The discrete periodic ambiguity function

Given $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$.

The discrete periodic ambiguity function, $A(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$, of $u$ is

$$A(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u[m + k] \overline{u[k]} e^{-2\pi i kn/N}.$$
CAZAC sequences

- $u : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ is
  
  *Constant Amplitude Zero Autocorrelation (CAZAC)* if

  $$\forall m \in \mathbb{Z}/N\mathbb{Z}, \quad |u[m]| = 1, \quad \text{(CA)}$$

  and

  $$\forall m \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}, \quad A(u)(m,0) = 0. \quad \text{(ZAC)}$$

- Are there only finitely many non-equivalent CAZAC sequences?

  - ”Yes” for $N$ prime and ”No” for $N = MK^2$,
  - Generally unknown for $N$ square free and not prime.
Let $A(b_p)$ be the Björck CAZAC discrete periodic ambiguity function defined on $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

**Theorem (J. and R. Benedetto and J. Woodworth)**

$$|A(b_p)(m, n)| \leq \frac{2}{\sqrt{p}} + \frac{4}{p}$$

for all $(m, n) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \setminus (0, 0)$.

- The proof is at the level of Weil’s proof of the Riemann hypothesis for finite fields and depends on Weil’s exponential sum bound.
- Elementary construction/coding and intricate combinatorial/geometrical patterns.
- The Welch bound is attained.
Figure: Absolute value of the ambiguity functions of the Alltop (non-CAZAC) and Björck (CAZAC) sequences with $N = 17$. 
Multi-sensor environments and vector sensor and MIMO capabilities and modeling.

Vector-valued DFTs

Discrete time data vector $u(k)$ for a $d$-element array,

$$k \mapsto u(k) = (u_0(k), \ldots, u_{d-1}(k)) \in \mathbb{C}^d.$$ 

We can have $\mathbb{R}^N \rightarrow GL(d, \mathbb{C})$, or even more general.
Ambiguity functions for vector-valued data

- Given $u : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}^d$.
- For $d = 1$, $A(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ is

$$A(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m + k)\overline{u(k)}e^{-2\pi i kn/N}.$$

Goal

Define the following in a meaningful, computable way:

- Generalized $\mathbb{C}$-valued periodic ambiguity function $A^1(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$
- $\mathbb{C}^d$-valued periodic ambiguity function $A^d(u)$.

The STFT is the guide and the theory of frames is the technology to obtain the goal.
Given $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}^d$.

If $d = 1$ and $e_n = e^{2\pi in/N}$, then

$$A(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} \langle u(m + k), u(k)e_{nk} \rangle.$$ 

To characterize sequences $\{\varphi_k\} \subseteq \mathbb{C}^d$ and compatible multiplications $\ast$ and $\cdot$ so that

$$A^1(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} \langle u(m + k), u(k) \ast \varphi_{nk} \rangle \in \mathbb{C}$$

is a meaningful and well-defined ambiguity function. This formula is clearly motivated by the STFT.
Given $u : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}^d$, $d \leq N$. Let $\{\varphi_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$, let $\ast$ be componentwise multiplication in $\mathbb{C}^d$ with a factor of $\sqrt{d}$, and let $\bullet = +$ in $\mathbb{Z}/N\mathbb{Z}$.

In this case $A^1(u)$ is well-defined by

$$A^1(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} \langle u(m + k), u(k) \ast \varphi_{n \bullet k} \rangle$$

$$= \frac{d}{N^2} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \langle \varphi_j, u(k) \rangle \langle u(m + k), \varphi_{j+nk} \rangle.$$
Take $*: \mathbb{C}^3 \times \mathbb{C}^3 \rightarrow \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$ and let $\{i, j, k\}$ be the standard basis.

$i * j = k$, $j * i = -k$, $k * i = j$, $i * k = -j$, $j * k = i$, $k * j = -i$.
$i * i = j * j = k * k = 0$. $\{0, i, j, k, -i, -j, -k, \}$ is a tight frame for $\mathbb{C}^3$ with frame constant 2. Let

$$\varphi_0 = 0, \varphi_1 = i, \varphi_2 = j, \varphi_3 = k, \varphi_4 = -i, \varphi_5 = -j, \varphi_6 = -k.$$ 

The index operation corresponding to the frame multiplication is the non-abelian operation $\bullet: \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \rightarrow \mathbb{Z}/7\mathbb{Z}$, where

$$1 \bullet 2 = 3, 2 \bullet 1 = 6, 3 \bullet 1 = 2, 1 \bullet 3 = 5, 2 \bullet 3 = 1, 3 \bullet 2 = 4,$$

etc.

Thus, $u: \mathbb{Z}/7\mathbb{Z} \rightarrow \mathbb{C}^3$ and we can write $u \times v \in \mathbb{C}^3$ as

$$u \times v = u * v = \frac{1}{2^2} \sum_{s=1}^{6} \sum_{t=1}^{6} \langle u, \varphi_s \rangle \langle v, \varphi_t \rangle \varphi_{s \bullet t}.$$ 

Consequently, $A^1(u)$ is well-defined.

Generalize to quaternion groups, order 8 and beyond.
Definition (Frame multiplication)

Let $\mathcal{H}$ be a finite dimensional Hilbert space over $\mathbb{C}$, and let $\Phi = \{\varphi_j\}_{j \in J}$ be a frame for $\mathcal{H}$. Assume $\bullet : J \times J \to J$ is a binary operation. The mapping $\bullet$ is a frame multiplication for $\Phi$ if there exists a bilinear product $* : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ such that

$$\forall j, k \in J, \quad \varphi_j \ast \varphi_k = \varphi_{j \cdot k}.$$ 

- The existence of frame multiplication allows one to define the ambiguity function for vector-valued data.
- There are frames with no frame multiplications.
Harmonic frames

- Slepian (1968) - *group codes.*
- Forney (1991) - *geometrically uniform* signal space codes.
Harmonic frames

\[(G, \bullet) = \{g_1, \ldots, g_N\}\] abelian group with \(\hat{G} = \{\gamma_1, \ldots, \gamma_N\}\).

\[N \times N\] matrix with \((j, k)\) entry \(\gamma_k(g_j)\) is character table of \(G\).

\(K \subseteq \{1, \ldots, N\}, |K| = d \leq N\), and columns \(k_1, \ldots, k_d\).

**Definition**

Given \(U \in \mathcal{U}(\mathbb{C}^d)\). The harmonic frame \(\Phi = \Phi_{g, K, U}\) for \(\mathbb{C}^d\) is

\[\Phi = \{U(\gamma_{k_1}(g_j), \ldots, \gamma_{k_d}(g_j)) : j = 1, \ldots, N\}.

Given \(G, K,\) and \(U = I\). \(\Phi\) is the DFT – FUNTF on \(G\) for \(\mathbb{C}^d\). Take \(G = \mathbb{Z}/N\mathbb{Z}\) for usual DFT – FUNTF for \(\mathbb{C}^d\).
Definition

Let \((G, \bullet)\) be a finite group, and let \(H\) be a finite dimensional Hilbert space. A finite tight frame \(\Phi = \{\varphi_g\}_{g \in G}\) for \(H\) is a group frame if there exists

\[\pi : G \to U(H),\]

a unitary representation of \(G\), such that

\[\forall g, h \in G, \quad \pi(g)\varphi_h = \varphi_{g \bullet h}.\]

Harmonic frames are group frames.
Abelian results

Theorem (Abelian frame multiplications – 1)

Let $(G, \bullet)$ be a finite abelian group, and let $\Phi = \{\varphi_g\}_{g \in G}$ be a tight frame for $\mathcal{H}$. Then $\bullet$ defines a frame multiplication for $\Phi$ if and only if $\Phi$ is a group frame.
Theorem (Abelian frame multiplications – 2)

Let $(\mathcal{G}, \bullet)$ be a finite abelian group, and let $\Phi = \{\varphi_g\}_{g \in \mathcal{G}}$ be a tight frame for $\mathbb{C}^d$. If $\bullet$ defines a frame multiplication for $\Phi$, then $\Phi$ is unitarily equivalent to a harmonic frame and there exists $U \in \mathcal{U}(\mathbb{C}^d)$ and $c > 0$ such that

$$cU\left(\varphi_g \ast \varphi_h\right) = cU\left(\varphi_g\right)cU\left(\varphi_h\right),$$

where the product on the right is vector pointwise multiplication and $\ast$ is defined by $(\mathcal{G}, \bullet)$, i.e., $\varphi_g \ast \varphi_h := \varphi_{g \bullet h}$. 
Remarks

- Given \( u : G \rightarrow H \), where \( G \) is a finite abelian group and \( H \) is a finite dimensional Hilbert space. The vector-valued ambiguity function \( A^d(u) \) exists if frame multiplication is well-defined for a given tight frame for \( H \).
- There is an analogous characterization of frame multiplication for non-abelian groups (T. Andrews).
- It remains to extend the theory to infinite Hilbert spaces and groups.
- It also remains to extend the theory to the non-group case, e.g., our cross product example.
That's all folks!
Woodward’s (1953) narrow band cross-correlation ambiguity function of $v, w$ defined on $\mathbb{R}^d$:

$$A(v, w)(t, \gamma) = \int v(s + t)\overline{w(s)}e^{-2\pi is \cdot \gamma} \, ds.$$  

The STFT of $v$: $V_w v(t, \gamma) = \int v(x)\overline{w(x - t)}e^{-2\pi ix \cdot \gamma} \, dx$.

$A(v, w)(t, \gamma) = e^{2\pi it \cdot \gamma}V_w v(t, \gamma)$.

The narrow band ambiguity function $A(v)$ of $v$:

$$A(v)(t, \gamma) = A(v, v)(t, \gamma) = \int v(s + t)\overline{v(s)}e^{-2\pi is \cdot \gamma} \, ds.$$
Let $p$ be a prime number, and $\left( \frac{k}{p} \right)$ the Legendre symbol. A Björck CAZAC sequence of length $p$ is the function $b_p : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ defined as

$$b_p[k] = e^{i\theta_p(k)}, \quad k = 0, 1, \ldots, p - 1,$$

where, for $p = 1 \pmod{4}$,

$$\theta_p(k) = \arccos \left( \frac{1}{1 + \sqrt{p}} \right) \left( \frac{k}{p} \right),$$

and, for $p = 3 \pmod{4}$,

$$\theta_p(k) = \frac{1}{2} \arccos \left( \frac{1 - p}{1 + p} \right) \left[ (1 - \delta_k) \left( \frac{k}{p} \right) + \delta_k \right].$$

$\delta_k$ is the Kronecker delta symbol.
For given CAZACs $u_p$ of prime length $p$, estimate minimal local behavior $|A(u_p)|$. For example, with $b_p$ we know that the lower bounds of $|A(b_p)|$ can be much smaller than $1/\sqrt{p}$, making them more useful in a host of mathematical problems, cf. Welch bound.

Even more, construct all CAZACs of prime length $p$.

Optimally small coherence of $b_p$ allows for computing sparse solutions of Gabor matrix equations by greedy algorithms such as OMP.