Constructing Tight Gabor Frames using CAZAC Sequences

Mark Magsino
mmagsino@math.umd.edu

Norbert Wiener Center for Harmonic Analysis and Applications
Department of Mathematics
University of Maryland, College Park

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CAZAC Definition

Let $\varphi \in \mathbb{C}^N$. φ is said to be a *constant amplitude zero autocorrelation (CAZAC) sequence* if

$$\forall j \in (\mathbb{Z}/N\mathbb{Z}), |\varphi_j| = 1 \tag{CA}$$

and

$$\forall k \in (\mathbb{Z}/N\mathbb{Z}), k \neq 0, \frac{1}{N}\sum_{j=0}^{N-1} \varphi_{j+k}\overline{\varphi_j} = 0. \tag{ZAC}$$

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Examples

Quadratic Phase Sequences

Let $\varphi \in \mathbb{C}^N$ and suppose for each j, φ_j is of the form $\varphi_j = e^{-\pi i p(j)}$ where p is a quadratic polynomial. The following quadratic polynomials generate CAZAC sequences:

- Chu: $p(j) = j(j-1)$
- P4: $p(j) = j(j-N)$, N is odd
- Odd-length Wiener: $p(j) = sj^2$, $\gcd(s, N) = 1$, N is odd
- Even-length Wiener: $p(j) = sj^2/2$, $\gcd(s, 2N) = 1$, N is even

Examples

Let p be prime. Then, the *Legendre symbol* is defined as follows,

$$\left(\frac{j}{p}\right) = \begin{cases} 0 & \text{if } j \equiv 0 \mod p, \\ 1 & \text{if } j \equiv k^2 \mod p \text{ has a solution,} \\ -1 & \text{if } j \equiv k^2 \mod p \text{ does not have a solution.} \end{cases}$$

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Examples

Björck Sequences

Let p be prime and $\varphi \in \mathbb{C}^p$ be of the form $\varphi_j = e^{i\theta(j)}$. Then φ will be CAZAC in the following cases:

- If $p \equiv 1 \mod 4$, then,

$$\theta(j) = \left(\frac{j}{p}\right) \arccos\left(\frac{1}{1+\sqrt{p}}\right)$$

- If $p \equiv 3 \mod 4$, then,

$$\theta_j = \begin{cases} \arccos\left(\frac{1-p}{1+p}\right), & \text{if } \left(\frac{j}{p}\right) = -1 \\ 0, & \text{otherwise} \end{cases}$$

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Connection to Hadamard Matrices

Theorem

Let $\varphi \in \mathbb{C}^N$ and let H be the circulant matrix given by

$$H = \begin{bmatrix} \text{——} & \varphi & \text{——} \\ \text{——} & \tau_1\varphi & \text{——} \\ \text{——} & \tau_2\varphi & \text{——} \\ & \cdots & \\ \text{——} & \tau_{N-1}\varphi & \text{——} \end{bmatrix}$$

Then, φ is a CAZAC sequence if and only if H is Hadamard, i.e. $H^*H = NId_N$ and $|H_{ij}| = 1$ for every (i,j). In particular there is a one-to-one correspondence between CAZAC sequences and circulant Hadamard matrices.

Connection to Cyclic N-roots

Definition

$x \in \mathbb{C}^N$ is a cyclic N-root if it satisfies

$$\begin{cases} x_0 + x_1 + \cdots + x_{N-1} = 0 \\ x_0 x_1 + x_1 x_2 + \cdots + x_{N-1} x_0 = 0 \\ \cdots \\ x_0 x_1 x_2 \cdots x_{N-1} = 1 \end{cases}$$

Connection to Cyclic N-roots

Theorem

(a) *If $\varphi \in \mathbb{C}^N$ is a CAZAC sequence then,*

$$\left(\frac{\varphi_1}{\varphi_0}, \frac{\varphi_2}{\varphi_1}, \cdots, \frac{\varphi_0}{\varphi_{N-1}}\right)$$

is a cyclic N-root.

(b) *If $x \in \mathbb{C}^N$ is a cyclic N-root then,*

$$\varphi_0 = x_0, \varphi_j = \varphi_{j-1} x_j$$

is a CAZAC sequence.

(c) *There is a one-to-one correspondence between CAZAC sequences which start with 1 and cyclic N-roots.*

Gabor Frames

Definition

(a) Let $\varphi \in \mathbb{C}^N$ and $\Lambda \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\widehat{\mathbb{Z}/N\mathbb{Z})}$. The *Gabor system*, (φ, Λ) is defined by

$$(\varphi, \Lambda) = \{e_\ell \tau_k \varphi : (k, \ell) \in \Lambda\}.$$

(b) If (φ, Λ) is a frame for $\mathbb{C}^N$ we call it a Gabor frame.

Time-Frequency Transforms

Definition

Let $\varphi, \psi \in \mathbb{C}^N$.

(a) The *discrete periodic ambiguity function* of φ, $A_p(\varphi)$, is defined by

$$A_p(\varphi)[k,\ell] = \frac{1}{N}\sum_{j=0}^{N-1} \varphi[j+k]\overline{\varphi[j]}e^{-2\pi i j\ell/N} = \frac{1}{N}\langle \tau_{-k}\varphi, e_\ell\varphi\rangle.$$

(b) The *short-time Fourier transform* of φ with window ψ, $V_\psi(\varphi)$, is defined by

$$V_\psi(\varphi)[k,\ell] = \langle \varphi, e_\ell \tau_k \psi\rangle.$$

Full Gabor Frames Are Always Tight

Theorem

Let $\varphi \in \mathbb{C}^N \setminus \{0\}$*. and* $\Lambda = (\mathbb{Z}/N\mathbb{Z}) \times \widehat{(\mathbb{Z}/N\mathbb{Z})}$*. Then,* (φ, Λ) *is always a tight frame with frame bound* $N\|\varphi\|_2^2$*.*

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Janssen's Representation

Definition

Let $\Lambda \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})\widehat{}$ be a subgroup. The *adjoint subgroup* of Λ, $\Lambda^\circ \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})\widehat{}$, is defined by

$$\Lambda^\circ = \{(m,n) : e_\ell \tau_k e_n \tau_m = e_n \tau_m e_\ell \tau_k, \forall (k,\ell) \in \Lambda\}$$

Theorem (Janssen '95)

Let Λ be a subgroup of $(\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})\widehat{}$ and $\varphi \in \mathbb{C}^N$. Then, the (φ, Λ) Gabor frame operator has the form

$$S = \frac{|\Lambda|}{N} \sum_{(m,n) \in \Lambda^\circ} \langle \varphi, e_n \tau_m \varphi \rangle e_n \tau_m.$$

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Λ°-sparsity and Tight Frames

Theorem (MM '17)

Let $\varphi \in \mathbb{C}^N \setminus \{0\}$ *and let* $\Lambda \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})\widehat{}$ *be a subgroup.* (φ, Λ) *is a tight frame if and only if*

$$\forall (m,n) \in \Lambda^\circ, \quad A_p(\varphi)[m,n] = 0.$$

The frame bound is $|\Lambda| A_p(\varphi)[0,0]$.

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DPAF of Chu Sequence

$$A_p(\varphi_{\text{Chu}})[k,\ell]:$$

$$\begin{cases} e^{\pi i(k^2-k)/N}, & k \equiv \ell \bmod N \\ 0, & \text{otherwise} \end{cases}$$

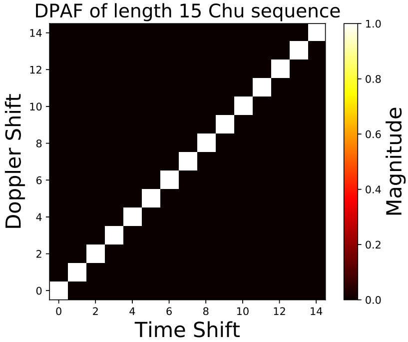

Figure: DPAF of length 15 Chu sequence.

Example: Chu/P4 Seqeunce

Proposition

Let $N = abN'$ where $\gcd(a, b) = 1$ and $\varphi \in \mathbb{C}^N$ be the Chu or P4 sequence. Define $K = \langle a \rangle$, $L = \langle b \rangle$ and $\Lambda = K \times L$.

(a) $\Lambda^\circ = \langle N'a \rangle \times \langle N'b \rangle$.

(b) (φ, Λ) *is a tight Gabor frame bound NN'.*

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DPAF of Even Length Wiener Sequence

$$A_p(\varphi_{\text{Wiener}})[k,\ell]:$$
$$\begin{cases} e^{\pi i s k^2/N}, & sk \equiv \ell \bmod N \\ 0, & \text{otherwise} \end{cases}$$

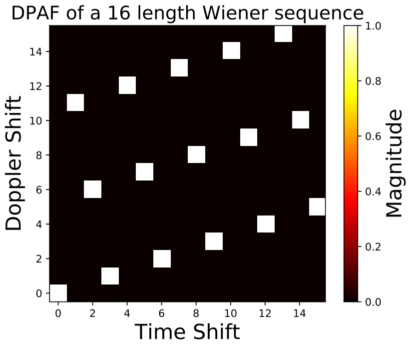

Figure: DPAF of length 16 P4 sequence.

DPAF of Björck Sequence

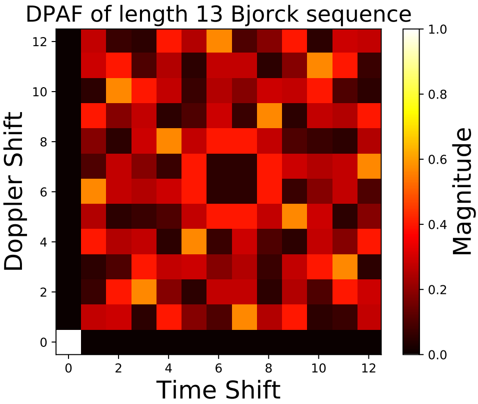

Figure: DPAF of length 13 Björck sequence.

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DPAF of a Kronecker Product Sequence

Kronecker Product:
Let $u \in \mathbb{C}^M, v \in \mathbb{C}^N$.
$(u \otimes v)[aM + b] = u[a]v[b]$

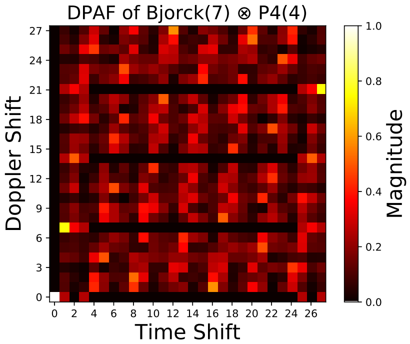

Figure: DPAF of Kroneker product of length 7 Bjorck and length 4 P4.

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Example: Kronecker Product Sequence

Proposition

Let $u \in \mathbb{C}^M$ be CAZAC, $v \in \mathbb{C}^N$ be CA, and $\varphi \in \mathbb{C}^{MN}$ be defined by the Kronecker product: $\varphi = u \otimes v$. If gcd $(M, N) = 1$ and $\Lambda = \langle M \rangle \times \langle N \rangle$, then (φ, Λ) is a tight frame with frame bound MN.

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Gram Matrices and Discrete Periodic Ambiguity Functions

Definition

Let $\mathcal{F} = \{v_i\}_{i=1}^{M}$ be a frame for $\mathbb{C}^N$. The *Gram matrix*, G, is defined by

$$G_{ij} = \langle v_i, v_j \rangle.$$

In the case of Gabor frames $\mathcal{F} = \{e_{\ell_m}\tau_{k_m}\varphi : m \in 0, \cdots, M-1\}$, we can write the Gram matrix in terms of the discrete periodic ambiguity function of φ:

$$G_{mn} = Ne^{-2\pi i k_n(\ell_n - \ell_m)/N} A_p(\varphi)[k_n - k_m, \ell_n - \ell_m]$$

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Gram Matrix of Chu and P4 Sequences

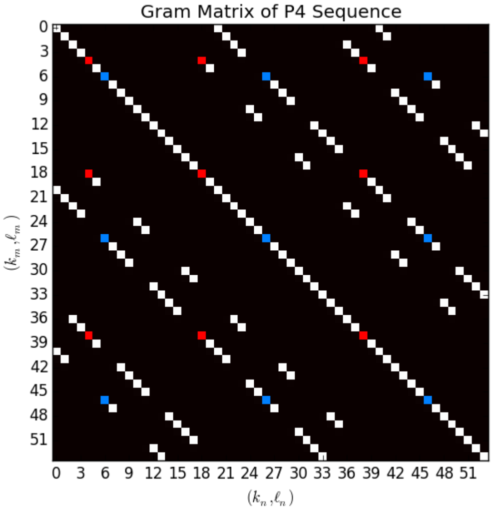
Lemma

Let $\varphi \in \mathbb{C}^N$ be the Chu or P4 sequence and let $N = abN'$ where $gcd(a, b) = 1$. Suppose G is the Gram matrix generated by the Gabor system $(\varphi, K \times L)$ where $K = \langle a \rangle$ and $L = \langle b \rangle$. Then,

(a) *The support of the rows (or columns) of G either completely conincide or are completely disjoint.*

(b) *If two rows (or columns) have coinciding supports, they are scalar multiples of each other.*

Example: P4 Gram Matrix



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Tight Frames from Gram Matrix

Theorem

Let $\varphi \in \mathbb{C}^N$ be the Chu or P4 sequence and let $N = abN'$ where $gcd(a, b) = 1$. Suppose G is the Gram matrix generated by the Gabor system $(\varphi, K \times L)$ where $K = \langle a \rangle$ and $L = \langle b \rangle$. Then,

(a) $rank(G) = N$.

(b) *G has exactly one nonzero eigenvalue, NN'.*

In particular (a) and (b) together imply that the Gabor system $(\varphi, K \times L)$ is a tight frame with frame bound NN'.

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