Regularization of the Inverse Laplace Transform with Applications in Nuclear Magnetic Resonance Relaxometry
Candidacy Exam

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  - FINDCORNER Algorithm
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Nuclear Magnetic Resonance (NMR) Relaxometry

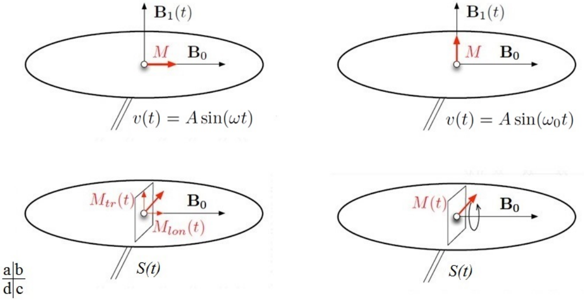

Figure: Clockwise from top left: a. Local magnetization M emerges from alignment with magnetic field B_0. b. With an RF pulse, M aligns with the magnetic field B_1 in the transversal plane. c. After the pulse, M begins to realign with B_0. d. Components $M_{lon}(t)$ and $M_{tr}(t)$, characterized by decay rates T_1 and T_2, respectively, describe $M(t)$ at time t. Images courtesy of Alfredo Nava-Tudela.

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Objective

A 1-dimensional continuous NMR relaxometry signal takes the form

$$y(t) = \int_0^\infty f(T_2)e^{-t/T_2}\,\mathrm{d}T_2 + n(t) \tag{1}$$

where T_2 is the transversal decay rate, $f(T_2)$ corresponds to the amplitude of the associated component, and $n(t)$ is additive noise.

Objective:
Recover the distribution of amplitudes $f(T_2)$ present in the signal via an inverse Laplace transform (ILT).

Toy Example

Consider a signal

$$y(t) = 0.6e^{-t/T_{2,1}} + 0.4e^{-t/T_{2,2}} + n(t) \tag{2}$$

where the exact distribution $f(T_2)$ is

$$f(T_2) = 0.6\ \delta_{T_{2,1}}(T_2) + 0.4\ \delta_{T_{2,2}}(T_2) \tag{3}$$

The recovery of $f(T_2)$ is unstable due to the sensitivity of the inversion to noise.

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Motivation

Celik et al [1] demonstrated stabilization of the ILT through the introduction of a second, indirect dimension.

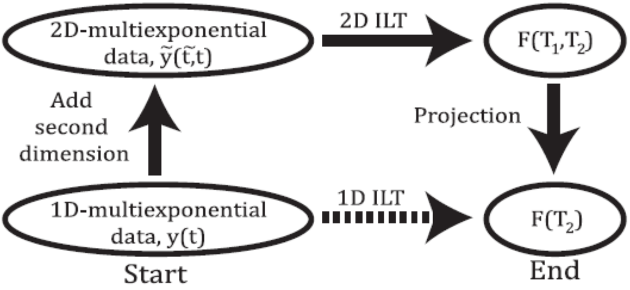

Figure: The experimental process applied by Celik [1]. The 2D ILT path illustrated by the solid arrows produced better resolution of peaks in a sparse signal than the 1D ILT path (dashed arrow).

Motivation

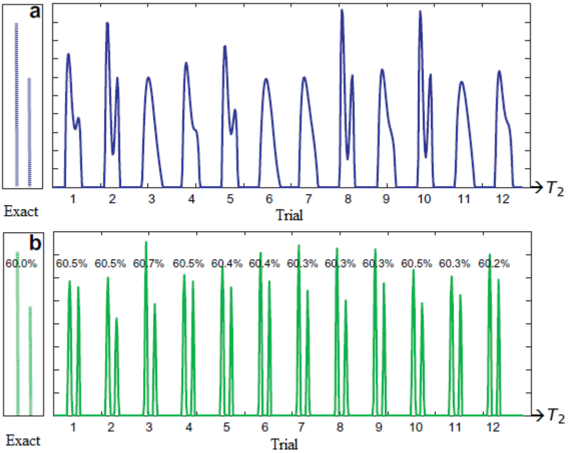

Figure: The experimental process applied by Celik [1]. Inversions from 12 different noise realizations are shown, demonstrating the stability of the 2D ILT. Top: 1D ILT. Bottom: 2D ILT projection.

Discrete Model

A 1D NMR signal takes discrete form

$$z(t_i) = \sum_{j=1}^{K} F(T_{2,j}) e^{-t_i / T_{2,j}} \tag{4}$$

Discrete Model

A 1D NMR signal takes discrete form

$$z(t_i) = \sum_{j=1}^{K} F(T_{2,j}) e^{-t_i/T_{2,j}} \tag{4}$$

In matrix form:

$$z = AF \tag{5}$$

where

$$[A]_{ij} = e^{-t_i/T_{2,j}} \tag{6}$$

$$z_i = z(t_i) \tag{7}$$

2D NMR Model

The 2D continuous NMR relaxometry signal takes the form

$$\tilde{z}(\tilde{t}, t) = \int_0^\infty \int_0^\infty F(T_1, T_2) e^{-\tilde{t}/T_1} e^{-t/T_2} dT_1 dT_2 \tag{8}$$

In discrete form,

$$\tilde{z}(\tilde{t}_i, t_j) = \sum_{k=1}^{K_1} \sum_{m=1}^{K_2} F(T_{1,k}, T_{2,m}) e^{-\tilde{t}_i/T_{1,k}} e^{-t_j/T_{2,m}} \tag{9}$$

2D NMR Model

Define $A_1, A_2, \tilde{F}$, and Z such that

$$\begin{aligned} [A_1]_{ik} &= e^{-\tilde{t}_i/T_{1,k}} & [\tilde{F}]_{km} &= F(T_{1,k}, T_{2,m}) \\ [A_2]_{jm} &= e^{-t_j/T_{2,m}} & [Z]_{ij} &= \tilde{z}(\tilde{t}_i, t_j) \end{aligned} \tag{10}$$

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2D NMR Model

Define $A_1, A_2, \tilde{F}$, and Z such that

$$[A_1]_{ik} = e^{-\tilde{t}_i/T_{1,k}} \qquad [\tilde{F}]_{km} = F(T_{1,k}, T_{2,m})$$
$$[A_2]_{jm} = e^{-t_j/T_{2,m}} \qquad [Z]_{ij} = \tilde{z}(\tilde{t}_i, t_j) \tag{10}$$

Then

$$Z = A_1 \tilde{F} A_2^T \tag{11}$$

$$(M_1 \times M_2) = (M_1 \times K_1)(K_1 \times K_2)(K_2 \times M_2)$$

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2D NMR Model

Define $A_1, A_2, \tilde{F}$, and Z such that

$$[A_1]_{ik} = e^{-\tilde{t}_i / T_{1,k}} \qquad [\tilde{F}]_{km} = F(T_{1,k}, T_{2,m})$$
$$[A_2]_{jm} = e^{-t_j / T_{2,m}} \qquad [Z]_{ij} = \tilde{z}(\tilde{t}_i, t_j) \tag{10}$$

Then

$$Z = A_1 \tilde{F} A_2^T \tag{11}$$

$$(M_1 \times M_2) = (M_1 \times K_1)(K_1 \times K_2)(K_2 \times M_2)$$

which becomes

$$vec(Z) = (A_2 \otimes A_1) vec(\tilde{F}) \tag{12}$$

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Ordinary Least Squares

With 1D ordinary least squares (OLS), we solve

$$\min_{F \in \mathbb{R}^K} ||AF - y||_2^2 \tag{13}$$

Ordinary Least Squares

Define the singular value decomposition of A as

$$A = \sum_{i=1}^{L} \sigma_i u_i v_i^T \tag{14}$$

where σ_i are the singular values and u_i, v_i are the left and right singular vectors, respectively. Then the OLS solution is

$$F_{OLS} = \sum_{i=1}^{N} \frac{u_i^T y \; v_i}{\sigma_i}. \tag{15}$$

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Regularization

To increase stability in the inversion, we add a penalty term tuned by the parameter α.

The most common form is Tikhonov regularization, aka ridge regression.

$$\min_{F \in \mathbb{R}^K} ||AF - y||_2^2 + \alpha^2 ||F||_2^2 \tag{16}$$

Regularization

Other common penalty forms include:

- L_p regularization, $p \geq 1$

$$\min_{F \in \mathbb{R}^K} ||AF - y||_2^2 + \alpha ||F||_p \qquad (16)$$

L_1 regularization is known as *LASSO*. L_p regularization for $1 < p < 2$ is called *bridge regression.*

Regularization

Other common penalty forms include:

- differential operator, L

$$\min_{F \in \mathbb{R}^K} ||AF - y||_2^2 + \alpha ||LF||_2^2 \qquad (16)$$

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Regularization

We will discuss Tikhonov regularization.

$$\min_{F \in \mathbb{R}^K} ||AF - y||_2^2 + \alpha^2 ||F||_2^2 \tag{16}$$

Regularized Least Squares

The regularized problem can be expressed as a normal least squares problem

$$\min_{F \in \mathbb{R}^K} ||\tilde{A}F - \tilde{y}||_2^2 \tag{17}$$

where

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} A \\ \alpha I_K \end{bmatrix} \\ \tilde{y} &= \begin{bmatrix} y \\ 0_K \end{bmatrix} \end{aligned}$$

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Regularized Least Squares

For an appropriate choice of α,

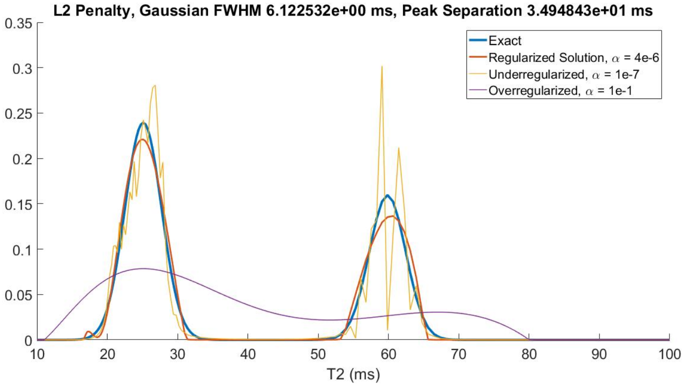
$$F_{Tikh} = \sum_{i=1}^{N} f_i \frac{u_i^T y \, v_i}{\sigma_i} \tag{18}$$

where, in the case of Tikhonov regularization, the filter factors f_i take the form

$$f_i = \frac{\sigma_i^2}{\alpha^2 + \sigma_i^2}. \tag{19}$$

Regularization Parameter

The choice of regularization parameter α strongly influences the character of the solution.



Methods to Choose the Regularization Parameter

Numerous methods have been proposed to choose the ideal regularization parameter. We will consider the following:

- Discrepancy Principle
- Generalized Cross Validation
- L-Curve Method

Discrepancy Principle

The discrepancy principle attempts to minimize the residual based on a prespecified error bound ϵ, such that

$$||AF_\alpha - y|| = \epsilon \tag{20}$$

for the optimal α.

Disadvantages:

- Requires *a priori* knowledge of the error
- Often oversmooths the solution

Generalized Cross Validation

Generalized Cross Validation (GCV) extends the idea of "leave-one-out" cross-validation, minimizing the function

$$G(\alpha) = \frac{||AF_\alpha - y||^2}{(\tau(\alpha))^2} \tag{21}$$

with

$$\tau(\alpha) = \text{trace}(I - A(A^TA + \alpha^2 L^T L)^{-1} A^T). \tag{22}$$

where L is a differential operator or the identity matrix (Hansen).

Disadvantages:

- $G(\alpha)$ is difficult to minimize numerically due to its flatness
- Does not perform well with correlated errors

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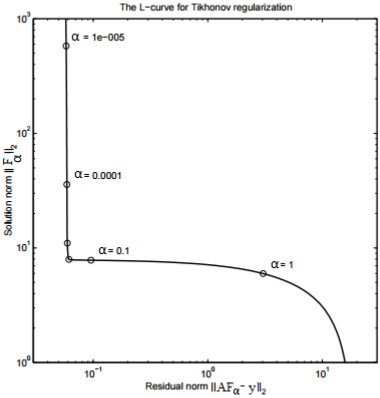
The L-Curve

Introduced by P.C. Hansen in 1993, the L-curve was originally used as an analytical tool.



It plots the residual $||AF_\alpha - y||_2$ against the size of the solution $||F_\alpha||_2$ as a function of α.

L-Curve Method

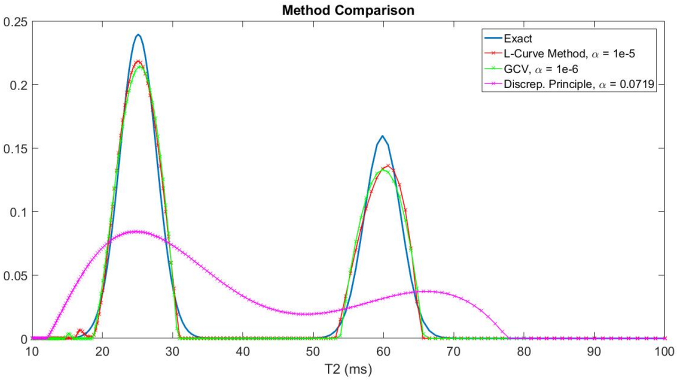
The L-curve method was proposed by Hansen and O'Leary as a means of choosing the regularization parameter α. Idea: Find the "corner" of the L-curve.

Advantages over the other methods:

- Well-defined numerically
- Requires no prior knowledge of the errors
- Not heavily influenced by large correlated errors when considered on a log scale

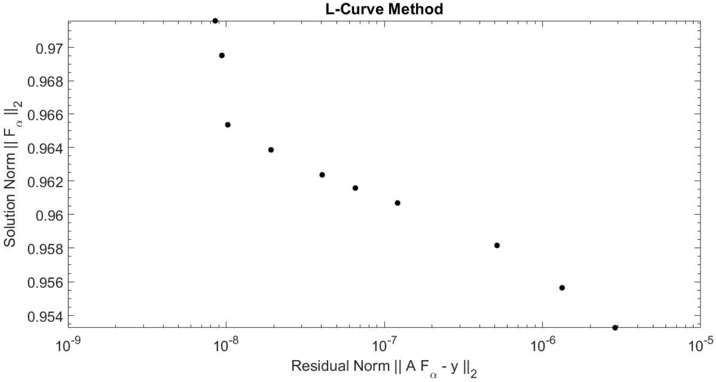
L-Curve Method

When GCV performs well, the chosen α value is close to the value chosen by the L-curve method.



L-Curve Method (Hansen, O'Leary)

Let (ρ, η) define a point on the L-curve in log scale.



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1. Calculate several points (ρ_i, η_i) on each side of the corner.

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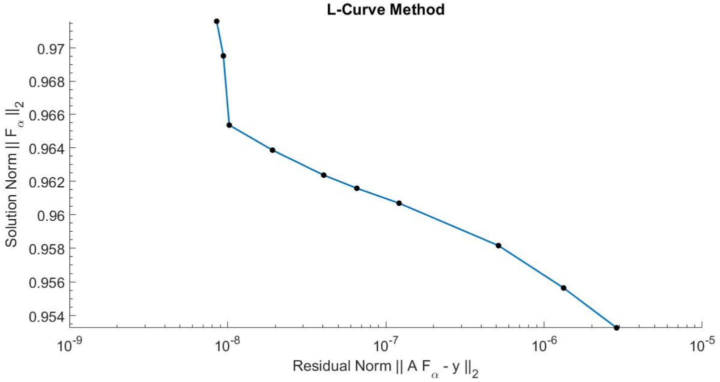
L-Curve Method (Hansen, O'Leary)

Let (ρ, η) define a point on the L-curve in log scale.



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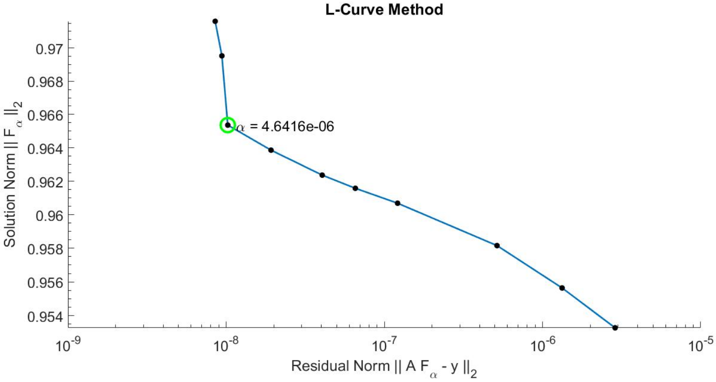
2. Fit a 3-dimensional cubic spline to those points $(\rho_i, \eta_i, \alpha_i)$ after first performing local smoothing.

L-Curve Method (Hansen, O'Leary)

Let (ρ, η) define a point on the L-curve in log scale.



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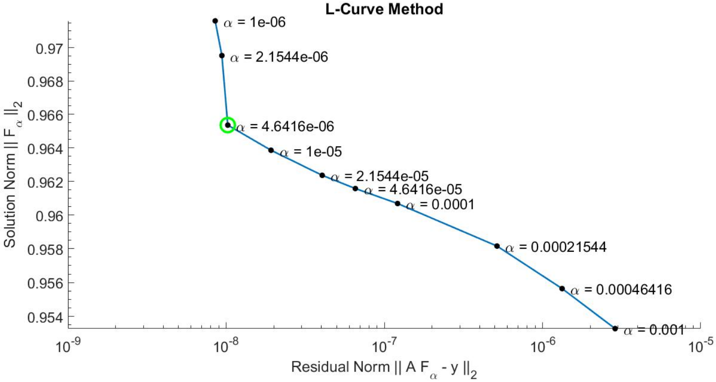
3. Compute the point of maximum curvature and find the corresponding regularization parameter α_0.

L-Curve Method (Hansen, O'Leary)

Let (ρ, η) define a point on the L-curve in log scale.



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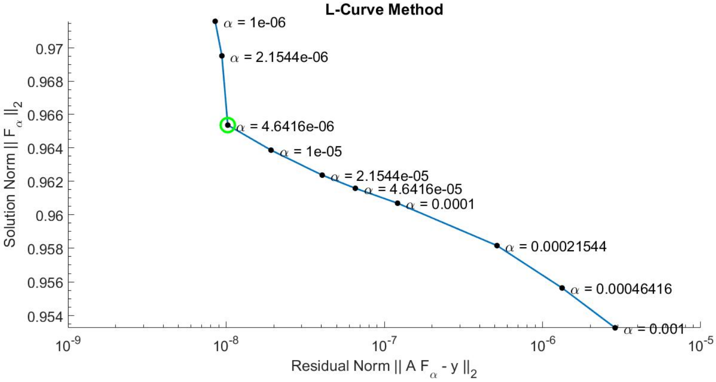
4. Solve the regularized problem and add the new point $(\rho(\alpha_0), \eta(\alpha_0))$ to the L-curve.

L-Curve Method (Hansen, O'Leary)

Let (ρ, η) define a point on the L-curve in log scale.



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5. Repeat until convergence.

Model Extensions

Characterize the "optimal" choice of *penalty term* using the L-curve method to optimize the regularization parameter.

- L_2 penalty:

$$\min_{F \in \mathbb{R}^K} ||AF - y||_2^2 + \alpha ||F||_2^2$$

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- L_1 penalty:

$$\min_{F \in \mathbb{R}^K} ||AF - y||_2^2 + \alpha ||F||_1$$

Model Extensions

Characterize the "optimal" choice of *penalty term* using the L-curve method to optimize the regularization parameter.

- L_2 penalty:

$$\min_{F \in \mathbb{R}^K} ||AF - y||_2^2 + \alpha ||F||_2^2$$

- L_1 penalty:

$$\min_{F \in \mathbb{R}^K} ||AF - y||_2^2 + \alpha ||F||_1$$

- Elastic Net:

$$\min_{F \in \mathbb{R}^K} ||AF - y||_2^2 + \alpha_1 ||F||_1 + \alpha_2 ||F||_2$$

Model Extensions

Characterize the "optimal" choice of *penalty term* using the L-curve method to optimize the regularization parameter.

- L_2 penalty:

$$\min_{F\in\mathbb{R}^K} ||AF - y||_2^2 + \alpha||F||_2^2$$

- L_1 penalty:

$$\min_{F\in\mathbb{R}^K} ||AF - y||_2^2 + \alpha||F||_1$$

- Elastic Net:

$$\min_{F\in\mathbb{R}^K} ||AF - y||_2^2 + \alpha_1||F||_1 + \alpha_2||F||_2$$

- L_p penalty:

$$\min_{F\in\mathbb{R}^K} ||AF - y||_2^2 + \alpha||F||_p$$

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Conclusion

An NMR relaxometry signal can be inverted via an inverse Laplace transform. The measurement of an additional, indirect dimension provides increased stability in the inversion.

- Regularization
  - Form a least-squares problem.
  - Add a Tikhonov regularization term for stability.
- L-Curve
  - Critical to the quality of the inversion is the choice of regularization parameter α.
  - Use parametric plot of $||AF_\alpha - y||_2$ versus $||F_\alpha||_2$ to find the optimal α.

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Primary Material:

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