|)$, then $|\chi_{gf}(0,s)|^2=|\chi_{vu}(0,s)|^2$.

X. If $f(t)=(2\pi)^{-\frac{1}{2}}U(t)$ and $g(t)=(2\pi)^{-\frac{1}{2}}V(t)$, then $|\chi_{gf}(0,s)|^2=|\chi_{vu}(0,1/s)|^2=|\chi_{uv}(0,s)|^2$. For auto-ambiguity functions, $|\chi_{UU}(0,s)|^2=(2\pi)^2|\chi_{uu}(0,s)|^2$, where $U(\omega)$ is the Fourier transform of $u(t)$. If $u(t)$ is Doppler tolerant and if

$$\max_\tau |\chi_{uu}(\tau,s)|^2=|\chi_{uu}(0,s)|^2,$$

then $U(t)$ is also Doppler tolerant. If

$$\max_\tau |\chi_{UU}(\tau,s)|^2=|\chi_{UU}(0,s)|^2$$

and if $u(t)$ is Doppler resolvent, then $U(t)$ is also Doppler resolvent.

Properties VII–X are especially useful when

$$\max_\tau |\chi_{vu}(\tau,s)|^2=|\chi_{vu}(0,s)|^2 \tag{14a}$$

and

$$\max_\tau |\chi_{gf}(\tau,s)|^2=|\chi_{gf}(0,s)|^2. \tag{14b}$$

A sufficient condition for Eq. 14a is that

$U(\omega)$ and $V(\omega)$ are real (or imaginery) (15a)

and

$U(\omega)$ and $V(\omega)$ are positive (or negative) semidefinite. (15b)

If Conditions 15a and 15b are true, then $|V(\omega)U^*(\omega/s)|=V(\omega)U^*(\omega/s)$. Since $|\exp(-j\omega\tau)|=1=\exp(-j\omega 0)$,

$$|\chi_{vu}(\tau,s)|^2 \le \left|\frac{1}{2\pi s^{\frac{1}{2}}}\int_0^\infty |V(\omega)U^*\left(\frac{\omega}{s}\right)e^{-j\omega\tau}|d\omega\right|^2 = |\chi_{vu}(0,s)|^2. \tag{16}$$

By Property II, Eq. 16 is also true if signals that satisfy Conditions 15 are multiplied by $\exp(jk\log\omega)$.

If Condition 15b is dropped, then Eq. 14a will hold true in the neighborhood of $s=1$, but not necessarily over the entire (τ,s) plane. If Condition 15a is true, then

$$u(t)=\pm u^*(-t) \quad\text{and}\quad v(t)=\pm v^*(-t), \tag{17}$$

a condition which implies that $|\chi_{vu}(\tau,s)|^2$ is symmetric about the s axis[10]; $|\chi_{vu}(\tau,s)|^2=|\chi_{vu}(-\tau, s)|^2$. Since

$$\max_\tau |\chi_{vu}(\tau,1)|^2=|\chi_{vu}(0,1)|^2$$

Eq. 14a will be true for $s\approx 1$, i.e., on the main lobe of the ambiguity function, if Condition 15a is satisfied. This conclusion also follows from the two-dimensional Taylor series expansion of $|\chi_{vu}(\tau,s)|^2$ about the point $(\tau,s)=(0,1)$.[7]

If Condition 15a or both Conditions 15a and 15b are satisfied by $U(\omega)$ and $V(\omega)$, then the same conditions will be satisfied by $F(\omega)$ and $G(\omega)$ in Property VII. Therefore, both Eqs. 14a and 14b are true for Property VII, if Condition 15 is satisfied.

If Condition 15a is satisfied, then Eqs. 17 are true, and Eqs. 17 are unaffected by the transformations in Property VIII. In the neighborhood of $s=1$, both Eqs. 14a and 14b are therefore true for Property VIII, if Condition 15a is satisfied.

If the signal and filter functions in Property IX are multiplied by $\exp(\pm jk\log|t|)$ for $t>0$ and by $\exp(\mp jk\log|t|)$ for $t<0$, then Eqs. 17 are unaffected by these transformations. For $s\approx 1$, Eqs. 14a and 14b are therefore true for the transformations in Property IX, provided that Condition 15a is satisfied.

Property X is applicable to real signals rather than to Analytic ones. If $U(t)=0$ for $t<0$, then $u(-\omega)$, the Fourier transform of $U(t)$, cannot be Analytic. For real signals that are either even or odd, Eqs. 17 are satisfied for both $u(t)$ and $U(t)$, so that Eqs. 14 hold for the transformations in Property X, for s in the neighborhood of unity.

An illustration of Property X, as applied to real, odd signals, is shown in Figs. 1 and 2. Figure 1(a) shows an optimum signal for velocity discrimination.[7] Figure 1(b) shows the ambiguity function of the signal in Fig. 1(a). Figure 1(c) shows the $\tau=0$ profile of the ambiguity function. Figure 2(a) shows the Fourier transform of the waveform in Fig. 1(a). The corresponding ambiguity function is shown in Fig. 2(b), and the $\tau=0$ profile of this function is shown in Fig. 2(c). Note that both ambiguity functions are symmetrical about the line $\tau=0$, that their maximum value occurs at $\tau=0$ for the range of s values shown, and that the Doppler axis behavior is identical for both signals.

In order to apply properties VII–X to an arbitrary signal $f(t)$, one can force the signal to satisfy Condition 17 by forming the function $u(t)=2^{-\frac{1}{2}}[f(t)+f^*(-t)]$.

XI. A large set of transformations do not affect the behavior of $|\chi_{vu}(\tau,s)|^2$ for the particular values of s described by $s=s_0^n$, where s_0 can be chosen by the designer. If $\beta_0=v_0/c\ll 1$ and $s_0=(1+\beta_0)/(1-\beta_0)$, then $s_0^n\approx(1+2\beta_0)^n\approx 1+2n\beta_0$. For $\beta_0\ll 1$, s_0^n corresponds to uniform velocity intervals for integer values of n.

Let $F(\omega)=U(\omega)\exp[jkH(\log\omega/\log s_0)]$ and $G(\omega)=V(\omega)\exp[jkH(\log\omega/\log s_0)]$, where $H(x)$ is a periodic function of x; $H(x\pm n)=H(x)$, and n is any integer. Then $|\chi_{gf}(\tau,s_0^n)|^2=|\chi_{vu}(\tau,s_0^n)|^2$ for any integer n.

Since a large class of functions are periodic, the above property provides a large set of transformations that leave $|\chi_{vu}(\tau,s_0^n)|^2$ invariant. If s_0 is close to unity ($\beta_0\ll 1$), then $H(\log\omega/\log s_0)$ oscillates very rapidly as ω changes.

The transformations given in Properties I–XI should prove useful for the derivation of optimum sonar waveforms. If a signal is optimally Doppler tolerant (or resolvent), then a transformation that preserves or improves Doppler tolerance (or resolution) will again produce an optimally Doppler tolerant (resolvent) waveform. If the optimum waveform is unique, a tolerance (resolution) invariant transformation will have no effect upon the signal. Optimum signals can therefore be found by looking for waveforms that are invariant to the various transformations listed above. For example, a first-order Bessel polynomial[11] with a logarithmic phase function is invariant to the transformations given in Properties VII and VIII.

Other eigenfunctions of the transformations in Properties VII and VIII are given by

$$U(\omega)=\omega^{-\frac{1}{2}}F(\log\omega), \tag{18}$$

where $F(x)$ is any even function of x, i.e., any function such that $F(x)=F(-x)$.

XII. The volume under the ambiguity function for $\tau\epsilon(-\infty,\infty)$, $s\epsilon(0,\infty)$ is not easily related to any particular resolution property. The volume of the narrow-band auto-ambiguity function depends only upon signal energy[12]; all signals with the same energy have the same volume. Narrow-band volume invariance is conceptually useful because, if the ambiguity is "pushed down" in one area, it is bound to "pop up" somewhere else.

Volume invariance, therefore, can sometimes be a helpful signal design concept. Let us consider some signal–filter transformations that do not affect wide-band cross-ambiguity volume.

XIIa. If a wide-band filter function has Fourier transform $V(\omega)$ and if the transmitted signal $U(\omega)=\omega^{-\frac{1}{2}}V(1/\omega)$, then the volume under the cross-ambiguity function is $2\pi E^2$, where

$$E=(1/2\pi)\int_0^\infty |V(\omega)|^2 d\omega.$$

This volume is exactly the same as the volume under the narrow-band auto-ambiguity function.

If a signal's Fourier transform satisfies the relation

$$\omega^{-\frac{1}{2}}U(1/\omega)=U(\omega), \tag{19}$$

then the signal's wide-band and narrow-band auto-ambiguity functions have the same volume. There are many functions that satisfy Eq. 19. For example, all signals of the form

$$U(\omega)=\omega^{-\frac{1}{2}}F(\log\omega)$$

satisfy Eq. 19 provided $F(x)=F(-x)$ as in Eq. 18.

XIIb. If $F(\omega)=\omega^{-\frac{1}{2}}U(1/\omega)$, then the volume under $|\chi_{ff}(\tau,s)|^2$ equals the volume under $|\chi_{uu}(\tau,s)|^2$.

XIIc. If $F(\omega)=\omega^{-\frac{1}{2}}U(1/\omega)$ and $G(\omega)=\omega^{-\frac{1}{2}}V(1/\omega)$, then the volume under $|\chi_{vu}(\tau,s)|^2$ equals the volume under $|\chi_{fg}(\tau,s)|^2$.

To prove these relations it is necessary to determine the volume, W_{vu}, under the wide-band cross-ambiguity function[2,13]:

$$\begin{aligned} W_{vu} &= \int_0^\infty \int_{-\infty}^\infty |\chi_{vu}(\tau,s)|^2 d\tau ds \\ &= \frac{1}{(2\pi)^2}\int_0^\infty \frac{1}{s}\int_0^\infty \int_0^\infty V(x)U^*\left(\frac{x}{s}\right)V^*(y)U(y/s) \\ &\quad \times\left[\int_{-\infty}^\infty e^{-j\tau(x-y)}d\tau\right]dxdyds \\ &= \frac{1}{2\pi}\int_0^\infty \frac{1}{s}\int_0^\infty |V(x)|^2\left|U\left(\frac{x}{s}\right)\right|^2 dxds. \end{aligned} \tag{20}$$

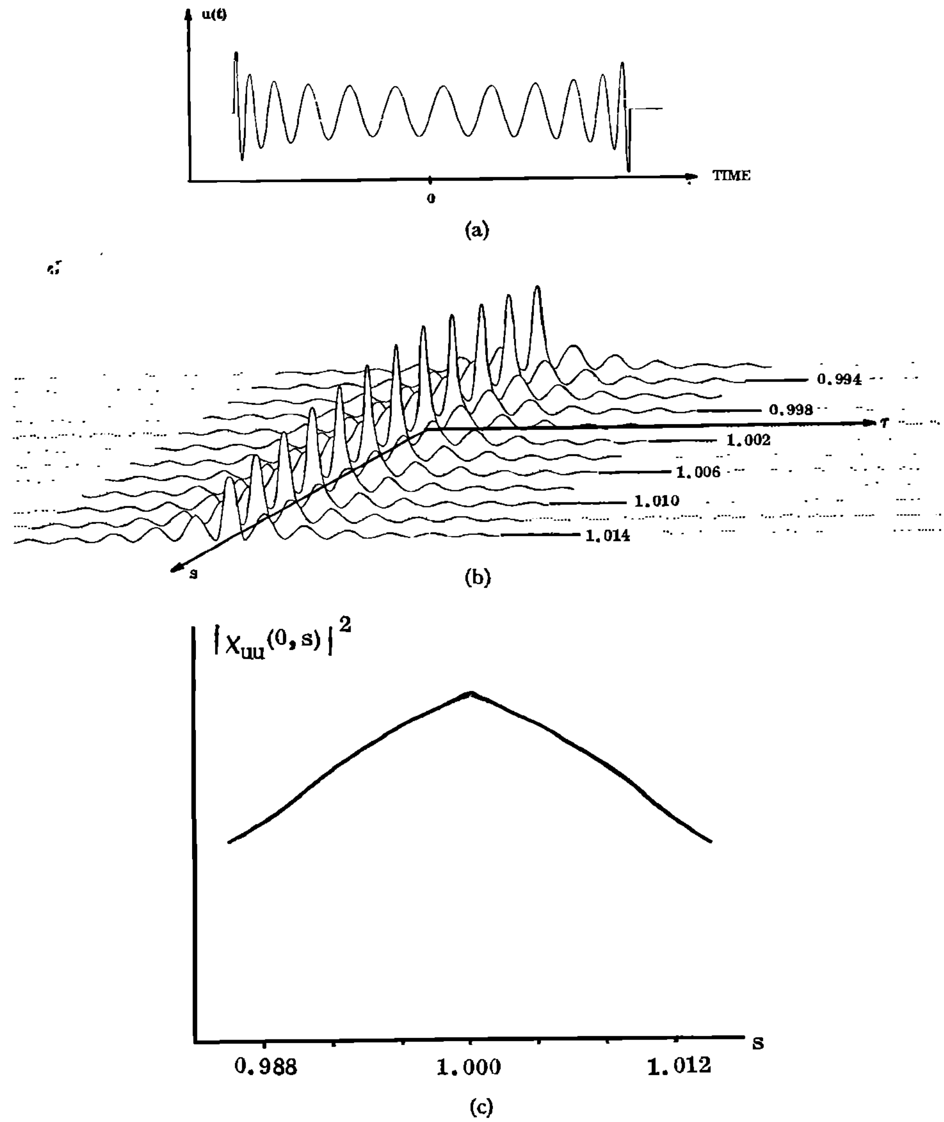

FIG. 1(a). A Legendre polynomial of order 25, truncated at its last zero crossings. (b) The wide-band auto-ambiguity function of the signal in (a). (c) The Doppler axis behavior of the ambiguity function in (b).

Letting $x=sz$, we have

$$W_{vu}=\frac{1}{2\pi}\int_0^\infty\int_0^\infty |V(sz)|^2|U(z)|^2dzds$$

$$=\int_0^\infty \frac{|U(z)|^2}{z}\left[\frac{1}{2\pi}\int_0^\infty |V(sz)|^2d(sz)\right]dz$$

$$=\frac{1}{2\pi}\int_0^\infty \frac{|U(\omega)|^2}{\omega}d\omega\int_0^\infty |V(\omega)|^2d\omega. \tag{21}$$

If $U(\omega)=\omega^{-\frac{1}{2}}V(1/\omega)$,

$$\int_0^\infty \frac{|U(\omega)|^2}{\omega}d\omega=\int_0^\infty |V(\omega)|^2d\omega, \tag{22}$$

and

$W_{vu}=2\pi E^2$, as in Property XIIa.

If $G(\omega)=\omega^{-\frac{1}{2}}V(1/\omega)$ and $F(\omega)=\omega^{-\frac{1}{2}}U(1/\omega)$,

$$\int_0^\infty |V(\omega)|^2d\omega=\int_0^\infty \frac{|G(1/\omega)|^2}{\omega}d\omega=\int_0^\infty \frac{|G(\omega)|^2}{\omega}d\omega \tag{23}$$

$$\int_0^\infty \frac{|U(\omega)|^2}{\omega}d\omega=\int_0^\infty \frac{|F(1/\omega)|^2}{\omega^2}d\omega=\int_0^\infty |F(\omega)|^2d\omega, \tag{24}$$

so that $W_{vu}=W_{fg}$, as in Property XIIc, and, if $V(\omega)=U(\omega)$, $W_{uu}=W_{ff}$ as in Property XIIb.

The above discussion has generally neglected range resolution invariance. For a correct velocity hypothesis range resolution is determined by $|\chi_{vu}(\tau,1)|^2$, a cross-

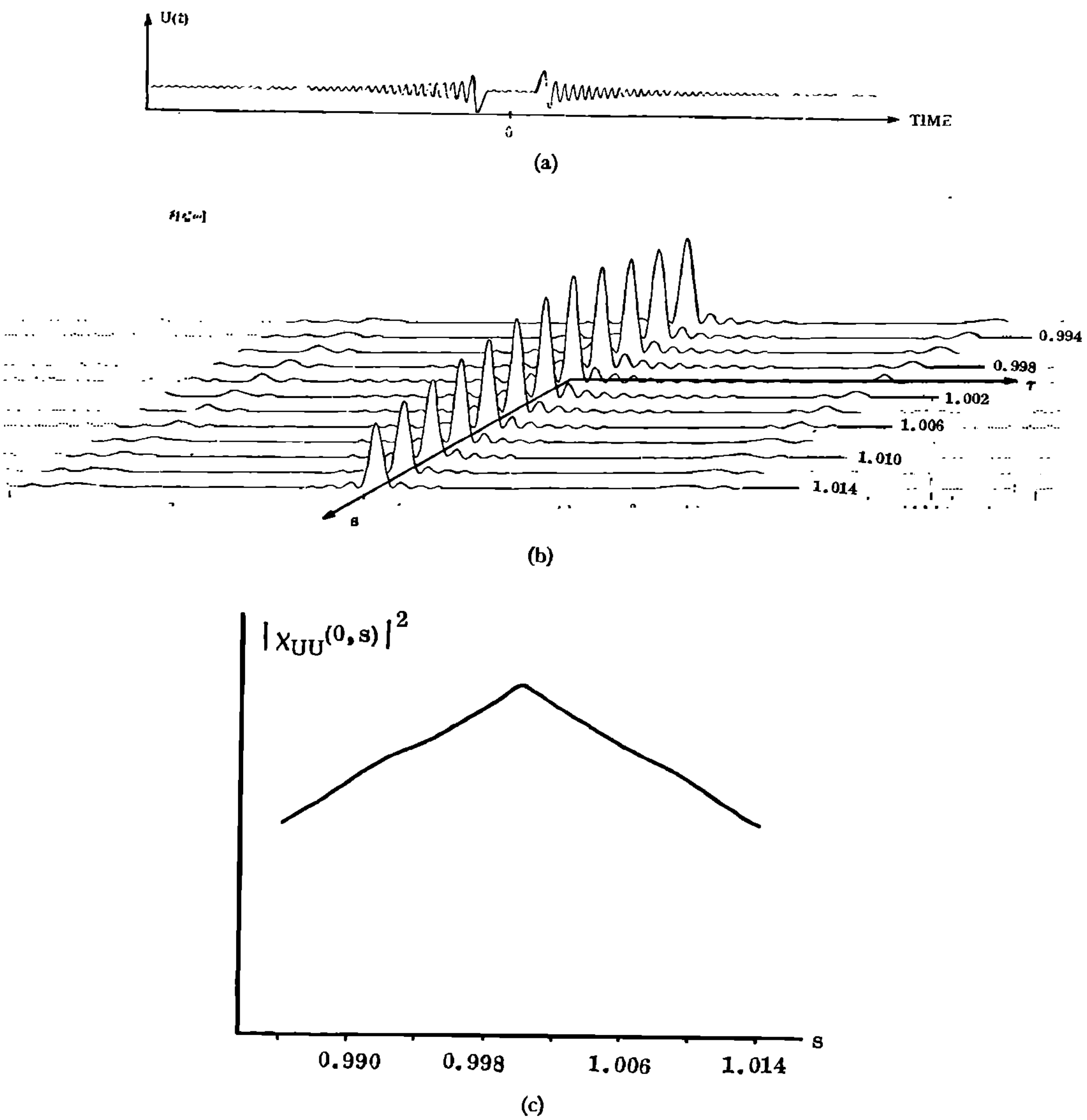

FIG. 2(a). The fourier transform of the signal in Fig. 1(a), written as a function of time. (b) The wide-band auto-ambiguity function of the signal in (a). (c) The Doppler axis behavior of the ambiguity function in (b).

correlation function. Two signal–filter pairs $(u,v)(f,g)$ can then be considered equally range resolvent if $|\chi_{vu}(\tau,1)|^2 = |\chi_{gf}(\tau,1)|^2$.

Many invariance properties of the cross-correlation function are already well known. If $V(\omega)$ and $U(\omega)$ have the same spectral phase function, then $|\chi_{vu}(\tau,1)|^2$ is invariant to changes in this phase function[12] and depends only upon the product $|V(\omega)||U(\omega)|$. The cross-correlation function is unaffected by spectral shifts; $u(t)\exp(j\omega_0 t)$ and $v(t)\exp(j\omega_0 t)$ have the same cross-correlation function as $u(t)$ and $v(t)$. If signal and filter are interchanged, $|\chi_{vu}(\tau,1)|^2 = |\chi_{uv}(-\tau,1)|^2$; the expected range resolution capability of the system is unaffected if positive and negative range errors are equally likely. Time reversing signal and filter functions has the same effect as interchanging them (see Property I). If the time-reversed waveforms $u(-t)$ anf $v(-t)$ are interchanged, the cross-correlation function remains invariant.[14]

Invariance properties of $|\chi_{vu}(\tau,1)|^2$ are important for Doppler resolution studies by virtue of J. Speiser's transformations.[15,16] If $U(\omega) = \omega^{-\frac{1}{2}}F(\log\omega)$ and $V(\omega) = \omega^{-\frac{1}{2}}G(\log\omega)$, then

$$|\chi_{vu}(0,s)|^2 = \left|\frac{1}{2\pi}\int_0^\infty G(\log\omega)F^*(\log\omega - \log s)\,d(\log\omega)\right|^2$$

$$= |\chi_{GF}(-\log s, 1)|^2$$

$$= \left|\int_{-\infty}^{\infty} g(t)f^*(t)e^{-jt\log_e s}dt\right|^2. \quad (25)$$

From Eq. 25, it can be seen that time reversing $f(t)$ has no effect upon $|\chi_{uu}(0,s)|^2$. The function $f(t)$ is the

Fourier transform of $F(\omega)$, i.e.,

$$f(t)=\frac{1}{2\pi}\int_0^\infty F(\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^\infty e^{\omega/2}U(e^\omega)e^{j\omega t}d\omega$$

$$=\frac{1}{2\pi}\int_0^\infty U(\omega)[\omega^{-\frac{1}{2}}e^{jt\log\omega}]d\omega. \tag{26}$$

Any transformation of $U(\omega)$ that time reverses $f(t)$ will not affect $|\chi_{uu}(0,s)|^2$. $U(\omega)\rightarrow\omega^{-1}U(1/\omega)$ is such a transformation (Property VII).

The right-hand side of Eq. 25 is also unaffected by a translation, $g(t)$, $f(t)\rightarrow g(t+k)$, $f(t+k)$. It follows that any transformation of $U(\omega)$ that makes $f(t)\rightarrow f(t+k)$ will not affect $|\chi_{uu}(0,s)|^2$. According to Eq. (26) $U(\omega)\rightarrow U(\omega)\exp(jk\log\omega)$ is such a transformation (Property II).

The transformations listed above can be combined to provide further invariant transformations. For example, if $f_1(t)=k_1^{\frac{1}{2}}u(-k_1t)$, then $f_1(t)$ and $u(t)$ are equally Doppler resolvent, by Properties I and III. Using Property V, $k_1^{\frac{1}{2}}u(T-k_1t)$ also maintains Doppler resolution as measured by the auto-ambiguity function. This transformation demonstrates the equivalence of two well-publicized Doppler tolerant phase functions, $\log(1-kt)$ and $\log kt$.[4–6] If

$$\max_\tau |\chi_{f_1f_1}(\tau,s)|^2$$

occurs at $\tau=0$ for velocity errors of interest, then

$$f_2(t)=t^{-1}f_1(1/t)\exp(jk_2\log|t|)$$
$$=k_1^{\frac{1}{2}}t^{-1}u(-k_1/t)\exp(jk_2\log|t|)$$

has the same (or better) Doppler tolerance as $u(t)$, by Properties VIII and IX.

III. CONCLUSION

This paper has presented some signal transformations with which equivalent sonar waveforms can be generated. Such transformations provide a choice of signals and filters for any given system specification. The results also prove useful for the identification of equivalent animal echolocation signals.

[1]Constraints of interest might include carrier frequency, bandwidth, time duration, maximum power, and clutter suppression capability.

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[9]This observation may explain the occasional utilization of "pulse pairs" by several species of echolocating cetaceans.

[10]Signals that satisfy Eq. 17 are unaffected by a time reversal (except for complex conjugation of both signal and filter impulse response, which does not affect the ambiguity function). Property I shows that the ambiguity functions of such waveforms are symmetrical about the line $\tau=0$.

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