# Super-resolution by means of Beurling minimal extrapolation

John J. Benedetto and Weilin Li

Norbert Wiener Center
Department of Mathematics
University of Maryland, College Park
*http://www.norbertwiener.umd.edu*

Acknowledgements
ARO W911NF-16-1-0008 and ARO W911NF17-1-0014

# Outline

# What is super-resolution?

Broadly speaking, super-resolution is concerned with recovering fine details (high-frequency) from coarse information (low-frequency).

There are two main categories of super-resolution:

- Spectral extrapolation – Optical, radar, geophysics, astronomy, medical imaging, e.g., MRI, problems;
- Spatial interpolation – Geometrical or image-processing, e.g., in-painting problems.

**Remark** We shall deal with spectral extrapolation. We shall not deal with the critical setting of noisy environments. Also, we shall not deal with the highly motivated spatial setting of super-resolution, where non-uniform sampling and multiple measurements can play an essential role.

Norbert Wiener Center
for Harmonic Analysis and Applications

# Background and notation

Our super-resolution model is based on the theory of Candès and Fernandez-Granda [5], [6] for discrete measures, *and* our main idea was inspired by classical work of Beurling [3], [4].

- $\mathbb{T}^d$ is the $d$-dimensional torus group.
- $M(\mathbb{T}^d)$ is the space of complex Radon measures on the torus.
- $\|\cdot\|$ is the total variation norm.
- The Fourier transform of $\mu$ is the function $\widehat{\mu}\colon \mathbb{Z}^d \to \mathbb{C}$, defined as

$$\widehat{\mu}(m) = \int_{\mathbb{T}^d} e^{-2\pi imx}\, d\mu(x).$$

- $\Lambda \subseteq \mathbb{Z}^d$ is a finite set.

# The super-resolution problem

The *unknown information* is modeled as $\mu \in M(\mathbb{T}^d)$, not only discrete measures. There are two reasons for $\mu \in M(\mathbb{T}^d)$:

- Objects (images) are not necessarily supported by discrete sets;
- Fine features can be supported in measure 0 non-discrete sets.

The *given low-frequency information* is modeled as spectral data, $F(n)$, $n \in \Lambda$, i.e., there is $\nu \in M(\mathbb{T}^d)$ such that $\widehat{\nu} = F$ on $\Lambda$. To recover $\mu$ from $F$, we pose the *super-resolution problem*,

$$\inf \|\nu\| \quad \text{subject to} \quad \nu \in M(\mathbb{T}^d) \quad \text{and} \quad \widehat{\nu} = F \quad \text{on} \quad \Lambda. \qquad \text{(SR)}$$

# The super-resolution problem, continued

**Remark**

*a.* Using weak-$*$ compactness arguments, we can show that Problem (SR) is well-posed (the inf can be replaced with a min), but not without significant theoretical and computational challenges.

*b.* Problem (SR) is a convex minimization problem, and we interpret a solution as a *least complicated* high resolution extrapolation of $F$.

*c.* Independently, DeCastro-Gamboa [7] also use Beurling [3], [4], but to super-resolve a discrete measure $\mu$, whose support is contained in the level set of a certain family of generalized polynomials, given partial generalized moments of $\mu$. In contrast to their problem and techniques, we use Beurling's ideas to obtain super-resolution reconstruction of an arbitrary bounded measure $\mu$, given a finite subset of its Fourier coefficients.

# Connection with compressed sensing

If the unknown measure $\mu$ is of the form,

$$\mu = \sum_{m=0}^{N-1} x_m \delta_{\frac{m}{N}} \in M(\mathbb{T}),$$

where $x \in \mathbb{C}^N$, $x = (x_0, \ldots, x_{N-1})$, then

$$\widehat{\mu}(n) = \sum_{m=0}^{N-1} x_m \, e^{-2\pi imn/N} = \mathcal{F}_N(x)(n),$$

the DFT of $x$. This shows that Problem (SR) is a generalization of the *basis pursuit algorithm* [9] for under-sampled DFT data $F$:

$$\text{For given } F(n),\ n \in \Omega \subseteq \mathbb{Z}/N\mathbb{Z}, \text{ solve}$$

$$\min \|y\|_{\ell^1} \quad \text{subject to} \quad y \in \mathbb{C}^N \quad \text{and} \quad \mathcal{F}_N y = F \quad \text{on} \quad \Omega \subseteq \mathbb{Z}/N\mathbb{Z},$$

For this reason, super-resolution is a continuous theory of *compressed sensing*.

# Outline

# A theorem of Candès and Fernandez-Granda, $d = 1$

The following theorem for $d = 1$ shows that one can reconstruct a discrete measure whose support satisfies a minimum separation condition.

**Theorem, Candès and Fernandez-Granda [6]**

Let $\Lambda_M = \{-M, -M+1, \ldots, M\}$ for some integer $M \geq 128$ and let $F = \widehat{\mu}$ on $\Lambda_M$, where $\mu \in M(\mathbb{T})$ is a discrete measure for which

$$\inf_{x,y \in \mathrm{supp}(\mu),\ x \neq y} |x - y| \geq \frac{2}{M}.$$

Then, $\mu$ is the unique solution to Problem (SR) given $F$ on $\Lambda_M$.

Norbert Wiener Center for Harmonic Analysis and Applications

# A theorem of Candès and Fernandez-Granda, $d > 1$

The following theorem for $d > 1$ shows that one can reconstruct a discrete measure whose support satisfies a minimum separation condition.

**Theorem, Candès and Fernandez-Granda [6]**

Given $S = \{s_j\}_{j=1}^J \subseteq \mathbb{T}^d$ and $\mu \in M(\mathbb{T}^d)$ for which $\operatorname{supp}(\mu) \subseteq S$. Let $\Lambda_M = \{-M, -M+1, \ldots, M\}^d$, let $F$ be spectral data on $\Lambda_M$, and let $\widehat{\mu} = F$ on $\Lambda_M$. There exist $C_d,\ M_d > 0$ such that if $M \geq M_d$ and

$$\inf_{1 \leq j < k \leq J} \|s_j - s_k\|_{\ell^\infty(\mathbb{T}^d)} \geq \frac{C_d}{M},$$

then $\mu$ is the *unique solution* to Problem (SR).

# Definitions based on Beurling's theory

- Let $\epsilon$ be the smallest value attained by Problem (SR), i.e.,

$$\epsilon = \epsilon(\Lambda, F) = \inf\{\|\nu\| : \widehat{\nu} = F \text{ on } \Lambda\}.$$

- Let $\mathcal{E}$ be the set of all solutions to Problem (SR), i.e.,

$$\mathcal{E} = \mathcal{E}(\Lambda, F) = \{\nu \in M(\mathbb{T}^d) \colon \|\nu\| = \epsilon \text{ and } \widehat{\nu} = F \text{ on } \Lambda\}.$$

  If $\nu \in \mathcal{E}$, then we say $\nu$ is a *minimal extrapolation* from $\Lambda$.

- Our theory depends essentially on the set,

$$\Gamma = \Gamma(\Lambda, F) = \{m \in \Lambda \colon |F(m)| = \epsilon\}.$$

# Functional analysis properties of $\epsilon = \epsilon(\Lambda, F)$ and $\mathcal{E} = \mathcal{E}(\Lambda, F)$

**Definitions** for $\mu \in M(\mathbb{T}^d)$, $\Lambda \subseteq \mathbb{Z}^d$ finite, and $\widehat{\mu} = F$ on $\Lambda$.

$$C(\mathbb{T}^d; \Lambda) = \{f \in C(\mathbb{T}^d) \colon f(x) = \sum_{m \in \Lambda} a_m \, e^{2\pi i m \cdot x}, \; a_m \in \mathbb{C}\}.$$

$$U = U(\mathbb{T}^d; \Lambda) = \{f \in C(\mathbb{T}^d; \Lambda) \colon \|f\|_\infty \leq 1\}.$$

$L_\mu \in C(\mathbb{T}^d; \Lambda)'$ defined as

$$\forall f \in C(\mathbb{T}^d; \Lambda), \quad L_\mu(f) = \int_{\mathbb{T}^d} f(x) \, \overline{d\mu(x)} = \sum_{m \in \Lambda} a_m F(m).$$

$$\|L_\mu\| = \sup_{f \in U} |L_\mu(f)|.$$

# Functional analysis properties of $\epsilon = \epsilon(\Lambda, F)$ and $\mathcal{E} = \mathcal{E}(\Lambda, F)$, continued

**Properties** for $\mu \in M(\mathbb{T}^d)$, $\Lambda \subseteq \mathbb{Z}^d$ finite, and $\widehat{\mu} = F$ on $\Lambda$.

- $\mathcal{E} \subseteq M(\mathbb{T}^d)$ is non-empty, weak-$*$ compact, and convex.
- $C(\mathbb{T}^d; \Lambda)$ is a closed subspace of $C(\mathbb{T}^d)$.
- $U$ is a compact subset of $C(\mathbb{T}^d; \Lambda)$.
- $\epsilon = \|L_\mu\| = \max_{f \in U} |\langle f, \mu \rangle|$.
- There exists $\varphi(x) = \sum_{m \in \Lambda} a_m \, e^{2\pi i m \cdot x} \in U$ such that $\langle \varphi, \mu \rangle = \epsilon$.
- If $\varphi \in U$ and $\langle \varphi, \mu \rangle = \epsilon$, then

$$\forall \nu \in \mathcal{E},\ \varphi = \mathrm{sign}(\nu)\ \nu\text{-a.e.,} \quad \text{and} \quad \mathrm{supp}(\nu) \subseteq \{x \in \mathbb{T}^d : |\varphi(x)| = 1\},$$

where $|\mathrm{sign}(\nu)| = 1$ arises in R-N Theorem.

Norbert Wiener Center for Harmonic Analysis and Applications

# Theorem

## Theorem [1]

Let $\Lambda \subseteq \mathbb{Z}^d$ be a finite set and let $F$ be spectral data defined on $\Lambda$.

(a) Suppose $\Gamma = \emptyset$. Then, there exists a closed set $S$ of $d$-dimensional Lebesgue measure zero such that each minimal extrapolation is a singular measure supported in $S$.

(b) Suppose $\#\Gamma \geq 2$. For each distinct pair $m, n \in \Gamma$, define $\alpha_{m,n} \in \mathbb{R}/\mathbb{Z}$ by $e^{2\pi i \alpha_{m,n}} = F(m)/F(n)$. Define the closed set,

$$S = \bigcap_{\substack{m,n\in\Gamma \\ m\neq n}} \{x \in \mathbb{T}^d : x \cdot (m-n) + \alpha_{m,n} \in \mathbb{Z}\},$$

which is an intersection of $\binom{\#\Gamma}{2}$ periodic hyperplanes. Then, each minimal extrapolation is a singular measure supported in $S$.

# Illustration of the theorem

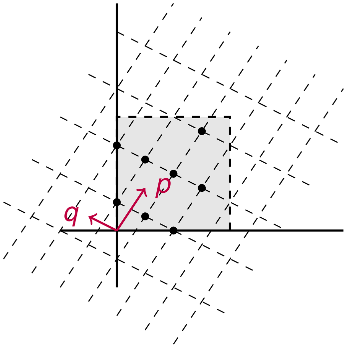


Figure: Second case of the Theorem for $d = 2$ and $\#\Gamma = 3$. Note $\binom{\#\Gamma}{2}$ takes the values $1, 3, 6, 10, \ldots$. Thus, for this case, the 3 hyperplanes of the Theorem are not unique, and are represented by the 2 periodic sets of dashed lines. The vectors $p = (1/4, 3/8)$ and $q = (-1/4, 1/8)$ are normal to the hyperplanes and their lengths determine the separation of the hyperplanes.

# Admissibility range

A numerical approximation of $\epsilon$ can be obtained by solving Problem (SR), but its exact value is typically unknown. On the other hand, if we are given finite $\Lambda \subseteq \mathbb{Z}^d$, spectral data $F$ on $\Lambda$, and $\mu \in \mathbb{T}^d$ for which $\widehat{\mu} = F$ on $\Lambda$, then

$$\sup_{m\in\Lambda} |F(m)| \leq \epsilon(\Lambda, F) \leq \|\mu\|.$$

- If the lower bound is attained, then $\Gamma \neq \emptyset$. Our theory is particularly strong for large $\#\Gamma$.
- The upper bound $\epsilon = \|\mu\|$ is a necessary condition for uniqueness of the super-resolution of $\mu$ from $F$.

# The role of uniqueness

Why uniqueness is important:

- If $\mu \in \mathcal{E}(\Lambda, F)$ is unique, then any numerical solution to Problem (SR) approximates $\mu$.
- Without uniqueness, even if $\mu \in \mathcal{E}(\Lambda, F)$, it is possible that a numerical solution to Problem (SR) does not approximate $\mu$.

Norbert Wiener Center for Harmonic Analysis and Applications

# Uniqueness and Meyer's theory (1970) of quasi-crystals

Define $\alpha \in (0, 1/2)$ and define the *sampling set*

$$\Lambda_\alpha = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : \exists r \in \mathbb{Z}, \text{ such that } |m\sqrt{2} + n\sqrt{3} - r| \leq \alpha\}.$$

Let $M_{d,+,N}(\mathbb{T}^2)$ be the set of positive discrete measures $\nu$ on $\mathbb{T}^2$, where $\operatorname{card}\operatorname{supp}(\nu) \leq N$ .

**Theorem (Basarab Matei 2014)**

Let $\mu \in M_{d,+,N}(\mathbb{T}^2)$. If $\nu$ is a positive measure on $\mathbb{T}^2$ and $\widehat{\nu} = \widehat{\mu}$ on $\Lambda_\alpha$, then $\nu = \mu$.

**Remark** Besides Matei's theorem, see the following collaborative work of Matei and Meyer: [11], [12], [13]

Norbert Wiener Center for Harmonic Analysis and Applications

# Outline

1. Introduction
2. Theorems
3. Examples
4. Conclusion

# Uniqueness and super-resolution reconstruction

Let $\Lambda = \{-1, 0, 1\}$. Define $F$ in the following ways.

- $F(0) = 0$, $F(\pm 1) = 2$. Define $\mu = \delta_0 - \delta_{1/2} \in M(\mathbb{T})$. $\Gamma = \{-1, 1\}$.
- $F(0) = 0, F(\pm 1) = 1 \pm i$. Define $\mu = \delta_0 - \delta_{1/4} \in M(\mathbb{T})$. $\Gamma = \{-1, 1\}$.
- $F(-1) = 0$, $F(0) = 1 + e^{\pi i/3}$, $F(1) = 1 + e^{-\pi i/3}$. Define $\mu = \delta_0 + e^{\pi i/3}\delta_{1/3} \in M(\mathbb{T})$. $\Gamma = \{0, 1\}$.

In each case $\mu$ can be proved to be the unique minimal extrapolation, and so super-resolution reconstruction of $\mu$ from the values of $F$ on $\Lambda$ is possible.

Norbert Wiener Center for Harmonic Analysis and Applications

# Cantor measures and $\#\Gamma = 1$

$C_q = \bigcap_{k=0}^{\infty} C_{q,k}$, integer $q \geq 3$, is the *middle* $1/q$*-Cantor set*, where

$$C_{q,0} = [0,1] \text{ and } C_{q,k+1} = \frac{C_{q,k}}{q} \cup (1-q) + \frac{C_{q,k}}{q},$$

and let $\sigma_q$ be the continuous singular Cantor-Lebesgue measure with

$$\widehat{\sigma_q}(m) = (-1)^m \prod_{k=1}^{\infty} \cos(\pi m q^{-k}(1-q)),$$

$$\forall n \in \mathbb{Z} \setminus \{0\}, \quad \widehat{\sigma_q}(q^n) \neq 0 \quad \text{takes the same constant value.}$$

Let $\Lambda \subseteq \mathbb{Z}$ be finite, assume $0 \in \Lambda$, and suppose $F$ defined on $\Lambda$ satisfies $F(0) = 1$, noting $\widehat{\sigma_q}(0) = \|\sigma_q\| = 1$. If $\sigma_q \in \mathcal{E}(\Lambda, F)$, then $\#\Gamma = 1$, and our present theory does not determine if $\sigma_q$ is the unique minimal extrapolation.

# Non-uniqueness: $\mu = \delta_0 + \delta_{1/2} \in M(\mathbb{T})$ and $\#\Gamma = 1$

- Given $\Lambda = \{-1, 0, 1\}$ and $F(0) = 2$, $F(\pm 1) = 0$. If $\mu = \delta_0 + \delta_{1/2} \in M(\mathbb{T})$, then $\widehat{\mu} = F$ on $\Lambda$.
- $\mu$ is a minimal extrapolation, $\epsilon = 2$, and $\Gamma = \{0\}$.
- There are uncountably many discrete minimal extrapolations. In fact, $x \in \mathbb{T}$ and any integer $N \geq 2$ define the discrete measure

$$\nu_{N,x} = \frac{2}{N} \sum_{n=0}^{N-1} \delta_{x+\frac{n}{N}},$$

and each $\nu_{N,x}$ is a minimal extrapolation.

# $\mu = \delta_0 + \delta_{1/2} \in M(\mathbb{T})$ and $\#\Gamma = 1$, continued

- There are also uncountably many positive absolutely continuous minimal extrapolations. In fact, for any integer $N \geq 2$ and constant $0 < c \leq (2N+2)/(3N+1)$, extend $F$ on $\Lambda$ to the sequence $\{(a_{N,c})_n\}_{n\in\mathbb{Z}}$, where

$$(a_{N,c})_n = \begin{cases} 2 & \text{if } n = 0, \\ c\left(1 - \frac{|n|}{N+1}\right) & \text{if } 2 \leq |n| \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

The non-negative real-valued function

$$f_{N,c}(x) = 2 + \sum_{n=-N}^{-2} (a_{N,c})_n \, e^{2\pi inx} + \sum_{n=2}^{N} (a_{N,c})_n \, e^{2\pi inx}$$

is a positive absolutely continuous minimal extrapolation.

# Optimality in higher dimensions

In higher dimensions, geometry plays an important role.

- Let $\Lambda = \{-1,0,1\}^2 \setminus \{(1,-1),(-1,1)\}$ and let $\mu = \delta_{(0,0)} + \delta_{(1/2,1/2)} \in M(\mathbb{T}^2)$.
- Then, $\mu$ is a minimal extrapolation, $\epsilon = 2$, and $\Gamma = \{(0,0),(1,1),(-1,-1)\}$.
- We can construct other discrete minimal extrapolations. For any $x \in \mathbb{R}$ and any integer $N \geq 2$, define the measure

$$\nu_{N,x} = \frac{2}{N} \sum_{n=0}^{N-1} \delta_{\left(x+\frac{n}{N},1-x-\frac{n}{N}\right)}.$$

  Then, each $\nu_{N,x}$ is a minimal extrapolation.

# Optimality in higher dimensions, continued

- For this example, with $d = 2$ and $\#\Gamma = 3$, we can also construct a continuous singular minimal extrapolation. According to the Theorem, each minimal extrapolation is supported in the set,

  $$S = \{x \in \mathbb{T}^2 \colon x_1 + x_2 = 1\}.$$

  In particular, all 3 hyperplanes are identical. Let $\sigma = \sqrt{2}\sigma_S$, where $\sigma_S$ is the *surface measure* of the Borel set $S$. We readily verify that $\sigma$ is a minimal extrapolation.

This example shows that the second statement of our theorem is optimal.

Norbert Wiener Center for Harmonic Analysis and Applications

# On minimum separation

In view of the CFG theorem, it natural to ask whether separation is necessary in order to recover a discrete measure. We show that if two Dirac masses are too close, super-resolution is impossible.

- Let $\Lambda \subseteq \mathbb{Z}^d$ be a finite set and let $\mu_y = \delta_0 - \delta_y$ for some non-zero $y \in \mathbb{T}^d$.
- Let $\nu_y$ be the absolutely continuous measure,

$$\nu_y(x) = \sum_{m \in \Lambda} \widehat{\mu_y}(m) e^{2\pi i m \cdot x}.$$

By construction, $\widehat{\nu_y} = \widehat{\mu_y}$ on $\Lambda$. As $y \to 0$,

$$\|\nu_y\| = \int_{\mathbb{T}^d} \left| \sum_{m \in \Lambda} \widehat{\mu_y}(m) e^{2\pi i m \cdot x} \right| dx \to 0.$$

For $|y|$ sufficiently small, we see that $\mu_y \notin \mathcal{E}$.

# Outline

# Epilogue

- And lest one thinks it is all 20th century spectral estimation theory – from the profound underlying harmonic analysis to MEM, MUSIC, ESPRIT, well . . . .
- Our theory shows that $\Gamma$ provides significant information about the minimal extrapolations. In particular, when $\#\Gamma \neq 1$, they are always singular measures, but when $\#\Gamma = 1$, they could be absolutely continuous.
- We have not discussed how to solve Problem (SR) computationally. Candès and Fernandez-Granda provided an algorithm, that is effective in some situations.
- The theorem opens up the possibility of the super-resolution of continuous singular measures. Since we are concerned with Fourier samples, medical imaging is a natural application of this theory.

Norbert Wiener Center
for Harmonic Analysis and Applications

# Epilogue, continued

- For example, consider the case of fast MRI signal reconstruction in the spatial domain using spectral data from spiral-scan echo planar imaging (SEPI), e.g., [8]. A new frame-based theoretical and computational methodology for fast data acquisition on interleaving spirals in $k$-space (the spectral domain) was developed with Alfredo Nava-Tudela, Alex Powell, Yang Wang, and Hui-Chuan Wu [2].
- In terms of super-resolution, this approach can be considered resolution by means of multiple spectral snapshots from bounded subsets of $k$-space, and because of the frame theoretic modeling there are inherent noise reduction and stability features.
- Dynamic MRI machines are now made by Siemens using David Donoho's patents on compressed sensing and sparsity to gather data 15 times faster than previous machines! See [10].

# Epilogue, continued

- Our theorem does not require additional assumptions on $\mu \in M(\mathbb{T}^d)$ or on the finite subset $\Lambda \subseteq \mathbb{Z}^d$. Since the theorem also describes the support set of the minimal extrapolations of $\mu$ from $\Lambda$, it is useful for determining whether a given $\mu$ can be recovered by solving the super-resolution problem.
- The second statement of the theorem provides sufficient conditions for when the minimal extrapolations are supported in a lattice. As we have seen, such measures correspond to vectors solving the discrete compressed sensing problem. Thus, our theorem is a continuous-discrete correspondence result.
- Our results are closely related to Beurling's work on minimal extrapolation. He dealt with $\mathbb{R}^1$ instead of $\mathbb{T}^d$, so our theorem is an adaptation to the torus and a generalization to higher dimensions. There are non-trivial technical differences between working with $\mathbb{R}^1$ and $\mathbb{T}^d$.

# References I


- John J. Benedetto and Weilin Li, *Super-resolution by means of Beurling minimal extrapolation*, submitted 2016.
- John J. Benedetto, Alfredo Nava-Tudela, Alex Powell, and Yang Wang, *MRI, the Beurling density theory, and finite frame algorithms*, (2016), invited chapter in Mayeli–Pesenson book.
- Arne Beurling, *Balayage of Fourier-Stieltjes transforms*, The Collected Works of Arne Beurling **2** (1989), 341–350.
- ______, *Interpolation for an interval in* $\mathbb{R}^1$, The Collected Works of Arne Beurling **2** (1989), 351–365.
- Emmanuel J. Candès and Carlos Fernandez-Granda, *Super-resolution from noisy data*, The Journal of Fourier Analysis and Applications **19** (2013), no. 6, 1229–1254.

# References II


- ______, *Towards a mathematical theory of super-resolution*, Communications on Pure and Applied Mathematics **LXVII** (2014), 906–956.
- Yohann De Castro and Fabrice Gamboa, *Exact reconstruction using Beurling minimal extrapolation*, Journal of Mathematical Analysis and Applications **395** (2012), no. 1, 336–354.
- B. M. Delattre, R. M. Heidemann, L. A. Crowe, J. P. Valle, and J. N. Hyacinthe, *Spiral demystified*, Magnetic Resonance Imaging **28 (6)** (2010), 862–881.
- David L. Donoho and Philip B. Stark, *Uncertainty principles and signal recovery*, SIAM Journal on Applied Mathematics **49** (1989), no. 3, 906–931.


Norbert Wiener Center
for Harmonic Analysis and Applications

# References III


- Michael Lustig, David Donoho, and John M. Pauly, *Sparse MRI: the application of compressed sensing to rapid MR imaging*, Magnetic Resonance in Medicine **58** (2007), 1182–1195.
- Basarab Matei and Yves Meyer, *A variant of compressed sensing*, Rev. Mat. Iberoam. **25** (2009), 669–692.
- ______, *Simple quasicrystals are sets of stable sampling*, Complex Variables and Elliptic Equations **55** (2010), no. 8-10, 947–964.
- Basarab Matei, Yves Meyer, and Ortega-Cerda, *Stable sampling and Fourier multipliers*, (2013).


Norbert Wiener Center
for Harmonic Analysis and Applications


Norbert Wiener Center
for Harmonic Analysis and Applications