Constructing Tight Gabor Frames using CAZAC Sequences

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Let $\varphi \in \mathbb{C}^N$. $\varphi$ is said to be a constant amplitude zero autocorrelation (CAZAC) sequence if

$$\forall j \in (\mathbb{Z}/N\mathbb{Z}), |\varphi_j| = 1$$

(CA)

and

$$\forall k \in (\mathbb{Z}/N\mathbb{Z}), k \neq 0, \frac{1}{N} \sum_{j=0}^{N-1} \varphi_{j+k} \bar{\varphi}_j = 0.$$  

(ZAC)
Examples

Quadratic Phase Sequences
Let $\varphi \in \mathbb{C}^N$ and suppose for each $j$, $\varphi_j$ is of the form $\varphi_j = e^{-\pi i p(j)}$ where $p$ is a quadratic polynomial. The following quadratic polynomials generate CAZAC sequences:

- **Chu:** $p(j) = j(j - 1)$
- **P4:** $p(j) = j(j - N)$, $N$ is odd
- **Odd-length Wiener:** $p(j) = sj^2$, $\gcd(s, N) = 1$, $N$ is odd
- **Even-length Wiener:** $p(j) = sj^2/2$, $\gcd(s, 2N) = 1$, $N$ is even
Let $p$ be prime. Then, the *Legendre symbol* is defined as follows,

$$\left( \frac{j}{p} \right) = \begin{cases} 
0 & \text{if } j \equiv 0 \pmod{p}, \\
1 & \text{if } j \equiv k^2 \pmod{p} \text{ has a solution}, \\
-1 & \text{if } j \equiv k^2 \pmod{p} \text{ does not have a solution}.
\end{cases}$$
Examples

**Björck Sequences**

Let $p$ be prime and $\varphi \in \mathbb{C}^p$ be of the form $\varphi_j = e^{i\theta(j)}$. Then $\varphi$ will be CAZAC in the following cases:

- If $p \equiv 1 \mod 4$, then,

  $$\theta(j) = \left(\frac{j}{p}\right) \arccos\left(\frac{1}{1 + \sqrt{p}}\right)$$

- If $p \equiv 3 \mod 4$, then,

  $$\theta_j = \begin{cases} \arccos\left(\frac{1-p}{1+p}\right), & \text{if } \left(\frac{j}{p}\right) = -1 \\ 0, & \text{otherwise} \end{cases}$$
Connection to Hadamard Matrices

Theorem

Let $\varphi \in \mathbb{C}^N$ and let $H$ be the circulant matrix given by

$$H = \begin{bmatrix}
\varphi \\
\tau_1 \varphi \\
\tau_2 \varphi \\
\vdots \\
\tau_{N-1} \varphi 
\end{bmatrix}$$

Then, $\varphi$ is a CAZAC sequence if and only if $H$ is Hadamard, i.e. $H^*H = N\text{Id}_N$ and $|H_{ij}| = 1$ for every $(i,j)$. In particular there is a one-to-one correspondence between CAZAC sequences and circulant Hadamard matrices.
Connection to Cyclic $N$-roots

**Definition**

$x \in \mathbb{C}^N$ is a cyclic $N$-root if it satisfies

\[
\begin{align*}
    x_0 + x_1 + \cdots + x_{N-1} &= 0 \\
    x_0x_1 + x_1x_2 + \cdots + x_{N-1}x_0 &= 0 \\
    \cdots \\
    x_0x_1x_2 \cdots x_{N-1} &= 1
\end{align*}
\]
Connection to Cyclic $N$-roots

Theorem

(a) If $\varphi \in \mathbb{C}^N$ is a CAZAC sequence then,

$$\left( \frac{\varphi_1}{\varphi_0}, \frac{\varphi_2}{\varphi_1}, \ldots, \frac{\varphi_0}{\varphi_{N-1}} \right)$$

is a cyclic $N$-root.

(b) If $x \in \mathbb{C}^N$ is a cyclic $N$-root then,

$$\varphi_0 = x_0, \varphi_j = \varphi_{j-1}x_j$$

is a CAZAC sequence.

(c) There is a one-to-one correspondence between CAZAC sequences which start with 1 and cyclic $N$-roots.
Gabor Frames

Definition

(a) Let $\varphi \in \mathbb{C}^N$ and $\Lambda \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$. The Gabor system, $(\varphi, \Lambda)$ is defined by

$$(\varphi, \Lambda) = \{ e_{\ell} \tau_k \varphi : (k, \ell) \in \Lambda \}.$$ 

(b) If $(\varphi, \Lambda)$ is a frame for $\mathbb{C}^N$ we call it a Gabor frame.
Definition

Let \( \varphi, \psi \in \mathbb{C}^N \).

(a) The \textit{discrete periodic ambiguity function} of \( \varphi \), \( A_p(\varphi) \), is defined by

\[
A_p(\varphi)[k, \ell] = \frac{1}{N} \sum_{j=0}^{N-1} \varphi[j + k] \overline{\varphi[j]} e^{-2\pi ij \ell/N} = \frac{1}{N} \langle \tau_{-k} \varphi, e_{\ell} \varphi \rangle.
\]

(b) The \textit{short-time Fourier transform} of \( \varphi \) with window \( \psi \), \( V_\psi(\varphi) \), is defined by

\[
V_\psi(\varphi)[k, \ell] = \langle \varphi, e_{\ell} \tau_k \psi \rangle.
\]
Theorem

Let $\varphi \in \mathbb{C}^N \setminus \{0\}$ and $\Lambda = (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$. Then, $(\varphi, \Lambda)$ is always a tight frame with frame bound $N\|\varphi\|_2^2$. 
Janssen’s Representation

Definition
Let $\Lambda \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$ be a subgroup. The adjoint subgroup of $\Lambda$, $\Lambda^\circ \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$, is defined by

$$\Lambda^\circ = \{(m, n) : e_\ell \tau_k e_n \tau_m = e_n \tau_m e_\ell \tau_k, \forall (k, \ell) \in \Lambda\}$$

Theorem (Janssen ’95)

Let $\Lambda$ be a subgroup of $(\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$ and $\varphi \in \mathbb{C}^N$. Then, the $(\varphi, \Lambda)$ Gabor frame operator has the form

$$S = \frac{|\Lambda|}{N} \sum_{(m,n) \in \Lambda^\circ} \langle \varphi, e_n \tau_m \varphi \rangle e_n \tau_m.$$
\( \Lambda^\circ \)-sparsity and Tight Frames

**Theorem (MM ’17)**

Let \( \varphi \in \mathbb{C}^N \setminus \{0\} \) and let \( \Lambda \subseteq (\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z}) \) be a subgroup. \( (\varphi, \Lambda) \) is a tight frame if and only if

\[
\forall (m, n) \in \Lambda^\circ, \quad A_p(\varphi)[m, n] = 0.
\]

The frame bound is \(|\Lambda|A_p(\varphi)[0, 0]|.\)
$A_p(\varphi_{\text{Chu}})[k, \ell] :$
\begin{align*}
e^{\pi i (k^2 - k)/N}, & \quad k \equiv \ell \mod N \\
0, & \quad \text{otherwise}
\end{align*}

**Figure:** DPAF of length 15 Chu sequence.
Example: Chu/P4 Sequence

Proposition

Let $N = abN'$ where $\gcd(a, b) = 1$ and $\varphi \in \mathbb{C}^N$ be the Chu or P4 sequence. Define $K = \langle a \rangle$, $L = \langle b \rangle$ and $\Lambda = K \times L$.

(a) $\Lambda^\circ = \langle N'a \rangle \times \langle N'b \rangle$.

(b) $(\varphi, \Lambda)$ is a tight Gabor frame bound $NN'$. 
DPAF of Even Length Wiener Sequence

\[ A_p(\varphi_{\text{Wiener}})[k, \ell] : \begin{cases} e^{\pi i sk^2 / N}, & sk \equiv \ell \mod N \\ 0, & \text{otherwise} \end{cases} \]

**Figure:** DPAF of length 16 P4 sequence.
DPAF of Björck Sequence

Figure: DPAF of length 13 Björck sequence.
Kronecker Product:
Let \( u \in \mathbb{C}^M, v \in \mathbb{C}^N \).

\[
(u \otimes v)[aM + b] = u[a]v[b]
\]

**Figure:** DPAF of Kronecker product of length 7 Bjorck and length 4 P4.
Example: Kronecker Product Sequence

Proposition

Let $u \in \mathbb{C}^M$ be CAZAC, $v \in \mathbb{C}^N$ be CA, and $\varphi \in \mathbb{C}^{MN}$ be defined by the Kronecker product: $\varphi = u \otimes v$. If $\gcd(M, N) = 1$ and $\Lambda = \langle M \rangle \times \langle N \rangle$, then $(\varphi, \Lambda)$ is a tight frame with frame bound $MN$. 
Definition

Let $\mathcal{F} = \{v_i\}_{i=1}^M$ be a frame for $\mathbb{C}^N$. The Gram matrix, $G$, is defined by

$$G_{ij} = \langle v_i, v_j \rangle.$$

In the case of Gabor frames $\mathcal{F} = \{e^{\ell_m \tau_k} \varphi : m \in 0, \ldots, M - 1\}$, we can write the Gram matrix in terms of the discrete periodic ambiguity function of $\varphi$:

$$G_{mn} = Ne^{-2\pi ik_n(\ell_n-\ell_m)/N}A_p(\varphi)[k_n - k_m, \ell_n - \ell_m]$$
Lemma
Let $\varphi \in \mathbb{C}^N$ be the Chu or P4 sequence and let $N = abN'$ where $\gcd(a, b) = 1$. Suppose $G$ is the Gram matrix generated by the Gabor system $(\varphi, K \times L)$ where $K = \langle a \rangle$ and $L = \langle b \rangle$. Then,

(a) The support of the rows (or columns) of $G$ either completely coincide or are completely disjoint.

(b) If two rows (or columns) have coinciding supports, they are scalar multiples of each other.
Example: P4 Gram Matrix
Theorem

Let \( \varphi \in \mathbb{C}^N \) be the Chu or P4 sequence and let \( N = abN' \) where \( \gcd(a, b) = 1 \). Suppose \( G \) is the Gram matrix generated by the Gabor system \( (\varphi, K \times L) \) where \( K = \langle a \rangle \) and \( L = \langle b \rangle \). Then,

(a) \( \text{rank}(G) = N \).

(b) \( G \) has exactly one nonzero eigenvalue, \( NN' \).

In particular (a) and (b) together imply that the Gabor system \( (\varphi, K \times L) \) is a tight frame with frame bound \( NN' \).