HRT Conjecture

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Introduction

- The HRT Conjecture states that any set of distinct time-frequency shifts of any nonzero $L^2(\mathbb{R})$ function, g,

$$\{e^{2\pi i b_k} g(t - a_k)\}_{k=1}^{N} = \{M_{b_k} T_{a_k} g\}_{k=1}^{N},$$

  is linearly independent.
- Such a set is called a Gabor system and is denoted as $\mathcal{G}(g, \Lambda)$, where $\Lambda = \{(a_k, b_k)\}_{k=1}^{N}$.
- This problem was introduced by Heil, Ramanathan and Topiwala in 1996 and remains largely unsolved even for strong decay and smoothness conditions.

Introduction

The initial results of Heil, Ramanathan and Topiwala included:

- g is compactly supported or supported on the half life for any N
- $g(x) = p(x)e^{-x^2}$, where p is a nonzero polynomial for any N
- g if $N \leq 3$
- If the HRT holds for a $g \in L^2(\mathbb{R})/0$ and Λ, then there exists an $\epsilon > 0$ such that the HRT holds for any $h \in L^2(\mathbb{R})/\{0\}$ satisfying $||g - h||_2 < \epsilon$ using the same set Λ
- If the HRT holds for a $g \in L^2(\mathbb{R})/0$ and Λ, then there exists an $\epsilon > 0$ such that the HRT holds for any set of N points within ϵ-Euclidean distance of Λ.

Linnell in 1999 was able to demonstrate the HRT for any Λ that is a translate of a full-rank lattice.

Outline

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Metaplectic Transformation

- One important method for addressing the HRT, is to take some unsolved frame Λ_1 and reducing it some other Λ_2 for which the HRT holds.
- For instance, if one could show that $\mathcal{G}(g, \Lambda_1)) = \mathcal{G}(A(U(A)(f), \Lambda_1) = \Lambda_2)$ where $A : \mathbb{R}^2 \to \mathbb{R}^2$ and $U(A) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ then one would have proven that the HRT holds for $\mathcal{G}(g, \Lambda)$ for all $g \in L^2(\mathbb{R})$
- If A is a linear transformation with determinant equal 1 then there exists U(A) which we call a metaplectic transform for which the above holds.

Outline

Pure Modulation

- To show that the HRT holds for $\mathcal{G}(g, \Lambda)$ where Λ is collinear, we will start with the case of pure modulaton.
- The case of pure modulation

$$\{e^{i\beta_k t} g(t)\}_{k=1}^{n}, \quad \forall g \in L^2(\mathbb{R})/\{0\}.$$

- Suppose that $\sum_{k=1}^{n} c_k e^{i\beta_k t} g(t) = 0$. Since g is nonzero, there is a set of positive measure on which g is nonzero. This implies that $m(t) = \sum_{k=1}^{n} c_k e^{i\beta_k t}$ is zero on that set.

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- $m(t)$ can be extended to $\mathbb{C}$ and the extension is analytic. Since it is zero on a set of positive measure, analyticity requires the function to be identically zero.
- We also see that for a sufficiently large imaginary number, it, $m(it)$ is nonzero since the term with the largest β_k that is paired with a nonzero c_k dominates the expression.
- This requires that all c_k are zero and we have proven the HRT for pure modulation.
- We have also demonstrated that trigonometric polynomials are nonzero, a result which will be used again in the lattice section.

Collinearity

- Suppose we have that Λ is colinear.
- Any collinear set of points by translation and rotation can be made to lie on the y-axis and since these transformations have unit determinant, we have $U(A)$ such that

$$U_A(\mathcal{G}(g,\Lambda)) = \{c_A(a,b)M_v T_u(U_A g)\}_{(u,v)\in A(\Lambda)}$$

- Since $A(\Lambda)$ lies on the y-axis, it represents pure modulation, a case already proven.
- We have demonstrated the HRT holds for any $\mathcal{G}(g,\Lambda)$ for which Λ is collinear.

Outline

Lattice

- Now we demonstrate that the HRT holds for $\mathcal{G}(g, \Lambda)$ where Λ is a subset of the lattice $A(\mathbb{Z} \times \mathbb{Z})$. By a metaplectic transform, this case reduces to $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$.
- We will need the Zak Transform

$$Z[f](t, \omega) = \sum_{k=-\infty}^{\infty} f(t+k)e^{-2\pi k\omega i}$$

- When $\alpha\beta = 1$, an application of the Zak transform yields

$$Z(M_k T_n g)(t, \omega) = e^{2\pi ik\omega} e^{-2\pi ik\omega} Zg(t, \omega), \ (k, n) \in \mathbb{Z}^2.$$

  Independence follows since trignometric polynomials are nonzero.
- The case for general $\alpha\beta$ is more complicated so let us restrict to when $n = 3$.

Lattice, n = 3

- By a metaplectic transform, we can assume that $\Lambda = \{(0,0),(a,0),(0,1)\}$. If the HRT fails for some nonzero c_1, c_2, c_3 for this combination, we can write

$$g(x-a) = m(x)e^{2\pi ix}g(x) \text{ a.e.},$$

where $m(x) = -\frac{1}{c_2}(c_1 + c_3 e^{2\pi ix})$.

- The 1-periodicity of m leads to

$$|g(x-na)| = |g(x)| \prod_{j=0}^{n-1} |m(x-ja)| = |g(x)| e^{n\frac{1}{n}\sum_{j=0}^{n-1} p(x-ja)} \text{ a.e.}$$

where $p(x) = \ln|m(x)|$.

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Lattice, n = 3

- The Birkhoff Ergodic Theorem states that if T is ergodic then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \int f \; dx \quad \text{x a.e.}$$

- Assuming the irrationality of a and defining $T(x)$ as $x - a \mod 1$ requires

$$\lim_{j\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} p(x - ja) = \int_0^1 p \; \mathrm{dx} = C$$

- This convergence implies $\exists \epsilon, \; \exists N$ s.t. $\forall n \geq N$,

$$g(x - na) > e^{Cn-\epsilon} |g(x)|, \text{a.e.}$$

Considering a subset of positive measure for which g is nonzero, this would demand g to have a nonfinite integral, unless $C - \epsilon < 0$ and thereby $C \leq 0$.

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- By a similar argument using $g(x) = m(x+a)g(x+a)$, we have $C \geq 0$
- The case of $C = 0$ is eliminated by another more subtle ergodic argument of Heil and we have demonstrated this for the $n = 3$ lattice case
- For when a is a rational number, $\frac{r}{q}$, we have

$$\lim_{j \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} p(x - ja) = \frac{1}{q} \sum_{l=0}^{q-1} p(x - ja) = C$$

- The same arguments apply to this new C we have defined.

Ratio-Limit Case

Definition

A measurable function g on $\mathbb{R}$ has the ratio-limit $l_g(\alpha) \in \mathbb{C} \cup \{\pm\infty\}$ at $\alpha \in \mathbb{R}$ if

$$\lim_{x \to \infty} \frac{g(x+\alpha)}{g(x)} = l_g(\alpha).$$

Theorem

Let $g \in L^2(\mathbb{R})$ have the ratio-limit $l_g(\alpha)$ at every $\alpha > 0$, and let $\Lambda = \{(\alpha_k, \beta_k)\}_{k=1}^N \subseteq \mathbb{R}^2$. The HRT conjecture holds for $\mathcal{G}(g, \Lambda)$ in the following cases:
(a) $l_g(1) = 0$ and Λ is any finite subset of $\mathbb{R}^2$; and
(b) $l_g(1) \neq 0$ and Λ satisfies the difference condition for the second variable. (Difference condition stated later.)

Proof of Ratio-Limit Case, Part (a)

- Assume the HRT is false.
- From this assumption, we can write

$$\sum_{k=1}^{M} c_k e^{2\pi i \beta_k x} g(x) = \sum_{k=M+1}^{N} c_k e^{2\pi i \beta_k x} g(x + \alpha_k) \text{ a.e.},$$

- Let $\{x_n\}$ be a positive sequence converging to infinity and such that $e^{2\pi i \beta_k x_n}$ converges to a limit $L_k, \forall k$.
- In order to establish a domain for which we can guarunte that the equality, we include the following lemma

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Proof of Ratio-Limit Case, Part (a)

Lemma

Let P be a property that holds for almost every $x \in \mathbb{R}$. For every sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, there exists $E \subseteq \mathbb{R}$ such that its completement is of measure zero and P holds for $x + u_n$ for each $(n, x) \in \mathbb{R} \times E$

Proof.

If $E = \cap_{n\in\mathbb{N}}\{x : P(x + u_n)$ holds for each n$\}$, then P holds for $x + u_n$ for each $(n, x) \in \mathbb{N} \times E$. We know that $|\{x : P(x + u_n) fails\}| = 0$ for each $n \in \mathbb{N}$, and so $| \cup_{n\in\mathbb{N}} \{x : P(x + u_n)$ fails $\}| = 0$, i,e.,$|\mathbb{R}/E| = 0$ $\square$

- Let E be the set obtained by applying this theorem to $\{x_n\}$ and the equality above.

Proof of Ratio-Limit Case, Part (a)

- Fixing a $x \in E$ and given that since g is ultimately nonzero, we have $n_0 > 0$ such that $g(x + x_0)$ is nonzero for all $n > n_0$ and thereby

$$\sum_{k=1}^{M} c_k e^{2\pi i \beta_k (x+x_n)} = \sum_{k=M+1}^{N} c_k e^{2\pi i \beta_k (x+x_n)} \frac{g(x + x_n + \alpha_k)}{g(x + x_n)}.$$

- Letting n tend to infinity, the RHS tends to zero, while the LHS becomes a sum of complex exponentials. Since this holds on all E, it holds a.e. and requiring thereby that the L_k are zero which they could not be from their unit moduli.

Proof of Ratio-Limit Case, Part (b)

- For part b, again assume the HRT is false. We have $l(1) = a \neq 0$ and the difference condition on the second variable, *i.e.*, at least one of the β_k is different from the others. This implies

$$g(x) = \sum_{k=1}^{N} c_k e^{2\pi i \beta_k x} g(x + \alpha_k) \text{a.e.}$$

- Taking a sequence $\{x_n\}$ with the same properties as before, dividing both sides by $g(x)$, and taking the limit, we have

$$\sum_{k=1}^{N-1} c_k l_g(\alpha_k) L_k e^{2\pi i \beta_k x} = 1 \text{ a.e.}$$

Since none of the coefficients are zero, we have a contradiction.

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Ratio-Limit Case

Corollary

Let $g \in L^2(\mathbb{R})\backslash\{0\}$ and let $\Lambda \subseteq \mathbb{R}^2$ have the property that card$(\Lambda) \leq 5$. If g and $\hat{g}$ have ratio limits at every $\alpha \in \mathbb{R}$, then the HRT conjecture holds for $\mathcal{G}(g, \Lambda)$.

Extension and Restriction Principles

Definition

Let $f, g \in L^2(\mathbb{R})$. The Short-Time Fourier Transform (STFT) of a function with respect to a window g is

$$V_g f(x,y) = \int_{\mathbb{R}} f(t)\bar{g}(t-x)e^{-2\pi i y t}\,dt.$$

Definition

The Grammian G_g of $\mathcal{G}(g,\Lambda) = \{e^{2\pi i b_k} g(t-a_k)\}_{k=1}^N$ is given by

$$G_g = (\langle e^{2\pi i b_k t} g(t-a_k), e^{2\pi i b_l t} g(t-a_l)\rangle)_{k,l=1}^N$$

Definition

$$F(a,b) = F_{N+1}(a,b) = \langle G_N^{-1} u_N(a,b), u_N(a,b)\rangle$$

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Extension and Restriction Principles

Theorem

Given that G_N is a positive definite matrix, the following statements hold. (i) $0 \leq F(a,b) \leq 1, \forall (a,b) \in \mathbb{R}^2$*, and moreover,* $F(a_k, b_k) = 1$ *for each* $k = 1, \ldots, N$*. (ii) F is uniformly continuous and* $\lim_{|(a,b)| \to \infty} F(a,b) = 0$

Proof.

(i) The assumption of positive definiteness on G_N requires that G_N^{-1} is positive definite since the eigenvalues of the inverse must be the reciprocal of the eigenvalues of G_N which are all positive. It is proven elsewhere that F will be greater than zero and the upper bound.
(ii) Since both coordinates are uniformly continuous and that the function tends to 0 as it approaches ∞ $\square$

Extension and Restriction Principles

Corollary

Let $g \in L^2(\mathbb{R})$ with $||g||_2 = 1$ and $\Lambda = \{(a_k, b_k)\}_{k=1}^N \cup \{(a, b)\}$. Then $\mathcal{G}(g, \Lambda')$ is linearly independent if and only if $F(a, b) < 1$. Furthermore, there exists $R := R(\Lambda, g) > 0$ such that for all $(a, b) \in \mathbb{R}^2$ with $|(a, b)| > R$, then $\mathcal{G}(g, \Lambda')$ is linearly independent where $\Lambda' = \Lambda \cup \{(a, b)\}$

Theorem

Let $g \in L^2(\mathbb{R})$ with $||g||_2 = 1$. Suppose Λ is a (3,2) configuration given by $\Lambda = \{(0, 0, (0, 1), (0, -1), (a, b), (a, -b))\}$ where $b \neq 0$. Then, the HRT holds for Λ and g whenever

(i) a,b are rationally dependent,

(ii) $a \in \mathbb{Q}$ but $b \notin \mathbb{Q}$,

(iii) $a = b \notin \mathbb{Q}$,

(iv) a,b $\notin \mathbb{Q}$ but $ab \notin \mathbb{Q}$

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Proof of (ii)

- By use of a metaplectic transform, we can write

$$\Lambda = \{(0,0),(0,a),(0,-a),(1,b'=ba),(1,-b')\}$$

- Suppose $\mathcal{G}(g,\Lambda)$ is linearly dependent. So there are c_k such that

$$c_1 g + c_2 M_a g + c_3 M_{-a} g = -c_4 M_b T_1 g - c_5 M_{-b} T_1 g.$$

- Taking the difference between this equation and its conjugate, we find

$$(c_2 - \bar{c}_3) M_a g + (c_3 - \bar{c}_2) M_{-a} g + (c_4 - \bar{c}_5) M_{-b} T_1 g + (c_5 - \bar{c}_4) M_{-b} T_1 g = 0$$

- This is as a (2,2) configuration so the HRT demands $c_3 = \bar{c}_2, c_5 = \bar{c}_4$.

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Proof of (ii)

- We can rewrite our initial statement of the HRT as

$$P(x)g(x) = Q(x)g(x-1) \text{ a.e.},$$

  where $P(x) = c + 2r\cos 2\pi(ax+\theta)$ and $Q(x) = 2r' cos 2\pi(bx+\theta')$.
- Furthermore, because $g \in L^2(\mathbb{R})$ we have that

$$\lim_{|n|\to\infty, n\in\mathbb{Z}} g(x-n) = 0 \text{ a.e.},$$

  and that $supp(g) \cap [0,1]$ has positive measure.
- Let S be a subset thereof of positive measure and such that $S+\mathbb{Z}$ does not have any zeros of P and Q.

- By Birchoff's pointwise ergodic theorem with 1_S,

$$\exists x_0 \ \exists n' \text{ s.t. } x_1 = \{-x_0 - \frac{2\theta'}{b} + \frac{n'}{b}\} \in S.$$

  Define $m = -x_0 - \frac{2\theta'}{b} + \frac{n'}{b} - x_1$
- By iterating our polynomial expression, for all $N > m$

$$\begin{cases} \mathrm{abs}(\mathrm{g}(\mathrm{x}_0 + N)) = |g(x_0 - 1)| \frac{\prod_{n=0}^{N} |Q(x_0+n)|}{\prod_{n=0}^{N} |P(x_0+n)|} \\ \mathrm{abs}(\mathrm{g}(\mathrm{x}_1 - N + m)) = |g(x_1 - 1)| \frac{\prod_{n=-N+m}^{-1} |P(x_1+n)|}{\prod_{n=-N+m}^{-1} |Q(x_1+n)|} \end{cases}$$

Proof of (ii)

- Let $T(x) := \prod_{n=1}^{s-1} |P(x+n)|$, and assume $T(x_1) \geq T(x_0)$
- With some additional algebra, we have $N > m$

$$|g(x_1 - N + m)| = |g(x_1 - 1)| \frac{\prod_{n=-N+m}^{-1} |P(x_1 + n)|}{\prod_{n=-N+m}^{-1} |Q(x_1 + n)|}$$

$$\geq |g(x_1 - 1)| \frac{\prod_{n=0}^{N} |P(x_0 + n)|}{\prod_{n=0}^{N} |Q(x_0 + n)|} \geq |g(x_1 - 1)||g(x_0 - 1)||g(x_0 + N)|^{-1}$$

contradicting the convergence of $g(x-n)$ to 0, a.e. x.

- Alternatively suppose $T(x_0) \geq T(x_1)$. By a similar argument, we have

$$|g(x_0 - N + m)| \geq |g(x_0 - 1)||g(x_1 - 1)||g(x_1 + N)|^{-1}.$$

This again contradicts the convergence of $g(x-n)$ to 0 and we have proven (ii).

Summary

- The cases for which the HRT which we have seen:
  - Λ is collinear or the subset of a lattice.
  - The ratio-limit exists for the function
  - Certain (3,2)-configurations of Λ

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