# A Characterization of Shift-invariant Spaces on LCA Groups

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# Outline

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# Shift-invariant Spaces on $\mathbb{R}$

- A shift-invariant space V is a closed subspace of $L^2(\mathbb{R})$ that is invariant under integer translation,
  i.e., if $\phi \in V$, then $\tau_k \phi = \phi(\cdot - k) \in V, \forall k \in \mathbb{Z}$.

- Define the mapping $\mathcal{T} : L^2(\mathbb{R}) \to L^2(\mathbb{T}, \ell^2(\mathbb{Z}))$ as

$$\mathcal{T} f(x) = \{\hat{f}(x + k)\}_{k \in \mathbb{Z}}.$$

  Then V is shift-invariant $\Leftrightarrow \mathcal{T} V$ is closed under integer modulation.
  Where modulation by $k$ is define as $e_k(x)\phi(x) = e^{2\pi i k \cdot x}\phi(x)$.

- Q: Can we extend this result to LCA groups?
  A: Yes.

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# Intro

## Definition

The sequence $\{u_i\}_{i\in I}$ is a frame for the Hilbert space $\mathcal{H}$ with constants $A > 0$ and $B > 0$ if

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, u_i\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in \mathcal{H}.$$

# Intro (cont'd)

**Theorem [Benedetto & Li (1994)]**

Let $\phi \in L^2(\mathbb{R}^d)$ and let

$$V \equiv \overline{Span}\{\tau_k \phi : k \in \mathbb{Z}^d\}$$

be a closed subspace of $L^2(\mathbb{R}^d)$. The sequence $\{\tau_k \phi\}$ is a frame for V if and only if there are positive constants A and B such that

$$A \leq \Phi(\gamma) \leq B \text{ a.e. on } \mathbb{T}^d \setminus N,$$

where $\Phi(\gamma) \equiv \sum_{m \in \mathbb{Z}^d} |\hat{\phi}(\gamma + m)|^2$ and $N \equiv \{\gamma \in \mathbb{T}^d : \Phi(\gamma) = 0\}$.

# Outline

# Assumptions and Notations

- G is a second countable, locally compact abelian, Hausdorff group.
- A uniform lattice $H$ in G is a discrete subgroup of G such that the quotient group $G/H$ is compact.
  Note: We only consider countable uniform lattice.
- A section of $G/H$ is a set of representatives of this quotient.

# Assumptions and Notations (cont'd)

- Dual group of G,

$$\hat{G} = \Gamma = \{\gamma : G \to \mathbb{C} : \gamma \textit{ is continuous character of } G\}.$$

Where a character is a function such that $|\gamma(x)| = 1, \forall x \in G$ and $\gamma(x+y) = \gamma(x)\gamma(y), \forall x, y \in G$.

- Denote $(x, \gamma) = \gamma(x)$.
- Annihilator of H,

$$\Delta = \{\gamma \in \Gamma : (h, \gamma) = 1, \forall h \in H\}.$$

**Theorem**

$\Delta$ is a countable uniform lattice in $\Gamma$.

# Example

If we consider the group to be $\mathbb{R}$, we have:

| | | |
|---|---|---|
| $G = \mathbb{R}$ | $H = \mathbb{Z}$ | $G/H = \mathbb{T}$ |
| $\Gamma = \mathbb{R}$ | $\Delta = \mathbb{Z}$ | $\Gamma/\Delta = \mathbb{T}$ |

# Haar Measure on LCA Groups

- A Haar measure exists for each G.
- There exist a Borel measurable section of $G/H$.
- $L^p(G)$ can be defined as

$$L^p(G) = \{f : G \to \mathbb{C} : f \textit{ is measurable and } \int_G |f(x)|^p dm_G(x) < \infty\}.$$

- We focus on $L^2(G)$.

# Fourier Transform

## Definition

Given $f \in L^1(G)$, the Fourier transform is

$$\hat{f}(\gamma) = \int_G f(x)(x, -\gamma) dm_G(x), \gamma \in \Gamma.$$

- The Haar measure of Γ and G can be chosen that the following inversion formula holds for certain class of functions
$$f(x) = \int_\Gamma \hat{f}(\gamma)(x, \gamma) dm_\Gamma(\gamma).$$
- the Fourier transform on $L^1(G) \cap L^2(G)$ can be extended to a unitary operator from $L^2(G)$ onto $L^2(\Gamma)$.
- $y \in G$, then $\widehat{\tau_y f}(\gamma) = (y, -\gamma)\hat{f}(\gamma)$.

# Hilbert Space Properties of $L^2(\Omega)$

- Fix $\Omega$ a Borel section of $\Gamma/\Delta$.
- Define $\eta_h(\gamma) = (h, -\gamma)\chi_\Omega(\gamma)$,
  then $\{\eta_h\}_{h\in H}$ is an orthogonal basis for $L^2(\Omega)$.
- $m_H$ and $m_{\Gamma/\Delta}$ can be chosen such that

$$\|a\|_{\ell^2(H)} = \frac{m_H(\{0\})^{1/2}}{m_H(\Omega)^{1/2}} \|\sum_{h\in H} a_h\eta_h\|_{L^2(\Omega)}$$

for each $a = \{a_h\}_{h\in H} \in \ell^2(H)$.

# Outline

# H-invariant Spaces

## Definition

A closed subspace $V \subseteq L^2(G)$ is H-invariant if

$$f \in V \Rightarrow \tau_h f \in V \quad \forall h \in H,$$

where $\tau_y f(x) = f(x-y)$ denotes the translation of $f$ by an element $y$ of G.

- For a subset $\mathcal{A} \subseteq L^2(G)$, denote
  $E_H(\mathcal{A}) = \{\tau_h \phi : \phi \in \mathcal{A}, h \in H\}$ and $S(\mathcal{A}) = \overline{span} E_H(\mathcal{A})$.
  Call $S(\mathcal{A})$ the H-invariant space generated by $\mathcal{A}$.
- If $\mathcal{A}$ contains only one element $\phi$, then we call $S(\mathcal{A}) = S\phi$ a principle H-invariant space.

# Fiber Map

- $L^2(\Omega, \ell^2(\Delta))$ is the space of all measurable functions $\Phi : \Omega \to \ell^2(\Delta)$ such that

$$\int_\Omega \|\Phi(\omega)\|^2_{\ell^2(\Delta)} dm_\Gamma(\omega) < \infty.$$

## Proposition

The mapping $\mathcal{T} : L^2(G) \to L^2(\Omega, \ell^2(\Delta))$ defined as

$$\mathcal{T} f(\omega) = \{\hat{f}(\omega + \delta)\}_{\delta \in \Delta},$$

is an isomorphism that satisfies $\|\mathcal{T} f\|_2 = \|f\|_{L^2(G)}$.

- $\mathcal{T} \tau_h f(\omega) = (h, -\omega) \mathcal{T} f(\omega)$.

# Range Function

## Definition

A range function is a mapping,

$$J : \Omega \to \{\textit{closed spaces of } \ell^2(\Delta)\}.$$

The subspace $J(\omega)$ is called the fiber space associated to $\omega$.

Note:

- This concept was first developed by Helson in [6] .
- Denote the orthogonal projection onto $J(\omega)$, $P_\omega : \ell^2(\Delta) \to J(\omega)$.
- J is a measurable range function if and only if for all $\Phi \in L^2(\Omega, \ell^2(\Delta))$ and all $b \in \ell^2(\Delta)$, $\omega \mapsto \langle P_\omega(\Phi(\omega)), b \rangle$ is measurable.

# Orthogonal Projection

- Define the set $M_J$ as

$$M_J = \{\Phi \in L^2(\Omega, \ell^2(\Delta)) : \Phi(\omega) \in J(\omega) \quad a.e.\, \omega \in \Omega\}.$$

## Proposition

Let J be a measurable range function and $P_\omega$ the associated orthogonal projections. Denote by $\mathcal{P}$ the orthogonal projection onto $M_J$. Then,

$$(\mathcal{P}\Phi)(\omega) = P_\omega(\Phi(\omega)),\ a.e.\, \omega \in \Omega,\ \forall \Phi \in L^2(\Omega, \ell^2(\Delta)).$$

# Proof of Proposition

- Define $\mathcal{Q} : L^2(\Omega, \ell^2(\Delta)) \to L^2(\Omega, \ell^2(\Delta))$ as $(\mathcal{Q}\Phi)(\omega) = P_\omega(\Phi(\omega))$, Claim: $\mathcal{Q} = \mathcal{P}$.
- $\mathcal{Q}$ is a well defined and has norm $\leq 1$ since

$$\|\mathcal{Q}\Phi\|_2^2 = \int_\Omega \|P_\omega(\Phi(\omega))\|_{\ell^2(\Delta)}^2 dm_\Gamma(\omega) \leq \|\Phi\|_2^2.$$

- $\mathcal{Q}$ satisfies $\mathcal{Q}^2 = \mathcal{Q}$ and $\mathcal{Q}^* = \mathcal{Q}$ by definition
$\Rightarrow$ It is an orthogonal projection.
- $M := Ran(\mathcal{Q})$ equals $M_J \Rightarrow \mathcal{Q}$ is orthogonal projection onto $M_J$.

# Main Result

**Theorem 1 [Cabrelli & Paternostro (2010)]**

Let $V \subseteq L^2(G)$ be a closed subspace. Then V is H-invariant if and only if there exist a measurable range function J such that

$$V = \{f \in L^2(G) : \mathcal{T}f(\omega) \in J(\omega) \quad a.e.\, \omega \in \Omega\}.$$

If two range functions which are equal almost everywhere are identified, the correspondence is one-to-one and onto.

If $V = S(\mathcal{A})$ where $\mathcal{A}$ is a countable subset of $L^2(G)$, then

$$J(\omega) = \overline{span}\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\}.$$

# Proof of Theorem 1

We will need the following lemma:

**Lemma**

If J and K are two measurable range functions such that $M_J = M_K$, then $J(\omega) = K(\omega)$ $a.e.$ $\omega \in \Omega$.

Proof:

- Denote $P_\omega$ and $Q_\omega$ projections correspond to J, K; $\mathcal{P}$ the orthogonal projection onto $M_J = M_K$.
- $P_\omega(\Phi(\omega)) = (\mathcal{P}\Phi)(\omega) = Q_\omega(\Phi(\omega))$.
- $P_\omega$ and $Q_\omega$ map basis of $\ell^2(\Delta)$ onto same image.

# Proof of Theorem 1

($\Rightarrow$) $L^2(G)$ is separable, then $\exists\mathcal{A}$ countable such that $V = S(\mathcal{A})$.
Define $J(\omega) = \overline{span}\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\}$.

Step 1: $V = \{f \in L^2(G) : \mathcal{T}f(\omega) \in J(\omega) \; a.e. \; \omega \in \Omega\}$

- Need: $M := \mathcal{T}V = M_J$.
- For $\Phi \in M$,
  $\exists\{g_j\}_{j\in\mathbb{N}} \subseteq spanE_H(\mathcal{A})$ such that $\mathcal{T}g_j = \Phi_j \to \Phi$ in $L^2(\Omega, \ell^2(\Delta))$.
  $\Phi_j(\omega) \in J(\omega) \Rightarrow \Phi(\omega) \in J(\omega)$.

# Proof of Theorem 1

- Suppose there exists a non-zero $\Psi \in L^2(\Omega, \ell^2(\Delta))$ orthogonal to M. Since V is H-invariant, for any $\Phi \in \mathcal{TA} \subseteq M$

$$0 = \int_\Omega (h, -\omega)\langle \Phi(\omega), \Psi(\omega)\rangle_{\ell^2(\Delta)} dm_\Gamma(\omega)$$

$\Psi(\omega) \perp J(\omega)$ a.e. $\omega \in \Omega$, thus $\Psi \perp M_J$.

# Proof of Theorem 1

Step 2: J is measurable

- Let $\mathcal{I}$ be identity mapping on $L^2(\Omega, \ell^2(\Delta))$;
  $\mathcal{P} : L^2(\Omega, \ell^2(\Delta)) \to M$ be the orthogonal projection onto M.

- For $\Psi \in L^2(\Omega, \ell^2(\Delta))$,$(\mathcal{I} - \mathcal{P})\Psi(\omega) \perp J(\omega)$, a.e. $\omega \in \Omega$, then

$$P_\omega((\mathcal{I} - \mathcal{P})\Psi(\omega)) = P_\omega(\Psi(\omega) - \mathcal{P}\Psi(\omega)) = 0.$$

- $P_\omega(\Psi(\omega)) = \mathcal{P}\Psi(\omega)$.

# Proof of Theorem 1

($\Leftarrow$)

- We need: $V := \mathcal{T}^{-1}(M_J)$ is H-invariant.
- For any $f \in V$, $\mathcal{T}(\tau_h f)(\omega) = (h, -\omega)\mathcal{T} f(\omega)$ for almost every $\omega \in \Omega$ $\Rightarrow (h, -\omega)\mathcal{T} f(\omega) \in J(\omega)$.
- $\mathcal{T}(\tau_h f) \in M_J \Rightarrow \tau_h f \in \mathcal{T}^{-1}(M_J)$.

# Shift-invariant Spaces on $L^2(\mathbb{R})$

**Theorem [Helson (1964)]**

The doubly invariant subspaces of $L^2_{\mathcal{H}}$ are precisely the subspace $M_J$, where J is a measurable range function.
The correspondence between J and $M_J$ is one-to-one, under the convention that range functions are identified if they are equal almost everywhere.

# From Shift-invariant Spaces to Frames

## Theorem [Bownik (2000)]

Suppose $\mathcal{A} \subseteq L^2(\mathbb{R}^n)$ is countable. Then the following are equivalent:

1. $E_H(\mathcal{A})$ is a frame for its close span $S(\mathcal{A})$ with constants A and B.
2. For a.e. $x \in \mathbb{T}^n$, $\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\} \subseteq \ell^2(\mathbb{Z}^n)$ is a frame for its closed span with constants A and B.

## Theorem [Gol & Tousi (2008)]

Let $\phi \in L^2(G)$. $E_H\{\phi\}$ form a Parseval frame for the space $S\phi$ if and only if $\|\mathcal{T}\phi(\omega)\|_2 = 1$ a.e. on $\Omega \backslash N$ where $N \equiv \{\omega \in \Omega : \|\mathcal{T}\phi(\omega)\|_2 = 0\}$.

# Outline

# Characterization of Frames for H-invariant Spaces

**Theorem 2 [Cabrelli & Paternostro (2010)]**

Let $\mathcal{A}$ be a countable subset of $L^2(G)$, J the measurable range function associated, and $A \leq B$ positive constants. Then the following are equivalent:

1. The set $E_H(\mathcal{A})$ is a frame for its closed span $S(\mathcal{A})$ with contants A and B.
2. For a.e. $\omega \in \Omega$, the set $\{\mathcal{T}\phi(\omega) : \phi \in \mathcal{A}\} \subseteq \ell^2(\Delta)$ is a frame for $J(\omega)$ with constants A and B.

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# Proof of Theorem 2

- Assuming either (i) or (ii), we have

$$\sum_{\phi\in\mathcal{A}}\sum_{h\in H}|\langle t_h\phi, f\rangle_{L^2(G)}|^2$$

$$= \sum_{\phi\in\mathcal{A}}\sum_{h\in H}|\int_\Omega (h,-\omega)\langle\mathcal{T}\phi(\omega),\mathcal{T}f(\omega)\rangle_{\ell^2(\Delta)}dm_\Gamma(\omega)|^2 \qquad (1)$$

$$= \sum_{\phi\in\mathcal{A}}\int_\Omega |\langle\mathcal{T}\phi(\omega),\mathcal{T}f(\omega)\rangle_{\ell^2(\Delta)}|^2 dm_\Gamma(\omega) \qquad (2)$$

# Proof of Theorem 2

$(ii) \Rightarrow (i)$

- We need: $A\|f\|^2 \leq \sum_{\phi\in\mathcal{A}} \sum_{h\in H} |\langle t_h\phi, f\rangle_{L^2(G)}|^2 \leq B\|f\|^2$ for $f \in S(\mathcal{A})$.
- For any $f \in S(\mathcal{A})$, we have $\mathcal{T}f \in J(\omega)$, then

$$A\|\mathcal{T}f(\omega)\|^2 \leq \sum_{\phi\in\mathcal{A}} |\langle \mathcal{T}\phi(\omega), \mathcal{T}f(\omega)\rangle|^2 \leq B\|\mathcal{T}f(\omega)\|^2.$$

- $\mathcal{T}$ is an isometry, by (1), we get $(ii) \Rightarrow (i)$.

# Proof of Theorem 2

$(i) \Rightarrow (ii)$

- Let $D$ be a dense countable subset of $\ell^2(\Delta)$, then (ii) is equivalent to: For all $d \in D$,

$$A\|P_\omega d\|^2 \leq \sum_{\phi \in \mathcal{A}} |\langle \mathcal{T}\phi(\omega), P_\omega d\rangle|^2 \leq B\|P_\omega d\|^2, \ a.e. \omega \in \Omega.$$

- Suppose above statement is not true, then $\exists d_0 \in D$ such that either

$$\sum_{\phi \in \mathcal{A}} |\langle \mathcal{T}\phi(\omega), P_\omega d_0\rangle|^2 > (B+\epsilon)\|P_\omega d_0\|^2 \tag{3}$$

or

$$\sum_{\phi \in \mathcal{A}} |\langle \mathcal{T}\phi(\omega), P_\omega d_0\rangle|^2 < (A-\epsilon)\|P_\omega d_0\|^2$$

on a measurable set $W \subseteq \Omega$ with positive measure.

# Proof of Theorem 2

- Suppose (3) holds,
  take $f \in S(\mathcal{A})$ such that $\mathcal{T}f(\omega) = \chi_W(\omega)P_\omega d_0$.
- By (i) and (1),

$$A\|\mathcal{T}f\|^2 \leq \sum_{\phi\in\mathcal{A}} \int_\Omega |\langle \mathcal{T}\phi(\omega), \mathcal{T}f(\omega)\rangle_{\ell^2(\Delta)}|^2 dm_\Gamma(\omega) \leq B\|\mathcal{T}f\|^2$$

- Integrate (3) we get

$$\sum_{\phi\in\mathcal{A}} \int_\Omega |\langle \mathcal{T}\phi(\omega), \chi_W(\omega)P_\omega d_0\rangle_{\ell^2(\Delta)}|^2 dm_\Gamma(\omega) \geq (B+\epsilon)\|\mathcal{T}f\|^2$$

  This gives a contradiction.

# Outline

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