Constant Amplitude and Zero Autocorrelation Sequences and Single Pixel Camera Imaging

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Frames

A *finite frame* for $\mathbb{C}^N$ is a set $\mathcal{F} = \{v_j\}_{j=1}^M$ such that there exists constants $0 < A \leq B < \infty$ where

$$A\|x\|_2^2 \leq \sum_{j=1}^{M} |\langle x, v_j \rangle|^2 \leq B\|x\|_2^2$$

for any $x \in \mathbb{C}^N$. $\mathcal{F}$ is called a *tight frame* if $A = B$ is possible.

Theorem

$\mathcal{F}$ is a frame for $\mathbb{C}^N$ if and only if $\mathcal{F}$ spans $\mathbb{C}^N$.

The Frame Operator

Let $\mathcal{F} = \{v_j\}_{j=1}^{M}$ be a frame for $\mathbb{C}^N$ and $x \in \mathbb{C}^N$.

(a) The *frame operator*, $S : \mathbb{C}^N \to \mathbb{C}^N$, is given by

$$S(x) = \sum_{j=1}^{M} \langle v_j, x \rangle v_j.$$

(b) Given any $x \in \mathbb{C}^N$ we can write x in terms of frame elements by

$$x = \sum_{j=1}^{M} \langle x, S^{-1} v_j \rangle v_j.$$

(c) If $\mathcal{F}$ is a tight frame with bound A, then $S = A\, Id_N$.

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Definition

(a) Let $\varphi \in \mathbb{C}^N$ and $\Lambda \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})\widehat{}$. The *Gabor system*, (φ, Λ) is defined by

$$(\varphi, \Lambda) = \{e_\ell \tau_k \varphi : (k, \ell) \in \Lambda\}.$$

(b) If (φ, Λ) is a frame for $\mathbb{C}^N$ we call it a Gabor frame.

Time-Frequency Transforms

Definition

Let $\varphi, \psi \in \mathbb{C}^N$.

(a) The *discrete periodic ambiguity function* of φ, $A_p(\varphi)$, is defined by

$$A_p(\varphi)[k,\ell] = \frac{1}{N}\sum_{j=0}^{N-1} \varphi[j+k]\overline{\varphi[j]}e^{-2\pi i j\ell/N} = \frac{1}{N}\langle \tau_{-k}\varphi, e_\ell\varphi\rangle.$$

(b) The *short-time Fourier transform* of φ with window ψ, $V_\psi(\varphi)$, is defined by

$$V_\psi(\varphi)[k,\ell] = \langle \varphi, e_\ell \tau_k \psi\rangle.$$

Full Gabor Frames Are Always Tight

Theorem

Let $\varphi \in \mathbb{C}^N \setminus \{0\}$. *and* $\Lambda = (\mathbb{Z}/N\mathbb{Z}) \times (\widehat{\mathbb{Z}/N\mathbb{Z}})$. *Then,* (φ, Λ) *is always a tight frame with frame bound* $N\|\varphi\|_2^2$.

Janssen's Representation

Definition

Let $\Lambda \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})\widehat{}$ be a subgroup. The *adjoint subgroup* of Λ, $\Lambda^\circ \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})\widehat{}$, is defined by

$$\Lambda^\circ = \{(m,n) : e_\ell \tau_k e_n \tau_m = e_n \tau_m e_\ell \tau_k, \forall (k,\ell) \in \Lambda\}$$

Theorem

Let Λ be a subgroup of $(\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})\widehat{}$ and $\varphi \in \mathbb{C}^N$. Then, the (φ, Λ) Gabor frame operator has the form

$$S = \frac{|\Lambda|}{N} \sum_{(m,n)\in\Lambda^\circ} \langle \varphi, e_n \tau_m \varphi \rangle e_n \tau_m.$$

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Λ°-sparsity and Tight Frames

Theorem

Let $\varphi \in \mathbb{C}^N \setminus \{0\}$ and let $\Lambda \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})\widehat{}$ be a subgroup. (φ, Λ) is a tight frame if and only if

$$\forall (m,n) \in \Lambda^\circ, (m,n) \neq 0, \quad A_p(\varphi)[m,n] = 0.$$

The frame bound is $|\Lambda| A_p(\varphi)[0,0]$.

Proof of Λ°-sparsity Theorem

By Janssen's representation we have

$$S = \frac{|\Lambda|}{N} \sum_{(m,n)\in\Lambda^\circ} \langle e_n \tau_m \varphi, \varphi \rangle e_n \tau_m = \sum_{(m,n)\in\Lambda^\circ} \langle \tau_m \varphi, e_{-n}\varphi \rangle e_n \tau_m$$

$$= |\Lambda| \sum_{(m,n)\in\Lambda^\circ} A_p(\varphi)[-m,-n] e_n\, \tau_m = |\Lambda| \sum_{(m,n)\in\Lambda^\circ} A_p(\varphi)[m,n] e_{-n}\tau_{-m}.$$

If $A_p(\varphi)[m,n] = 0$ for every $(m,n) \in \Lambda^\circ, (m,n) \neq 0$, then S is $|\Lambda| A_p(\varphi)[0,0]$ times the identity. and so (φ, Λ) is a tight frame.

Proof of Λ°-sparsity Theorem (con't)

To show this is a necessary condition, we observe that for S to be tight we need

$$S = |\Lambda| \sum_{(m,n)\in\Lambda^\circ} A_p(\varphi)[m,n] e_n \tau_m = A\, Id$$

which can be rewritten as

$$\sum_{(m,n)\in\Lambda^\circ\setminus\{(0,0)\}} |\Lambda| A_p(\varphi)[m,n] e_n \tau_m + (|\Lambda| A_p(\varphi)[0,0] - A) Id = 0.$$

CAZAC Definition

Let $\varphi \in \mathbb{C}^N$. φ is said to be a *constant amplitude zero autocorrelation (CAZAC) sequence* if

$$\forall j \in (\mathbb{Z}/N\mathbb{Z}), |\varphi_j| = 1 \tag{CA}$$

and

$$\forall k \in (\mathbb{Z}/N\mathbb{Z}), k \neq 0, \frac{1}{N}\sum_{j=0}^{N-1} \varphi_{j+k}\overline{\varphi_j} = 0. \tag{ZAC}$$

Examples

Quadratic Phase Sequences

Let $\varphi \in \mathbb{C}^N$ and suppose for each j, φ_j is of the form $\varphi_j = e^{-\pi i p(j)}$ where p is a quadratic polynomial. The following quadratic polynomials generate CAZAC sequences:

- Chu: $p(j) = j(j-1)$
- P4: $p(j) = j(j-N)$, N is odd
- Odd-length Wiener: $p(j) = sj^2$, $\gcd(s, N) = 1$, N is odd
- Even-length Wiener: $p(j) = sj^2/2$, $\gcd(s, 2N) = 1$, N is even

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Björck Sequences

Let p be prime and $\varphi \in \mathbb{C}^p$ be of the form $\varphi_j = e^{i\theta(j)}$. Then φ will be CAZAC in the following cases:

- If $p \equiv 1 \mod 4$, then,

$$\theta(j) = \left(\frac{j}{p}\right) \arccos\left(\frac{1-p}{1+\sqrt{p}}\right)$$

- If $p \equiv 3 \mod 4$, then,

$$\begin{cases} \arccos\left(\frac{1-p}{1+p}\right), & \text{if } \left(\frac{j}{p}\right) = -1 \\ 0, & \text{otherwise} \end{cases}$$

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Connection to Hadamard Matrices

Theorem

Let $\varphi \in \mathbb{C}^N$ and let H be the circulant matrix given by

$$H = \begin{bmatrix} \text{———} \ \varphi \ \text{———} \\ \text{———} \ \tau_1\varphi \ \text{———} \\ \text{———} \ \tau_2\varphi \ \text{———} \\ \cdots \\ \text{——} \ \tau_{N-1}\varphi \ \text{——} \end{bmatrix}$$

Then, φ is a CAZAC sequence if and only if H is Hadamard. In particular there is a one-to-one correspondence between CAZAC sequences and circulant Hadamard matrices.

Connection to Cyclic N-roots

Definition

$x \in \mathbb{C}^N$ is a cyclic N-root if it satisfies

$$\begin{cases} x_0 + x_1 + \cdots + x_{N-1} = 0 \\ x_0 x_1 + x_1 x_2 + \cdots + x_{N-1} x_0 = 0 \\ \cdots \\ x_0 x_1 x_2 \cdots x_{N-1} = 1 \end{cases}$$

Connection to Cyclic N-roots

Theorem

(a) *If $\varphi \in \mathbb{C}^N$ is a CAZAC sequence then,*

$$\left(\frac{\varphi_1}{\varphi_0}, \frac{\varphi_2}{\varphi_1}, \dots, \frac{\varphi_0}{\varphi_{N-1}}\right)$$

is a cyclic N-root.

(b) *If $x \in \mathbb{C}^N$ is a cyclic N-root then,*

$$\varphi_0 = x_0, \varphi_j = \varphi_{j-1} x_j$$

is a CAZAC sequence.

(c) *There is a one-to-one correspondence between CAZAC sequences which start with 1 and cyclic N-roots.*

5-Operation Equivalence

Proposition

Let $\varphi \in \mathbb{C}^N$ be a CAZAC sequence. Then, the following are also CAZAC sequences:

(a) $\forall c \in \mathbb{C}, |c| = 1,\ c\varphi$

(b) $\forall k \in \mathbb{Z}/N\mathbb{Z},\ \tau_k\varphi$

(c) $\forall \ell \in \widehat{(\mathbb{Z}/N\mathbb{Z})}, e_\ell\varphi$

(d) $\forall m \in \mathbb{Z}/N\mathbb{Z}, gcd(m, N) = 1,\ \pi_m\varphi[j]$

(e) $n = 0, 1,\ c_n\varphi$

The operation π_m is defined by $\pi_m\varphi[j] = \varphi[mj]$ and c_0, c_1 is defined by $c_0\varphi = \varphi$ and $c_1\varphi = \overline{\varphi}$.

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5-Operation Group Action

Let G be the set given by

$$\{(a,b,c,d,f) : a \in \{0,1\}, b,c,d,f \in \mathbb{Z}/p\mathbb{Z}, c \neq 0\}$$

and define the operation $\cdot : G \times G \to G$ by

$$(a,b,c,d,f) \cdot (h,j,k,\ell,m)$$

$$= (a+h, cj+b, ck, \ell + (-1)^h kd, m + (-1)^h (f - jc)).$$

Then, $(G, \cdot)$ is a group of size $2p^3(p-1)$.

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5-Operation Group Action (cont'd)

- To each element $(a, b, c, d, f) \in G$ we associate the operator $\omega_f e_d \pi_c \tau_b c_a$, which form a group under composition, where $\omega = e^{2\pi i/p}$.
- The composition is computed by the operation for $(G, \cdot)$ and obtaining the operator associated with the computed result.
- The group of operators under composition forms a group action for U_p^p, the set of p-length vectors comprised entirely of p-roots of unity.
- There are $p(p-1)$ many CAZACs in U_p^p which start with 1. Adding in any scalar multiples of roots of unity, there are $p^2(p-1)$ many.

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5-Operation Orbits

Theorem

Let p be an odd prime and let $\varphi \in U_p^p$ be the Wiener sequence $\varphi[n] = e^{2\pi i s n^2/p}$, where $s \in \mathbb{Z}/p\mathbb{Z}$. Denote the stabilizer of φ under the group $(G, \cdot)$ as G_φ. If $p \equiv 1 \mod 4$, then $|G_\varphi| = 4p$. If $p \equiv 3 \mod 4$, then $|G_\varphi| = 2p$. In particular, the orbit of φ has size $p^2(p-1)/2$ if $p \equiv 1 \mod 4$ and size $p^2(p-1)$ if $p = 3 \mod 4$.

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Sketch of Proof

- Take a linear operator $\omega_f e_d \pi_c \tau_b c_a$ and write the system of equations that would describe fixing each term of $\varphi[n]$.
- Use the $n = 0, 1$ cases to get expressions for f and d in terms of the other variables.
- Use any other $n > 1$ and substitute the expressions for f and d to obtain the condition

$$c^2 \equiv (-1)^a \mod p.$$

Sketch of Proof (con't)

- In the case $a = 0$, there are always the solutions $c \equiv \pm 1 \mod p$. If $a = 1$, then it depends if -1 is a quadratic residue modulo p. It is if $p \equiv 1 \mod 4$ and is not if $p \equiv 3 \mod 4$.
- All variables are solved for except b and the solutions leave it as a free parameter. Thus there are $4p$ and $2p$ stabilizers for the $p \equiv 1 \mod 4$ and $p \equiv 3 \mod 4$ cases respectively.

DPAF of Chu Sequence

$$A_p(\varphi_{\text{Chu}})[k,\ell]:$$

$$\begin{cases} e^{\pi i(k^2-k)/N}, & k \equiv \ell \bmod N \\ 0, & \text{otherwise} \end{cases}$$

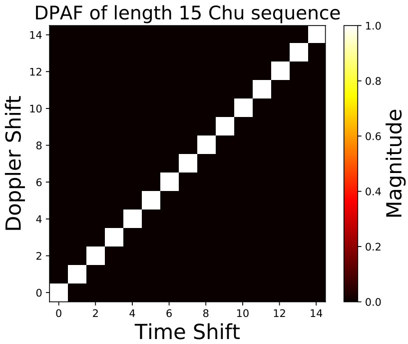

Figure: DPAF of length 15 Chu sequence.

Example: Chu/P4 Seqeunce

Proposition

Let $N = abN'$ where $\gcd(a, b) = 1$ and $\varphi \in \mathbb{C}^N$ be the Chu or P4 sequence. Define $K = \langle a \rangle$, $L = \langle b \rangle$ and $\Lambda = K \times L$.

(a) $\Lambda^\circ = \langle N'a \rangle \times \langle N'b \rangle$.

(b) (φ, Λ) *is a tight Gabor frame bound* NN'.

DPAF of Even Length Wiener Sequence

$$A_p(\varphi_{\text{Wiener}})[k,\ell]:$$
$$\begin{cases} e^{\pi i s k^2/N}, & sk \equiv \ell \bmod N \\ 0, & \text{otherwise} \end{cases}$$

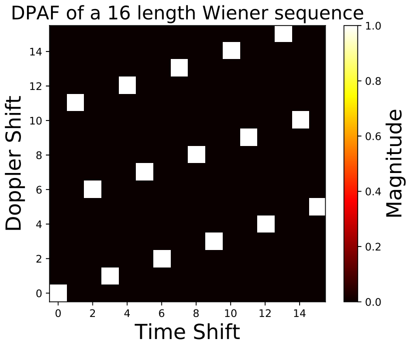

Figure: DPAF of length 16 P4 sequence.

DPAF of Björck Sequence

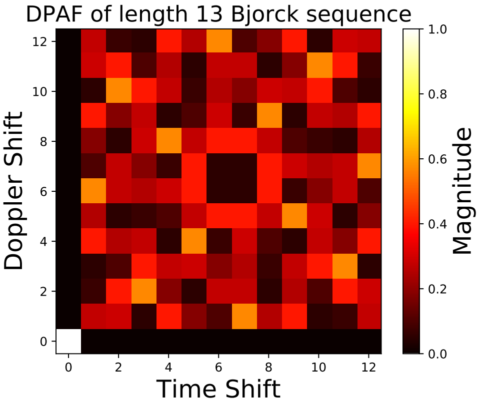

Figure: DPAF of length 13 Björck sequence.

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DPAF of a Kronecker Product Sequence

Kronecker Product:
Let $u \in \mathbb{C}^M, v \in \mathbb{C}^N$.
$(u \otimes v)[aM + b] = u[a]v[b]$

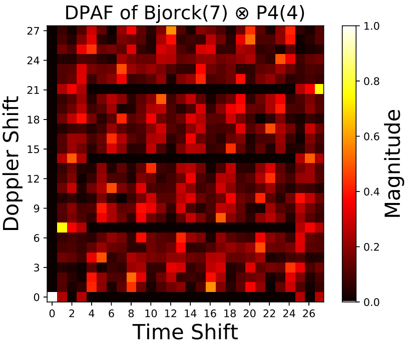


Figure: DPAF of Kroneker product of length 7 Bjorck and length 4 P4.

Example: Kronecker Product Sequence

Proposition

Let $u \in \mathbb{C}^M$ be CAZAC, $v \in \mathbb{C}^N$ be CA, and $\varphi \in \mathbb{C}^{MN}$ be defined by the Kronecker product: $\varphi = u \otimes v$. If $\gcd(M, N) = 1$ and $\Lambda = \langle M \rangle \times \langle N \rangle$, then (φ, Λ) is a tight frame with frame bound MN.

Gram Matrices and Discrete Periodic Ambiguity Functions

Definition

Let $\mathcal{F} = \{v_i\}_{i=1}^{M}$ be a frame for $\mathbb{C}^N$. The *Gram matrix*, G, is defined by

$$G_{i,j} = \langle v_i, v_j \rangle.$$

In the case of Gabor frames $\mathcal{F} = \{e_{\ell_m} \tau_{k_m} \varphi : m \in 0, \cdots, M-1\}$, we can write the Gram matrix in terms of the discrete periodic ambiguity function of φ:

$$G_{m,n} = N e^{-2\pi i k_n (\ell_n - \ell_m)/N} A_p(\varphi)[k_n - k_m, \ell_n - \ell_m]$$

Gram Matrix of Chu and P4 Sequences

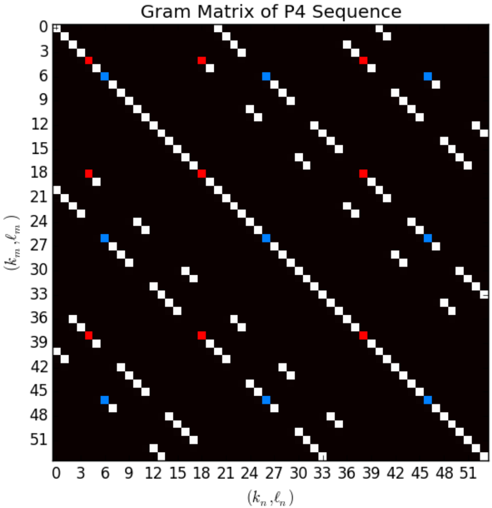
Lemma

Let $\varphi \in \mathbb{C}^N$ be the Chu or P4 sequence and let $N = abN'$ where $gcd(a, b) = 1$. Suppose G is the Gram matrix generated by the Gabor system $(\varphi, K \times L)$ where $K = \langle a \rangle$ and $L = \langle b \rangle$. Then,

(a) *The support of the rows (or columns) of G either completely conincide or are completely disjoint.*

(b) *If two rows (or columns) have coinciding supports, they are scalar multiples of each other.*

Example: P4 Gram Matrix



Tight Frames from Gram Matrix

Theorem

Let $\varphi \in \mathbb{C}^N$ be the Chu or P4 sequence and let $N = abN'$ where $gcd(a, b) = 1$. Suppose G is the Gram matrix generated by the Gabor system $(\varphi, K \times L)$ where $K = \langle a \rangle$ and $L = \langle b \rangle$. Then,

(a) $rank(G) = N$.

(b) G *has exactly one nonzero eigenvalue,* NN'.

In particular (a) and (b) together imply that the Gabor system $(\varphi, K \times L)$ is a tight frame with frame bound NN'.

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Proof

- $K \cap L = \langle ab \rangle$.
- $G_{mn} \neq 0$ if and only if $(\ell_n - \ell_m) \equiv (k_n - k_m) \mod N$.
- This can only occur at the intersection of K and L, i.e.,

$$\forall j \in (\mathbb{Z}/N'\mathbb{Z}),\ (\ell_n - \ell_m) \equiv (k_n - k_m) \equiv jab \mod N$$

- Fix an m, we can write k_n as $k_n = a(j_m + jb)$ for some j
- By the column ordering of G we can write the n-th column by

$$n = k_n N' + \ell_n / b.$$

Proof (con't)

- We want to look at the first N columns so we want $n < N$ and thus require $k_n < ab$.
- Thus, $(j_m + jb) < b$.
- There is exactly one such $j \in (\mathbb{Z}/N'\mathbb{Z})$ and it is $j = -\lfloor j_m/b \rfloor$.
- Consequently, for each row m, there is exactly 1 column $n \leq N$ where $G_{mn} \neq 0$ and the first N columns of G are linearly independent.
- $\text{rank}(G) = N$.

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Proof (con't)

- Let g_n be the n-th column of G, $n < N$.
- The goal is to show that $Gg_n = NN'g_n$.
- $Gg_n[m]$ is given by the inner product of row m and column n.
- The n-th column is the conjugate of the n-th row.
- If $Gg_n[m] \neq 0$, then $G_{mn} \neq 0$ and $G_{nn} \neq 0$.

Proof (con't)

- Lemma implies row m and n have coinciding supports and are constant multiples of each other thus, $G_{m(\cdot)} = C_m g_n^*$ where $|C_m| = 1$.
- Therefore, $Gg_n[m] = C_m \|g_n\|_2^2 = N^2 N' C_m$.
- $G_{nn} = N$, so $g_n[m] = NC_m$.
- Finally, $Gg_n[m] = (NN')(NC_m)$ and we conclude the first N columns of G are eigenvectors of G with eigenvalue NN'.

Future work: Continuous CAZAC Property

Is it possible to generalize CAZAC to the real line? The immediate problem is the natural inner product to use for autocorrelation is the $L^2(\mathbb{R})$ inner product, but if $|f(x)| = 1$ for every $x \in \mathbb{R}$, then clearly $f \notin L^2(\mathbb{R})$.

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Future work: Continuous CAZAC Property

Alternatives and ideas:

- Define a continuous autocorrelation on $L^2(\mathbb{T})$ and push torus bounds to infinity.
- Distribution theory, esp. using tempered distributions.
- Wiener's Generalized Harmonic Analysis, which includes a theory of mean autocorrelation on $\mathbb{R}$.

Future work: Single Pixel Camera

- My work on this is with John Benedetto and Alfredo Nava-Tudela and is an ongoing project.
- The original concept is due to Richard Baraniuk.
- The idea is to construct a camera using only a single light receptor or sensor.
- This is accomplished by filtering through a pixel grid that either admits or blocks light.
- Baraniuk's original design does this with digital micromirror devices.
- Several collections with different pixel grids are required but compressed sensing theory allows this to be done efficiently.

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Questions?

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