Group frames and the theory of frame multiplication

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Acknowledgements
DTRA 1-13-1-0015, ARO W911NF-15-1-0112,
ARO W911NF-16-1-0008, ARO W911NF-17-1-0014

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The discrete periodic ambiguity function

- Given $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$.
- The *discrete periodic ambiguity function*,

$$A(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C},$$

of u is

$$A(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} u[m+k]\overline{u[k]}e^{-2\pi ikn/N}.$$

CAZAC sequences

- $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$ is
  Constant Amplitude Zero Autocorrelation (CAZAC) if

$$\forall m \in \mathbb{Z}/N\mathbb{Z}, \quad |u[m]| = 1, \text{ (CA)}$$

  and

$$\forall m \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}, \quad A(u)(m,0) = 0. \text{ (ZAC)}$$

- Are there only finitely many non-equivalent CAZAC sequences?
  - "Yes" for N prime and "No" for $N = MK^2$,
  - Generally unknown for N square free and not prime.

Björck CAZAC discrete periodic ambiguity function

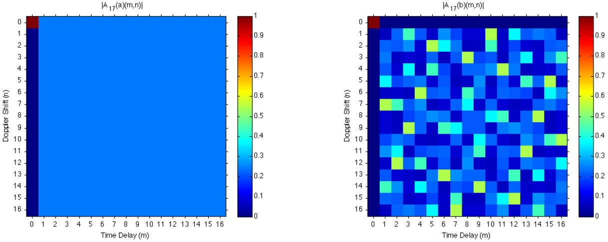
Let $A(b_p)$ be the Björck CAZAC discrete periodic ambiguity function defined on $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Theorem (J. and R. Benedetto and J. Woodworth)

$$|A(b_p)(m,n)| \leq \frac{2}{\sqrt{p}} + \frac{4}{p}$$

for all $(m,n) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \setminus (0,0)$.

- The proof is at the level of Weil's proof of the Riemann hypothesis for finite fields and depends on Weil's exponential sum bound.
- Elementary construction/coding and intricate combinatorial/geometrical patterns.
- The Welch bound is attained.



(a) (b)

Figure: Absolute value of the ambiguity functions of the Alltop (non-CAZAC) and Björck (CAZAC) sequences with $N = 17$.

Modeling for multi-sensor environments

- Multi-sensor environments and vector sensor and MIMO capabilities and modeling.
- Vector-valued DFTs
- Discrete time data vector $u(k)$ for a d-element array,

$$k \longmapsto u(k) = (u_0(k), \ldots, u_{d-1}(k)) \in \mathbb{C}^d.$$

We can have $\mathbb{R}^N \to GL(d, \mathbb{C})$, or even more general.

Ambiguity functions for vector-valued data

- Given $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}^d$.
- For $d = 1$, $A(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$ is

$$A(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} u(m+k)\overline{u(k)}e^{-2\pi ikn/N}.$$

Goal

Define the following in a meaningful, computable way:

- Generalized $\mathbb{C}$-valued periodic ambiguity function $A^1(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$
- $\mathbb{C}^d$-valued periodic ambiguity function $A^d(u)$.

The STFT is the *guide* and the *theory of frames* is the technology to obtain the goal.

Preliminary multiplication problem

- Given $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}^d$.
- If $d = 1$ and $e_n = e^{2\pi i n/N}$, then

$$A(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k)e_{nk}\rangle.$$

Preliminary multiplication problem

To characterize sequences $\{\varphi_k\} \subseteq \mathbb{C}^d$ and compatible multiplications $*$ and $\bullet$ so that

$$A^1(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * \varphi_{n\bullet k}\rangle \in \mathbb{C}$$

is a meaningful and well-defined *ambiguity function.* This formula is clearly motivated by the STFT.

$A^1(u)$ for DFT frames

- Given $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}^d, d \leq N$.
- Let $\{\varphi_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$, let $*$ be componentwise multiplication in $\mathbb{C}^d$ with a factor of $\sqrt{d}$, and let $\bullet = +$ in $\mathbb{Z}/N\mathbb{Z}$.

In this case $A^1(u)$ is well-defined by

$$\begin{aligned} A^1(u)(m,n) &= \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * \varphi_{n\bullet k}\rangle \\ &= \frac{d}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\langle \varphi_j, u(k)\rangle\langle u(m+k), \varphi_{j+nk}\rangle. \end{aligned}$$

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$A^1(u)$ for cross product frames

- Take $* : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$ and let $\{i, j, k\}$ be the standard basis.
- $i * j = k$, $j * i = -k$, $k * i = j$, $i * k = -j$, $j * k = i$, $k * j = -i$, $i * i = j * j = k * k = 0$. $\{0, i, j, k, -i, -j, -k, \}$ is a tight frame for $\mathbb{C}^3$ with frame constant 2. Let

$$\varphi_0 = 0, \varphi_1 = i, \varphi_2 = j, \varphi_3 = k, \varphi_4 = -i, \varphi_5 = -j, \varphi_6 = -k.$$

- The index operation corresponding to the frame multiplication is the non-abelian operation $\bullet : \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \longrightarrow \mathbb{Z}/7\mathbb{Z}$, where $1 \bullet 2 = 3, 2 \bullet 1 = 6, 3 \bullet 1 = 2, 1 \bullet 3 = 5, 2 \bullet 3 = 1, 3 \bullet 2 = 4$, etc.
- Thus, $u : \mathbb{Z}/7\mathbb{Z} \longrightarrow \mathbb{C}^3$ and we can write $u \times v \in \mathbb{C}^3$ as

$$u \times v = u * v = \frac{1}{2^2} \sum_{s=1}^{6} \sum_{t=1}^{6} \langle u, \varphi_s \rangle \langle v, \varphi_t \rangle \varphi_{s \bullet t}.$$

- Consequently, $A^1(u)$ is well-defined.

Generalize to quaternion groups, order 8 and beyond.

Frame multiplication

Definition (Frame multiplication)

Let $\mathcal{H}$ be a finite dimensional Hilbert space over $\mathbb{C}$, and let $\Phi = \{\varphi_j\}_{j \in J}$ be a frame for $\mathcal{H}$. Assume $\bullet : J \times J \to J$ is a binary operation. The mapping $\bullet$ is a *frame multiplication* for Φ if there exists a bilinear product $* : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ such that

$$\forall j, k \in J, \quad \varphi_j * \varphi_k = \varphi_{j \bullet k}.$$

- The existence of frame multiplication allows one to define the ambiguity function for vector-valued data.
- There are frames with no frame multiplications.

Harmonic frames

- Slepian (1968) - *group codes*.
- Forney (1991) - *geometrically uniform* signal space codes.
- Bölcskei and Eldar (2003) - *geometrically uniform* frames.
- Han and Larson (2000) - *frame bases and group representations*.
- Zimmermann (1999), Pfander (1999), Casazza and Kovacević (2003), Strohmer and Heath (2003), Vale and Waldron (2005), Hirn (2010), Chien and Waldron (2011) - *harmonic frames*.
- Han (2007), Vale and Waldron (2010) - *group frames, symmetry groups*.

Harmonic frames

- $(\mathcal{G}, \bullet) = \{g_1, \ldots, g_N\}$ abelian group with $\widehat{G} = \{\gamma_1, \ldots, \gamma_N\}$.
- $N \times N$ matrix with (j, k) entry $\gamma_k(g_j)$ is *character table* of $\mathcal{G}$.
- $K \subseteq \{1, \ldots, N\}, |K| = d \leq N$, and columns $k_1, \ldots, k_d$.

Definition

Given $U \in \mathcal{U}(\mathbb{C}^d)$. The *harmonic frame* $\Phi = \Phi_{\mathcal{G},K,U}$ for $\mathbb{C}^d$ is

$$\Phi = \{U\left((\gamma_{k_1}(g_j), \ldots, \gamma_{k_d}(g_j))\right) : j = 1, \ldots, N\}.$$

Given $\mathcal{G}, K$, and $U = I$. Φ is the $DFT - FUNTF$ on $\mathcal{G}$ for $\mathbb{C}^d$. Take $\mathcal{G} = \mathbb{Z}/N\mathbb{Z}$ for usual $DFT - FUNTF$ for $\mathbb{C}^d$.

Group frames

Definition

Let $(\mathcal{G}, \bullet)$ be a finite group, and let $\mathcal{H}$ be a finite dimensional Hilbert space. A finite tight frame $\Phi = \{\varphi_g\}_{g \in \mathcal{G}}$ for $\mathcal{H}$ is a *group frame* if there exists

$$\pi : \mathcal{G} \to \mathcal{U}(\mathcal{H}),$$

a unitary representation of $\mathcal{G}$, such that

$$\forall g, h \in \mathcal{G}, \quad \pi(g)\varphi_h = \varphi_{g \bullet h}.$$

Harmonic frames are group frames.

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Abelian results

Theorem (Abelian frame multiplications – 1)

Let $(\mathcal{G}, \bullet)$ be a finite abelian group, and let $\Phi = \{\varphi_g\}_{g \in \mathcal{G}}$ be a tight frame for $\mathcal{H}$. Then $\bullet$ defines a frame multiplication for Φ if and only if Φ is a group frame.

Abelian results

Theorem (Abelian frame multiplications – 2)

Let $(\mathcal{G}, \bullet)$ be a finite abelian group, and let $\Phi = \{\varphi_g\}_{g \in \mathcal{G}}$ be a tight frame for $\mathbb{C}^d$. If $\bullet$ defines a frame multiplication for Φ, then Φ is unitarily equivalent to a harmonic frame and there exists $U \in \mathcal{U}(\mathbb{C}^d)$ and $c > 0$ such that

$$cU\left(\varphi_g * \varphi_h\right) = cU\left(\varphi_g\right) cU\left(\varphi_h\right),$$

where the product on the right is vector pointwise multiplication and $*$ is defined by $(\mathcal{G}, \bullet)$, i.e., $\varphi_g * \varphi_h := \varphi_{g \bullet h}$.

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Remarks

- Given $u : \mathcal{G} \longrightarrow \mathcal{H}$, where $\mathcal{G}$ is a finite abelian group and $\mathcal{H}$ is a finite dimensional Hilbert space. The vector-valued ambiguity function $A^d(u)$ exists if frame multiplication is well-defined for a given tight frame for $\mathcal{H}$.
- There is an analogous characterization of frame multiplication for non-abelian groups (T. Andrews).
- It remains to extend the theory to infinite Hilbert spaces and groups.
- It also remains to extend the theory to the non-group case, e.g., our cross product example.

Thats all folks!

Ambiguity function and STFT

- Woodward's (1953) *narrow band cross-correlation ambiguity function* of v, w defined on $\mathbb{R}^d$:

$$A(v,w)(t,\gamma) = \int v(s+t)\overline{w(s)}e^{-2\pi i s\cdot\gamma}ds.$$

- The *STFT* of v : $V_w v(t,\gamma) = \int v(x)\overline{w(x-t)}e^{-2\pi i x\cdot\gamma}dx.$
- $A(v,w)(t,\gamma) = e^{2\pi i t\cdot\gamma}V_w v(t,\gamma).$
- The *narrow band ambiguity function* $A(v)$ of v :

$$A(v)(t,\gamma) = A(v,v)(t,\gamma) = \int v(s+t)\overline{v(s)}e^{-2\pi i s\cdot\gamma}ds$$

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Björck CAZAC sequences

Let p be a prime number, and $(\frac{k}{p})$ the *Legendre symbol.*
A *Björck CAZAC sequence* of length p is the function $b_p : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ defined as

$$b_p[k] = e^{i\theta_p(k)}, \quad k = 0, 1, \ldots, p-1,$$

where, for $p = 1 \pmod 4$,

$$\theta_p(k) = \arccos\left(\frac{1}{1+\sqrt{p}}\right)\left(\frac{k}{p}\right),$$

and, for $p = 3 \pmod 4$,

$$\theta_p(k) = \frac{1}{2}\arccos\left(\frac{1-p}{1+p}\right)\left[(1-\delta_k)\left(\frac{k}{p}\right) + \delta_k\right].$$

δ_k is the Kronecker delta symbol.

Problems and remarks

- For given CAZACs u_p of prime length p, estimate minimal local behavior $|A(u_p)|$. For example, with b_p we know that the lower bounds of $|A(b_p)|$ can be much smaller than $1/\sqrt{p}$, making them more useful in a host of mathematical problems, cf. Welch bound.
- Even more, construct all CAZACs of prime length p.
- Optimally small coherence of b_p allows for computing sparse solutions of Gabor matrix equations by greedy algorithms such as OMP.