LOCAL AND GLOBAL STABILITY OF FUSION FRAMES

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Outline

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Definition

A **frame** $\mathcal{F} = \{f_i\}_{i \in I}$ in a Hilbert space $\mathcal{H}$ is a countable sequence $\{f_i\} \subseteq \mathcal{H}$ for which there exist $A, B > 0$ such that

$$\forall f \in \mathcal{H}, \qquad A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2.$$

- Developed by Richard Duffin and Albert Schaeffer in 1952
- Generalization of orthonormal bases which allow for possibly redundant decompositions
- Frames have varied use in applications including image processing, wireless communication, and digital signal quantization because they are naturally robust to erasures

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Frame Basics

- **Frame operator** $S_{\mathcal{F}} : \mathcal{H} \to \mathcal{H}$ by $S_{\mathcal{F}}(f) = \sum_{i \in I} \langle f, f_i \rangle f_i$,
- A **dual frame** $\{\tilde{f}_i\}_{i \in I}$ for $\mathcal{F}$ is a frame which satisfies

  $f = \sum_{i \in I} \langle f, f_i \rangle \tilde{f}_i = \sum_{i \in I} \langle f, \tilde{f}_i \rangle f_i$ for all $f \in \mathcal{H}$.
- By invertibility of the frame operator $S_{\mathcal{F}}$, $\{S_{\mathcal{F}}^{-1} f_i\}$ is a dual frame, known as the **canonical dual frame**.

- In 2004, Peter Casazza and Shidong Li developed *frames of subspaces* for simple constructions of frames

Definition

A **fusion frame** $\mathcal{W} = \{(W_i, c_i)\}_{i\in I}$ in a separable Hilbert space $\mathcal{H}$ is a countable sequence of closed subspaces $W_i \subseteq \mathcal{H}$ and a sequence of weights $\{c_i\}_{i\in I} \subseteq \mathbb{R}$, $c_i > 0$ for all $i \in I$ for which there exist $C, D > 0$ such that

$$\forall f \in \mathcal{H}, \qquad C\|f\|^2 \leq \sum_{i\in I} c_i^2 \|P_{W_i}(f)\|^2 \leq D\|f\|^2,$$

where P_{W_i} is the orthogonal projection onto W_i.

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Distributed Sensing

- Casazza, Li, and Gitta Kutyniok give an example of a sensor network spread over a forest to measure temperature, where the sensors are divided into smaller sub-networks for processing. Within the fusion frame paradigm, the sub-networks form a set of redundant subspaces, so the signals can be processed globally, at one central processing center, or locally, at stations for each network.
- This example suggests that it's useful to consider fusion frames as a special case of frames rather than a generalization.

- **Tight fusion frames** are defined to be fusion frames for which the bounds C and D can be chosen to be equal.
- A **fusion frame system** $\{(W_i, c_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ is a fusion frame $\{(W_i, c_i)\}_{i\in I}$ for which $\mathcal{F}_i = \{f_{ij}\}_{j\in J_i}$ is a frame for W_i for each $i \in I$, known as a local frame.

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- For a fusion frame $\mathcal{W}$, define the *analysis operator* by $T_{\mathcal{W}}(f) := \{c_i P_{W_i} f\}_{i \in I}$ and its adjoint, the *synthesis operator*, by $T_{\mathcal{W}}^*(\{v_i\}_{i \in I}) := \sum_{i \in I} c_i v_i$, where $f \in \mathcal{H}$ and $v_i \in W_i$ for each $i \in I$.
- Similar to the case of frames, the **fusion frame operator** $S_{\mathcal{W}} : \mathcal{H} \to \mathcal{H}$ given by $S_{\mathcal{W}}(f) = T_{\mathcal{W}}^* T_{\mathcal{W}}(f) = \sum_{i \in I} c_i^2 P_{W_i}(f)$ is a positive, self-adjoint, and invertible operator.

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Outline

Perturbation Theory

- Let $T : \mathcal{X} \to \mathcal{X}$ be a bounded linear map on a Banach space $\mathcal{X}$.
- $\|(I - T)x\| \leq \lambda(\|x\| + \|Tx\|) \implies \frac{1-\lambda}{1+\lambda}\|Tx\| \leq \|x\| \leq \frac{1+\lambda}{1-\lambda}\|Tx\|$ for $\lambda < 1$.
- Paley and Wiener (1934) showed that if $\|Sx - Tx\| \leq \lambda_1\|Sx\| + \lambda_2\|Tx\|$ then their codimensions are equal.

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Fusion Frame Perturbations

- If P and Q are orthogonal projections on $\mathcal{H}$ with $0 \leq \lambda_1, \lambda_2 < 1$ such that $\|Pf - Qf\| \leq \lambda_1 \|Pf\| + \lambda_2 \|Qf\|$ for all $f \in \mathcal{H}$, then $P = Q$.

Definition

Let $\{W_i\}_{i \in I}$ and $\{V_i\}_{i \in I}$ be collections of closed subspaces in $\mathcal{H}$ with $\{c_i\}$ a positive real sequence, and let $0 \leq \lambda_1, \lambda_2 < 1$ and $\epsilon > 0$. $\{(W_i, c_i)\}$ is a $(\lambda_1, \lambda_2, \epsilon)-$perturbation of $\{(V_i, c_i)\}$ if

$$\|(P_{W_i} - P_{V_i})f\| \leq \lambda_1 \|P_{W_i} f\| + \lambda_2 \|P_{V_i} f\| + \epsilon \|f\|$$

for all $f \in \mathcal{H}$.

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Perturbation Fusion Frame Bounds

Proposition (Casazza, Li, Kutyniok)

Let $\{(W_i, c_i)\}_{i \in I}$ *be a fusion frame for* $\mathcal{H}$ *with bounds* C, D. *Let* $\{(V_i, c_i)\}_{i \in I}$ *be a* $(\lambda_1, \lambda_2, \epsilon)-$*perturbation of* $\{(W_i, c_i)\}_{i \in I}$ *where* $0 \leq \lambda_1, \lambda_2 < 1$, $\epsilon > 0$, *and* $(1 - \lambda_1)\sqrt{C} - \epsilon(\sum_{i \in I} c_i^2)^{1/2} > 0$. *Then* $\{(V_i, c_i)\}_{i \in I}$ *is a fusion frame of* $\mathcal{H}$ *with fusion frame bounds*

$$\left[\frac{(1 - \lambda_1)\sqrt{C} - \epsilon(\sum_{i \in I} c_i^2)^{1/2}}{1 + \lambda_2}\right]^2 \text{ and } \left[\frac{(1 + \lambda_1)\sqrt{D} + \epsilon(\sum_{i \in I} c_i^2)^{1/2}}{1 - \lambda_2}\right]^2.$$

- In the case of a fusion frame system, we can also perturb the local frames:

  For $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are sequences in $\mathcal{H}$ and $0 \leq \lambda_1, \lambda_2 < 1$, $\{g_i\}_{i\in I}$ is a (λ_1, λ_2)-perturbation of $\{f_i\}_{i\in I}$ if

  $$\|\sum_{i\in I} a_i(f_i - g_i)\| \leq \lambda_1 \|\sum_{i\in I} a_i f_i\| + \lambda_2 \|\sum_{i\in I} a_i g_i\| \text{ for all } \{a_i\}_{i\in I} \in \ell^2(I).$$

- Useful lemma: Let $W = span_{i\in I}\{f_i\}$ and $V = span_{i\in I}\{g_i\}$, then $\|P_W P_V(f)\| \geq \left(\frac{1-\lambda_1}{1+\lambda_2} - \lambda_1 \frac{1+\lambda_2}{1-\lambda_1} - \lambda_2\right) \|P_V(f)\|$ for all $f \in \mathcal{H}$.

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Theorem 1

Theorem (Casazza, Li, Kutyniok)

Let $\{(W_i, c_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ *be a fusion frame system for* $\mathcal{H}$ *with fusion frame bounds* C, D*. Choose* $0 \le \lambda_1, \lambda_2 < 1$ *and* $\epsilon > 0$ *such that* $(1-\lambda_1)\sqrt{C} - \epsilon(\sum_{i\in I} c_i^2)^{1/2} > 0$ *and* $1 - \frac{\epsilon^2}{2} = \frac{1-\lambda_1}{1+\lambda_2} - \lambda_1 \frac{1+\lambda_2}{1-\lambda_1} - \lambda_2$*. For every* i*, let* $\{g_{ij}\}_{j\in J_i}$ *be a* (λ_1, λ_2)*-perturbation of* $\{f_{ij}\}_{j\in J_i}$ *and let* $V_i = span\{g_{ij}\}_{j\in J_i}$*. Then* $\{(V_i, c_i,)\}_{i\in I}$ *is a fusion frame for* $\mathcal{H}$ *with fusion frame bounds*

$$\left[\sqrt{C} - \epsilon\Big(\sum_{i\in I} c_i^2\Big)^{1/2}\right]^2 \text{ and } \left[\sqrt{D} + \epsilon\Big(\sum_{i\in I} c_i^2\Big)^{1/2}\right]^2.$$

THEOREM 1 PROOF

1. Use the useful lemma to get two estimates: $\|(I - P_{V_i})(P_{W_i} f)\|^2 \leq \frac{\epsilon^2}{2}\|P_{W_i} f\|^2$ and vice versa.
2. $$\|(P_{W_i} - P_{V_i})f\|^2 = \langle (P_{W_i} - P_{V_i})^2 f, f\rangle = \langle (P_{W_i} - P_{V_i}P_{W_i} + P_{V_i} - P_{W_i}P_{V_i})f, f\rangle \leq$$
$$\leq \|(I - P_{V_i})(P_{W_i} f) + (I - P_{W_i})(P_{V_i} f)\| \cdot \|f\| \leq$$
$$\leq \frac{\epsilon^2}{2}\|P_{W_i} f\| \cdot \|f\| + \frac{\epsilon^2}{2}\|P_{V_i} f\| \cdot \|f\| \leq \epsilon^2 \|f\|^2.$$
3. Hence, $\mathcal{W}$ is a $(0, 0, \epsilon)$-perturbation of $\mathcal{V}$.

Theorem 1 Proof

1. Use the useful lemma to get two estimates: $\|(I-P_{V_i})(P_{W_i}f)\|^2 \leq \frac{\epsilon^2}{2}\|P_{W_i}f\|^2$ and vice versa.

2. $$\|(P_{W_i}-P_{V_i})f\|^2 = \langle (P_{W_i}-P_{V_i})^2 f, f\rangle = \langle (P_{W_i}-P_{V_i}P_{W_i}+P_{V_i}-P_{W_i}P_{V_i})f, f\rangle \leq$$
$$\leq \|(I-P_{V_i})(P_{W_i}f)+(I-P_{W_i})(P_{V_i}f)\|\cdot\|f\| \leq$$
$$\leq \frac{\epsilon^2}{2}\|P_{W_i}f\|\cdot\|f\| + \frac{\epsilon^2}{2}\|P_{V_i}f\|\cdot\|f\| \leq \epsilon^2\|f\|^2.$$

3. Hence, $\mathcal{W}$ is a $(0,0,\epsilon)$-perturbation of $\mathcal{V}$.

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1. Intro
2. Global Frame Stability
3. Dual Fusion Frames
4. Local Frame Stability
5. Potential Applications

Fusion Frame Dual

- *Canonical fusion frame dual* is $\widetilde{\mathcal{W}} = \{(S_{\mathcal{W}}^{-1} W_i, c_i \| S_{\mathcal{W}}^{-1} | W_i \|)\}_{i \in I}$, where $S_{\mathcal{W}}^{-1} | W_i$ is the inverse of the fusion frame operator restricted to W_i.

Weighted Fusion Frame Perturbation

Definition

Let $\mu > 0$ and let $\mathcal{W} = \{(W_i, c_i)\}_{i \in I}$ and $\mathcal{V} = \{(V_i, d_i)\}_{i \in I}$ be fusion frames on $\mathcal{H}$. $\mathcal{V}$ is said to be a $\mu-$perturbation of $\mathcal{W}$ if $\|T_{\mathcal{W}} - T_{\mathcal{V}}\| \leq \mu$, where $T_{\mathcal{W}}$ and $T_{\mathcal{V}}$ are the analysis operators of $\mathcal{W}$ and $\mathcal{V}$, respectively.

- This implies $\|c_i P_{W_i} - d_i P_{V_i}\| \leq \mu$ for every $i \in I$.
- In finite dimensions, when the sequences of weights are the same, both definitions are equivalent.

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Theorem 2

Theorem (Kutyniok, Paternostro, Philipp)

Let $\mathcal{W} = \{(W_i, c_i)\}_{i \in I}$ be a fusion frame for $\mathcal{H}$ with frame bounds $A \leq B$ and let $\mathcal{V} = \{(V_i, d_i)\}_{i \in I}$ be a $\mu-$perturbation of $\mathcal{W}$, where $0 < \mu < \sqrt{A}$. If there exists $\tau > 0$ such that τ is a lower bound for both $\{c_i\}_{i \in I}$ and $\{d_i\}_{i \in I}$, then the canonical fusion frame dual $\widetilde{\mathcal{V}}$ of $\mathcal{V}$ is a $C\mu-$perturbation of the canonical fusion frame dual $\widetilde{\mathcal{W}}$ of $\mathcal{W}$, where

$$C = \frac{\alpha^2 + \beta^2}{A}\Big[\frac{1 + (A^{-1} + B)^2}{\sqrt{A}}\Big(\frac{\sqrt{2}}{\tau} + \alpha\beta^2\Big) + \beta^2(1 + \alpha^2\beta^2)\Big]$$

with $\alpha := 2\sqrt{B} + \mu$ and $\beta := (\sqrt{A} - \mu)^{-1}$.

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- Perhaps only one or a few subspaces are perturbed.
- What about when an entire subspace is erased? Major strength of frames is robustness to erasure.
- Robert Calderbank, Taotao Liu, Gitta Kutyniok, and Ali Pezeshki considered conditions for finite fusion frames that optimize recovery.

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Finite Fusion Frames

- For the rest of the talk, all finite dimensional real Hilbert spaces $\mathcal{H} = \mathbb{R}^M$.
- We also want to consider equally weighted spaces to simplify the problem.
- Let m_i denote the dimension of subspace W_i.

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Model of Vector Recovery

- $z_i = U_i^T x + n_i$, for $i = 1, \ldots, N$ is the fusion frame measurement at each subspace.
- x is a random zero-mean vector with variance σ_x^2, n_i is a realization of an additive white noise vector with variance σ_n^2
- U_i is a left orthogonal $N \times m_i$-matrix such that $U_i^T U_i = I_{m_i}$ and $U_i U_i^T = P_{W_i}$.

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Mean-squared Recovery Error

- Want to minimize mean-squared error (MSE) in linearly estimating x from z_i.
- Let MSE_k denote the MSE when k subspaces are erased.
- Minimum is achieved when fusion frame is tight, given by

$$MSE_0 = \frac{M\sigma_n^2\sigma_x^2}{\sigma_n^2 + \frac{L\sigma_x^2}{M}}.$$

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One Erasure, MSE_1

- We consider where $m_i = m$ is constant for all i, since MSE_1 naturally increases with m_i.
- $MSE_1 = MSE_0 +$ Erasure Term
- MSE_1 is then a function of m which has a maximum at m^*

$$m^* = \begin{cases} m_{min}, & \text{if } m_{max} \leq \tilde{m} \text{ or} \\ & \text{if } m_{min} \leq \tilde{m} \leq m_{max} \text{ and } MSE(m_{min}) \leq MSE(m_{max}), \\ m_{max}, & \text{otherwise.} \end{cases}$$

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Two Erasures, MSE_2

- $MSE_2 = MSE_1$ + Cross Term from mixed projections $P_{W_i}\ P_{W_j}$
- Want to minimize the trace of $P_{W_i}P_{W_j}$, which is equivalent to minimizing eigenvalues
- Define *principal angles* $\theta_k(i,j)$ for $1 \leq k \leq M$ as inverse cosines of eigenvalues and *chordal distance* between W_i and W_j, as $d_c(i,j) := \left(\sum_{k=1}^{M} \sin^2 \theta_k(i,j)\right)^{1/2}$
- MSE_2 is minimized when MSE_1 is minimized and chordal distance is maximized

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Two Erasures, MSE_2

- $MSE_2 = MSE_1$ + Cross Term from mixed projections P_{W_i} P_{W_j}
- Want to minimize the trace of $P_{W_i} P_{W_j}$, which is equivalent to minimizing eigenvalues
- Define *principal angles* $\theta_k(i,j)$ for $1 \leq k \leq M$ as inverse cosines of eigenvalues and *chordal distance* between W_i and W_j, as $d_c(i,j) := \left(\sum_{k=1}^{M} \sin^2 \theta_k(i,j)\right)^{1/2}$
- MSE_2 is minimized when MSE_1 is minimized and chordal distance is maximized

More Than two Erasures

- For equidistant tight fusion frame with constant m^* dimension, $MSE_k = MSE_3$ for $k \geq 3$.
- It is not known if such a fusion frame exists. If it does, it's an optimal Grassmannian packing of m^*-dimensional subspaces in $\mathbb{R}^N$.

Outline

Heterogeneous Data Fusion

- We want to combine heterogenous information, not homogeneous like in distributed processing example.
- How do we combine different information to make up for gaps in knowledge of one?
- Studying stability explains the applicability of fusion frames as a data fusion model.

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