The Phase Retrieval Problem for the Radar Ambiguity Function and the Radar Ambiguity Function for the Phase Retrieval Problem

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***Abstract*—This talk is based on a joint paper with A. Bonami and G. Garrigós [3] in which the phase retrieval problem for the Radar Ambiguity Function (*i.e.* the Radar Ambiguity Problem) has been tackled. In particular it was shown that for wide classes of signals, the radar ambiguity problem has a unique solution, up to trivial transformations.**

In the second part of the talk, we report on ongoing work by the author where the radar ambiguity function is used as a tool to solve other phase retrieval problems.

I. Introduction

In this paper, we report on our work on the phase retrieval problem for the Radar Ambiguity Function and for Fractional Fourier Transforms. Let us recall that the ambiguity function of $u \in L^2(\mathbb{R})$ is defined by

$$A(u)(x,y)=\int_{\mathbb{R}} u\left(t+\frac{x}{2}\right)\overline{u\left(t-\frac{x}{2}\right)}e^{-2i\pi ty}\mathrm{d}t.$$

This function has been introduced by Woodward [11] in Radar theory and has been studied by many authors. The properties that we use here are well documented *see e.g.* [1], [10]. The first problem under consideration here is the phase retrieval problem for the ambiguity function:

Problem 1: **Radar ambiguity Problem.**
Given u in some subset $\mathcal{D}\subset L^2(\mathbb{R})$, find all $v\in\mathcal{S}$ such that

$$|A(v)(x,y)|=|A(u)(x,y)| \qquad \text{for all } x,y\in\mathbb{R}.$$

We then say that v is a partner *of u (in $\mathcal{D}$).*

It is not hard to see that v defined by $v(x)=ce^{ibx}u(\varepsilon x-a)$, $\varepsilon=\pm1$, $a,b\in\mathbb{R}$ $c\in\mathbb{C}$, $|c|=1$ is a partner of u, in which case we say that v is a *trivial partner* of u. The question we ask is thus whether non-trivial partners exist and to have as large as possible classes $\mathcal{D}$ of functions which have only trivial partners. The classes we consider are mainly the two following:

i) *Hermite functions* that is $\mathcal{D}$ is the set of all functions of the form $P(x)e^{-\pi x^2}$, P a polynomial;
ii) *rectangular pulse trains*, that is $\mathcal{D}$ is the set of all functions of the form $\sum_{k=1}^N a_k\chi_{(k,k+\eta)}$ where the a_k's are complex numbers and $\chi_{(k,k+\eta)}$ is the characteristic function of the interval $(k,k+\eta)$ where $0<\eta\leq 1/2$.

The second family of problems we deal with is the *Phase Retreival Problem* for the Fractional Fourier Transform (FrFT). This is defined as follows: let $\alpha\in\mathbb{R}\setminus\pi\mathbb{Z}$, let $c_\alpha=\dfrac{\exp\frac{i}{2}\left(\alpha-\frac{\pi}{2}\right)}{\sqrt{|\sin\alpha|}}$. For $u\in L^1(\mathbb{R}^d)$ and $\alpha\notin\pi\mathbb{Z}$, define

$$\begin{aligned}\mathcal{F}_\alpha u(\xi) &= c_\alpha e^{-i\pi|\xi|^2\cot\alpha}\times &(1)\\ &\quad\times\int_{\mathbb{R}} u(t)e^{-i\pi t^2\cot\alpha}e^{-2i\pi t\xi/\sin\alpha}\mathrm{d}t. &(2)\end{aligned}$$

Note that $\mathcal{F}_{\pi/2}$ is just $\mathcal{F}_{\pi/2}=\mathcal{F}$, the usual Fourier Transform. As $\|\mathcal{F}_\alpha u\|_{L^2(\mathbb{R})}=\|u\|_{L^2(\mathbb{R})}$ this transform can be extended to $u\in L^2(\mathbb{R})$. The problem we address here is the following:

Problem 2: **Phase Retrieval Problem for the fractional Fourier transform.**
Let $u,v\in L^2(\mathbb{R})$ and let $\tau\subset[0,\pi)$ be a set of indices (finite or not). Assume that $|\mathcal{F}_\alpha v|=|\mathcal{F}_\alpha u|$ for every $\alpha\in\tau$.

i) *Does this imply that $v=cu$ for some constant $c\in\mathbb{C}$, $|c|=1$?*

ii) *If we restrict $u \in \mathcal{D}$ for some set $\mathcal{D} \subset L^2(\mathbb{R})$ do we then have $v = cu$ for some constant $c \in \mathbb{C}$, $|c| = 1$?*

iii) *If we further restrict both $u, v \in \mathcal{D}$ for some set $\mathcal{D} \subset L^2(\mathbb{R})$ do we then have $v = cu$ for some constant $c \in \mathbb{C}$, $|c| = 1$?*

In the first two cases we say that u is uniquely determined (up to constant multiples or up to a constant phase factor) from $\{|\mathcal{F}_\alpha u|, \alpha \in \tau\}$. In the last case we say that u is uniquely determined (up to a constant phase factor) from $\{|\mathcal{F}_\alpha u|, \alpha \in \tau\}$ within the class $\mathcal{D}$.

This problem appears in diffraction optics and quantum mechanics. We will reformulate this problem in terms of the radar ambiguity function. This will allow us to bring a solution to the problem for various classes $\mathcal{D}$ of functions. We will again focus on the class of Hermite functions and of rectangular pulse trains and also give some results for more general compactly supported functions.

The remaining of this paper is organized as follows: the next section is devoted to the radar ambiguity functions and presents results that already appeared in [3]. Section 3 is then devoted to the announcement of new results from [9].

II. The radar ambiguity problem

The radar ambiguity problem as stated is not yet fully solved. Early results can be found in [5], [4]. The author tackled this problem in [8], [6] and [3]. The aim of this section is to present the two main results from this last paper which may so far have stayed unnoticed outside the mathematical community and all results presented in this section come from that paper. The main object of [3] was to study the Radar Ambiguity Problem in a "discrete" setting by restricting attention to Hermite functions and to pulse trains.

A. Hermite functions

In this section, we study Problem 1 when u is of the form $u(x) = P(x)e^{-\pi x^2}$ where P is a polynomial. As a consequence of Hardy's Uncertainty Principle for the ambiguity function (*see e.g.* [2], [7]), it is not hard to see that every partner v is of the form $v(x) = Q(x)e^{(a+ib)x-\pi x^2}$ with Q a polynomial. Thus, up to replacing v by a trivial partner, we may assume that $v(x) = Q(x)e^{-\pi x^2}$. Moreover, it is easy to see that P and Q have same degree.

If we write $H_j = (-1)^j e^{\pi t^2}\partial_t^j e^{-2\pi t^2}$ for the Hermite basis, then we may write u and v in that basis, $u = \sum_{j=0}^{n} \alpha_j H_j$ and $v = \sum_{j=0}^{n} \beta_j H_j$. Define $\mathcal{P} = \sum_{j=0}^{n} \alpha_j t^j$, $\mathcal{Q} = \sum_{j=0}^{n} \beta_j t^j$, and let us write $x, y \in \mathbb{R}$, $Z = x + iy$, $\bar{Z} = x - iy$. One may then show that

$$A(u)(x,y) = \sum_{j=0}^{n} \frac{2^{-j}}{j!} \mathcal{P}^{(j)}(Z)\overline{\mathcal{P}^{(j)}(-Z)}e^{|Z|^2/4}.$$

Next, expending $|A(u)|^2$, the radar ambiguity problem then amounts to determining β such that

$$\begin{aligned} &\sum \frac{2^{-j-k}}{j!k!}\mathcal{P}^{(k)}(-Z)\mathcal{P}^{(j)}(Z)\overline{\mathcal{P}^{(j)}(-Z)\mathcal{P}^{(k)}(Z)} \quad (3)\\ =&\sum \frac{2^{-j-k}}{j!k!}\mathcal{Q}^{(k)}(-Z)\mathcal{Q}^{(j)}(Z)\overline{\mathcal{Q}^{(j)}(-Z)\mathcal{Q}^{(k)}(Z)}. \end{aligned}$$

Expending this polynomial in Z and $\bar{Z}$ and comparing highest order terms leads to $|\beta_n| = |\alpha_n|$. Up to replacing v by a trivial partner, we may thus assume that $\beta_n = \alpha_n$.

Expending (3) further, we then obtain

$$\mathcal{P}(Z)\mathcal{P}(-Z) = \mathcal{Q}(Z)\mathcal{Q}(-Z)$$

so that the zeroes of $\mathcal{Q}$ are obtained from those of $\mathcal{P}$ via a symmetry, so that we may factor $\mathcal{P}(Z) = A(Z)B(Z)C(Z)$ and $\mathcal{Q}(Z) = A(Z)B(-Z)C(Z)$ where $C(Z)$ is of the form $Z^k \prod(Z^2 - \lambda_j^2)$. Our aim is to show that neither A nor B are even or odd.

For a polynomial P, write $\tilde{P}(Z) = P(Z)P'(-Z) + P(-Z)P'(Z)$. One easily sees that $\tilde{P} = 0$ if and only if P has a definite parity.

Expending further (3), we now obtain

$$\begin{aligned} &\mathcal{P}'(Z)\mathcal{P}'(-Z) + \frac{2}{n}\overline{\alpha_{n-1}}\tilde{\mathcal{P}}(Z)\\ =\;\; &\mathcal{Q}'(Z)\mathcal{Q}'(-Z) + \frac{2}{n}\overline{\beta_{n-1}}\tilde{\mathcal{Q}}(Z). \end{aligned}$$

We then reformulate this in terms of A and B and notice that the equations obtained that way generically do not have a solution. This leads to the following result:

Theorem 2.1: [3] *For almost all and quasi-all polynomials P, $u(t) = P(t)e^{-\pi t^2}$ has only trivial partners*

Here, for a fixed degree n, almost-all refers to 0 Lebesgue measure in $\mathbb{C}^{n+1}$, while quasi-all refers to Baire theory. Actually we showed that the set of polynomials for which there exists a non trivial partner is included in a lower dimensional sub-manifold of $\mathbb{C}^{n+1}$. We conjecture that this manifold is actually empty so that the above result is true for *all* polynomials.

There are some polynomials P for which one can assert that $u(t) = P(t)e^{-\pi t^2}$ has only trivial partners, for instance those that have no term of degree $n-1$. From this, it is not hard to deduce the following:

Corollary 2.2: [3] *The set of functions in $L^2(\mathbb{R})$ that has only trivial partners is dense in $L^2(\mathbb{R})$.*

B. Pulse trains

In this section, we consider signals of the form $u(t) = \sum a_j H(t-j)$ where H à support danshas support in $[0,1/2]$. One then easily shows that, for $y \in \mathbb{R}$, $k \in \mathbb{Z}$ and $k - \frac{1}{2} \leq x \leq k + \frac{1}{2}$,

$$A(u)(x,y) = \left(\sum_{j\in\mathbb{Z}} a_j \overline{a_{j-k}} e^{2i\pi jy}\right) A(H)(x-k,y). \quad (4)$$

This lead us to propose the following *discrete radar ambiguity problem* in [6]:

Problem 3: **Discrete Radar Ambiguity Problem.** *Given $a = \{a_j\} \in \ell^2(\mathbb{Z})$, find all sequences $b \in \ell^2(\mathbb{Z})$ such that, for every $k \in \mathbb{Z}$ and $y \in \mathbb{R}$,*

$$|\mathcal{A}(a)(k,y)| = |\mathcal{A}(b)(k,y)|$$

where $\mathcal{A}(a)(k,y) = \sum_{j\in\mathbb{Z}} a_j \overline{a_{j-k}} e^{2i\pi jy}$.

We will then say that a and b are partners and that they are trivial partners if $b_j = e^{i\beta+ij\omega} a_{\pm j-\ell}$ for some $\beta, \omega \in \mathbb{R}$ and $\ell \in \mathbb{Z}$.

One easily sees that, as $\mathcal{A}(a)$ and $\mathcal{A}(b)$ have same support, if the support of a has finite length N then so has b. More precisely, we will say that $a \in \mathcal{S}(N)$ if a has support $\{0,\ldots,N\}$ with $a_0 a_N \neq 0$. Then, if b is a partner of a, up to replacing it by a trivial partner, we may also assume that $b \in \mathcal{S}(N)$.

By adapting a method by Bueckner [4] from the continuous case, the discrete radar ambiguity problem can be reformulated as a combinatorial problem on matrices. More precisely, let

$$d_{j,k} = \begin{cases} a_{\frac{j+k}{2}} a_{\frac{j-k}{2}} & \text{if } j,k \text{ have same parity} \\ 0 & \text{otherwise} \end{cases},$$

and let $K_a = [d_{j,k}]_{-N\leq j,k\leq N}$ and call this the *ambiguity matrix* of a. One then shows that

Proposition 2.3: [3] *Two sequences $a, b \in \mathcal{S}(N)$ are partners if and only if $K_a^* K_a = K_b^* K_b$. In other words, if V_i (resp. W_i) is the i-th column of K_a (resp. K_b) this is equivalent to the following identity for all (i,j)'s:*

$$(i,j) \qquad \langle V_i, V_j\rangle = \langle W_i, W_j\rangle.$$

Recall that the *Kronecker product* of two matrices $A = [a_{i,j}]_{-N\leq i,j\leq N}$ and B is the matrix defined blockwise by

$$A \otimes B = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \ldots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \ldots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}B & a_{n,2}B & \ldots & a_{n,n}B \end{bmatrix}.$$

The main result of [3] for pulse type signals can be stated as follows:

Theorem 2.4: [3] *Let $a \in \mathcal{S}(N)$ and $b \in \mathcal{S}(M)$ $(N, M$ integers) then there is a $c \in \mathcal{S}(M(2N+1)+N)$ such that $K_a \otimes K_b = K_c$. Let us denote $c = a \otimes b$. Then, if a, a' are partners and b, b' are partners, then $a \otimes b$ and $a' \otimes b'$ are partners. Moreover, even if a, a' (resp. b, b') are trivial partners, $a \otimes b$ and $a' \otimes b'$ need not be trivial partners.*

Actually $c = a \otimes b$ may be constructed as follows. Write $P(z) = \sum_{k=0}^{N} a_k z^k$ and $Q(z) = \sum_{k=0}^{M} b_k z^k$. Then $P(z)Q(z^{2N+1}) = \sum_{k=0}^{M(2N+1)+N} c_k z^k$. For example, if $a = (1,2)$, $b = (1,2)$ and $a' = (2,1)$ (a trivial partner of a) then $a \otimes b = (1,2,0,2,4)$ while $a' \otimes b = (2,4,0,1,2)$ and these two sequences are not trivial partners.

Further, based on this construction, we are able to prove the following which has to be compared to Corollary 2.2:

Corollary 2.5: [3] *The set of functions in $L^2(\mathbb{R})$ that has non-trivial partners is dense in $L^2(\mathbb{R})$.*

Finally, we were also able to obtain a result for the ambiguity problem itself, without assuming that v is itself of pulse type. More precisely:

Theorem 2.6: [3] *Let $0 < \eta \leq \frac{1}{3}$ and $u(t) = \sum_{j=0}^{N} a_j \chi_{[j,j+\eta]}(t)$ where $a = (a_0, a_1, \ldots, a_N) \in \mathbb{C}^{N+1}$. Let $v \in L^2(\mathbb{R})$ be a trivial partner of u. Then, up to replacing v by a* trivial partner, *v is of the form $v = \sum_{j=0}^{N} b_j \chi_{[j,j+\eta]}$, where $b = (b_0, b_1, \ldots, b_N) \in \mathbb{C}^{N+1}$ is a partner of a.*

III. The radar ambiguity function and the phase retrieval problem for Fractional Fourier transforms

The aim of this section is to report on ongoing work [9] on the phase retrieval for the Fractional Fourier Transform (FrFT). Note that, mathematically, there is no difference between the phase retrieval problem for the Fractional Fourier Transform and the same problem for the classical Fourier Transform. We here concentrate on the question whether several phase-less measurements can lead to uniqueness results.

The link between the FrFT and the ambiguity function comes from the following property:

$$A(\mathcal{F}_\alpha u, \mathcal{F}_\alpha v)(x,y) = A(u,v)(x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha).$$

In particular,

$$\begin{aligned} A(u)(-y\sin\alpha, y\cos\alpha) &= A(\mathcal{F}_\alpha u)(0,y) \\ &= \mathcal{F}[|\mathcal{F}_\alpha u|^2](y). \quad (5) \end{aligned}$$

It follows that Problem 2 amounts to being able to reconstruct u from the knowledge of its ambiguity function on certain lines going through the origin.

The two main results from [9] are the following:

Theorem 3.1: In the following cases, exact reconstruction can be obtained.

1) *Let* $u, v \in L^2(\mathbb{R})$ *such that, for every* $\alpha \in [-\pi/2, \pi/2]$, $|\mathcal{F}_\alpha v| = |\mathcal{F}_\alpha u|$. *Then there exists* $c \in \mathbb{C}$ *with* $|c| = 1$ *such that* $v = cu$.
2) *Let* $\tau \subset [-\pi/2, \pi/2]$ *be either a set of positive measure or a set with an accumulation point* $\alpha_0 \neq 0$. *Let* $u, v \in L^2(\mathbb{R})$ *with compact support such that, for every* $\alpha \in \tau$, $|\mathcal{F}_\alpha v| = |\mathcal{F}_\alpha u|$. *Then there exists* $c \in \mathbb{C}$ *with* $|c| = 1$ *such that* $v = cu$.
3) *Let* $a > 0$ *and define* $(\alpha_k)_{k \in \mathbb{Z}}$ *by* $\alpha_0 = \pi/2$ *and, for* $k \in \mathbb{Z} \setminus \{0\}$, $\alpha_k = \arctan \dfrac{a^2}{k}$. *Then, given* $u, v \in L^2(\mathbb{R})$ *with compact support included in* $[-a, a]$, *if* $|\mathcal{F}_{\alpha_k} v| = |\mathcal{F}_{\alpha_k} u|$ *for every* $k \in \mathbb{Z}$, *then there exists* $c \in \mathbb{C}$ *with* $|c| = 1$ *such that* $v = cu$.

The last point is based on Shannon's Sampling Formula. In this case, there is even a reconstruction formula. However, to obtain a reasonably accurate reconstruction, we expect that the number of measures $|\mathcal{F}_{\alpha_k} u|$ be rather large, making this reconstruction formula non practicable.

If the signal is assumed to be more "structured", then one or two well chosen measures actually suffice. In the following theorem we group the main results from [9] in that direction.

Theorem 3.2: In the following cases, exact reconstruction can be obtained.

1) *Assume* u, v *are Hermite functions and let* $\alpha \in \mathbb{R} \setminus \mathbb{Q}\pi$. *If* $|v| = |u|$ *and* $|\mathcal{F}_\alpha v| = |\mathcal{F}_\alpha u|$ *then* $v = cu$ *with* $|c| = 1$.
2) *Assume* u, v *are pulse trains* $\sum c_i H(t - t_i)$ *where* $H \in L^2(\mathbb{R})$ *has* $\operatorname{supp} H \subset [0, 1/2]$ *(the same* H *for* u *and* v*) and let* $\alpha \in \mathbb{R} \setminus \frac{\pi}{2}\mathbb{Z}$. *If* $|\mathcal{F}_\alpha v| = |\mathcal{F}_\alpha u|$ *then* $v = cu$ *with* $|c| = 1$.
3) *Assume* u, v *are of the form* $\sum_{finite} c_i e^{-\pi(t-t_i)^2}$, $c_i \in \mathbb{C}$ *and* $t_i \in \mathbb{R}$, *and let* $\alpha \in \mathbb{R} \setminus \frac{\pi}{2}\mathbb{Z}$. *If* $|\mathcal{F}_\alpha v| = |\mathcal{F}_\alpha u|$ *then* $v = cu$ *with* $|c| = 1$.

The assumptions on α in this theorem are sharp. Note that so far we have no algorithm to obtain u from the measures, an issue we plan to tackle in the near future.

IV. CONCLUSION

In this paper we have partially solved the phase retrieval problem for the radar ambiguity function. More precisely, we concentrated our attention on the two common cases (gaussian signals and rectangular pulse trains). In both cases, we have proved that most signals have only trivial partners, if one restricts the problem to these classes of functions. In the case of pulse type signals, we have both the rareness of functions with strange partners, some criteria to have only trivial solutions and various ways to construct functions that have strange partners. Moreover, if the pulses are short enough, then the signal can be reconstructed among all signals (not just pulse trains).

In the second part of the paper, we have used the radar ambiguity function to reconstruct a signal from the modulus of its Fractional Fourier Transforms of various orders. We have shown that, if the orders are well chosen, then uniqueness occurs in several classes of functions, including gaussian signals and pulse trains.

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