Preconditioning of frames

Kasso Okoudjou
joint with X. Chen, Y. Hur, G. Kutyniok, F. Philipp, R. Wang

Department of Mathematics
&
Norbert Wiener Center
University of Maryland, College Park

Department of Mathematics Colloquium
Central Michigan University, Mount Pleasant, MI
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Outline

A standard problem

Question

Let $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{R}^N$ be a complete set. Recover x from $\hat{y}$:

$$\hat{y} = \Phi^T x + \eta,$$

where η is an error (noise).

Solution

Need to design "good" measurement matrix Φ, e.g., Φ should lead to reconstruction methods that are robust to erasures and noise.

Minimal requirements on the measurement matrix

Fact

$\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{K}^N$ *is complete* $\iff \exists A > 0:$

$$A\|x\|^2 \leq \sum_{i=1}^M |\langle x, \varphi_i\rangle|^2 \quad \text{for all } x \in \mathbb{K}^N$$

Clearly, there exists $B > 0$, *e.g.,* $B = \sum_{i=1}^M \|\varphi_i\|^2$ *such that*

$$\sum_{i=1}^M |\langle x, \varphi_i\rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathbb{K}^N.$$

Definition of finite frames

Definition

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. $\{\varphi_i\}_{i=1}^M \subset \mathbb{K}^N$ is called a *finite frame for* $\mathbb{K}^N$ if $\exists\, 0 < A \leq B$:

$$A\|x\|^2 \leq \sum_{i=1}^{M} |\langle x, \varphi_i\rangle|^2 \leq B\|x\|^2, \quad \text{for all } x \in \mathbb{K}^N. \tag{1}$$

If $A = B$, then $\{\varphi_i\}_{i=1}^M \subset \mathbb{K}^N$ is called *a finite tight frame for* $\mathbb{K}^N$.

Frame operator & Reconstruction formulas

- For $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{K}^N$ let $\Phi = \begin{bmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_M \end{bmatrix}$.
- Φ is a frame $\iff$ $S = \Phi\Phi^*$ is positive definite.

$$x = S(S^{-1}x) = \sum_{i=1}^{M} \langle x, S^{-1}\varphi_i \rangle \varphi_i = \sum_{i=1}^{M} \langle x, \varphi_i \rangle S^{-1}\varphi_i$$

- $\widetilde{\Phi} = \{\tilde{\varphi}_i\}_{i=1}^M = \{S^{-1}\varphi_i\}_{i=1}^M$ is the *canonical dual frame*.
- $A_{opt} = \lambda_{min}(S)$ and $B_{opt} = \lambda_{max}(S)$. The condition number of the frame is

$$\kappa(\Phi) = \lambda_{max}(S)/\lambda_{min}(S) \geq 1.$$

The canonical dual frame

Lemma

Assume that $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{K}^N$ *is a frame, and that* $\{\tilde{\varphi}_i\}_{i=1}^M \subset \mathbb{K}^N$ *is the canonical dual frame. For each* $x \in \mathbb{K}^N$, $\sum_{i=1}^M |\langle x, \tilde{\varphi}_i\rangle|^2$ *minimizes* $\sum_{i=1}^M |c_i|^2$ *for all* $\{c_i\}_{i=1}^M$ *such that* $x = \sum_{i=1}^M c_i\varphi_i$.

Why frames?

Question

Let $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{R}^N$ *be a unit norm frame. Recover* x *from*

$$\hat{y} = \Phi^* x + \eta.$$

Solution

If no assumption is made about η *we can just minimize* $\|\Phi^* x - \hat{y}\|_2$. *This leads to*

$$\hat{x} = (\Phi^\dagger)^* \hat{y} = \sum_{i=1}^M (\langle x, \varphi_i \rangle + \eta_i) \tilde{\varphi}_i.$$

Finite unit norm tight frames

Definition

A tight frame $\{\varphi_i\}_{i=1}^{M} \subset \mathbb{K}^N$ with $\|\varphi_k\| = 1$ for each k is called a *finite unit norm tight frame (FUNTF)* for $\mathbb{K}^N$. In this case, the frame bound is $A = M/N$.

Remark

Tight frames and FUNTFs can be considered optimally conditioned frames since the condition number of their frame operator is unity.

Reconstruction formulas for tight frames

- If Φ is a tight frame then $S = AI$ and $x = \frac{1}{A}\sum_{k=1}^{M}\langle x, \varphi_k\rangle\varphi_k$.
- If $\Phi = \{\varphi_k\}_{k=1}^{M} \subset \mathbb{K}^N$ is a frame then $\{S^{-1/2}\varphi_k\}_{k=1}^{M}$ is a tight frame.

Example of FUNTFs

Example

Let $\omega = e^{2\pi i/M}$

$$\frac{1}{\sqrt{M}}\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{M-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(M-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \omega^{M-1} & \omega^{2(M-1)} & \dots & \omega^{(M-1)^2} \end{bmatrix}$$

Any (normalized) N rows from the $M \times M$ DFT matrix is a tight frame for $\mathbb{C}^N$.
Every tight frame of M vectors in $\mathbb{K}^N$ is obtained from an orthogonal projection of an ONB in $\mathbb{K}^M$ onto $\mathbb{K}^N$.

Examples of frames

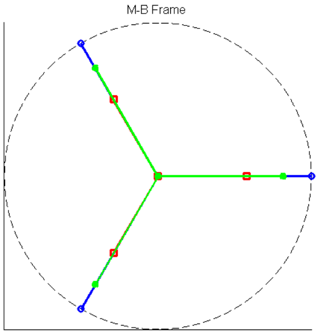

Figure : The MB-Frame

Why tight frames?

Assume that $\eta = (\eta_i)$ is iid $\mathcal{N}(0, \sigma^2)$. Then

$$x - \hat{x} = \sum_{i=1}^{M} \langle x, \varphi_i \rangle \tilde{\varphi}_i - \sum_{i=1}^{M} (\langle x, \varphi_i \rangle + \eta_i) \tilde{\varphi}_i = - \sum_{i=1}^{M} \eta_i \tilde{\varphi}_i .$$

Consequently,

$$MSE = \frac{1}{N} E \|x - \hat{x}\|^2 = \frac{1}{N} \operatorname{Trace}(S^{-1}) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i}$$

where $\{\lambda_i\}_{i=1}^{N}$ is the spectrum of S.

Theorem (Goyal, Kovačević, and Kelner (2001))

The MSE is minimum if and only if the frame Φ is tight.

Frames in applications

Example

- Quantum computing: construction of POVMs
- Spherical t-designs
- Classification of hyper-spectral data
- Quantization
- Phase-less reconstruction
- Compressed sensing.

Existence and characterization of FUNTFs

Theorem (Benedetto and Fickus, 2003)

For each $\Phi = \{\varphi_k\}_{k=1}^{M} \subset \mathbb{R}^N$, *such that* $\|\varphi_k\| = 1$ *for each* k, *we have*

$$FP(\Phi) = \sum_{j=1}^{M} \sum_{k=1}^{M} |\langle \varphi_j, \varphi_k \rangle|^2 \geq \frac{M}{N} \max(M, N). \qquad (2)$$

Furthermore,
- *If* $M \leq N$, *the minimum of FP is* M *and is achieved by orthonormal systems for* $\mathbb{R}^N$ *with* M *elements.*
- *If* $M \geq N$, *the minimum of FP is* $\frac{M^2}{N}$ *and is achieved by FUNTFs.*

Proof

Proof.

$$FP(\{\varphi_k\}_{k=1}^M) = M + \sum_{k\neq\ell=1}^{M} |\langle\varphi_k, \varphi_\ell\rangle|^2 \geq M.$$

- If $M \leq N$ the minimizers are exactly orthonormal systems and the minimum is M.
- Now assume $M \geq N$ and let $G = \Phi^*\Phi$. Then,

$$FP(\{\varphi_k\}_{k=1}^M) = Tr(G^2) = \sum_{k=1}^{N} \lambda_k^2$$

and, $trace(G) = \sum_{k=1}^{N} \lambda_k = M$. □

Proof

Proof.

$$FP(\{\varphi_k\}_{k=1}^M) = M + \sum_{k\neq\ell=1}^{M} |\langle\varphi_k, \varphi_\ell\rangle|^2 \geq M.$$

• If $M \leq N$ the minimizers are exactly orthonormal systems and the minimum is M.

• Now assume $M \geq N$ and let $G = \Phi^*\Phi$. Then,

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and, $trace(G) = \sum_{k=1}^N \lambda_k = M$. □

Proof (continued)

Proof.

Minimizing $FP(\{\varphi_k\}_{k=1}^M)$ is equivalent to minimizing

$$\sum_{k=1}^N \lambda_k^2 \quad such\, that \quad \sum_{k=1}^N \lambda_k = M.$$

Solution: $\lambda_k = M/N$ for all k.
Hence $S = \frac{M}{N} I_N$ where I_N is the identity matrix. The corresponding minimizers $\{\varphi_k\}_{k=1}^M$ are FUNTFs

$$x = \frac{N}{M} \sum_{k=1}^M \langle x, \varphi_k \rangle \varphi_k \quad \forall x \in \mathbb{K}^N.$$

Construction of FUNTFs

Fact

- *Numerical schemes such as gradient descent can be used to find minimizers of the frame potential and thus find FUNTFs.*
- *The spectral tetris method was proposed by Casazza, Fickus, Mixon, Wang, and Zhou (2011) to construct all FUNTFs. Further contributions by Krahmer, Kutyniok, Lemvig, (2012); Lemvig, Miller, Okoudjou (2012).*
- *Other methods (algebraic geometry) have been proposed by Cahill, Fickus, Mixon, Strawn.*

Optimally conditioned frames

Remark

1. *FUNTFs can be considered "optimally conditioned" frames. In particular the condition number of the frame operator is* 1.
2. *There are many preconditioning methods to improve the condition number of a matrix, e.g., Matrix Scaling.*
3. *A matrix* A *is (row/column) scalable if there exit diagonal matrices* D_1, D_2 *with positive diagonal entries such that* $D_1 A$, AD_2, *or* $D_1 AD_2$ *have constant row/column sum.*

Main question

Question

Can one transform a (non-tight) frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ into a tight one?

Solution

1. *A solution: The canonical tight frame*

$$\{S^{-1/2}\varphi_k\}_{k=1}^M.$$

2. *What "transformations" are allowed?*

Choosing a transformation

Question

Given a (non-tight) frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ can one find nonnegative numbers $\{c_k\}_{k=1}^M \subset [0, \infty)$ such that $\widetilde{\Phi} = \{c_k\varphi_k\}_{k=1}^M$ becomes a tight frame?

Definition

Definition

A frame $\Phi = \{\varphi_k\}_{k=1}^M$ in $\mathbb{R}^N$ is *scalable*, if $\exists \{c_k\}_{k=1}^M \subset [0, \infty)$ such that $\{c_k \varphi_k\}_{k=1}^M$ is a tight frame for $\mathbb{R}^N$.
The set of scalable frames is denoted by $\mathcal{SC}(M, N)$.
In addition, if $\{c_k\}_{k=1}^M \subset (0, \infty)$, the frame is called *strictly scalable* and the set of strictly scalable frames is denoted by $\mathcal{SC}_+(M, N)$.

A more general definition

Definition

Given, $N \leq m \leq M$, a frame $\Phi = \{\varphi_k\}_{k=1}^{M}$ is said to be m-*scalable*, respectively, *strictly* $m-$*scalable*, if $\exists\, \Phi_I = \{\varphi_k\}_{k\in I}$ with $I \subseteq \{1, 2, \ldots, M\}$, $\#I = m$, such that $\Phi_I = \{\varphi_k\}_{k\in I}$ is scalable, respectively, strictly scalable.
We denote the set of m-scalable frames, respectively, strictly m-scalable frames in $\mathcal{F}(M, N)$ by $\mathcal{SC}(M, N, m)$, respectively, $\mathcal{SC}_+(M, N, m)$.

An observation

Fact

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N \setminus \{0\}$ *be a frame with* $\varphi_k \neq \pm\varphi_\ell$ *for* $k \neq \ell$. Φ *is scalable if and only if* $\widetilde{\Phi} = \{\pm\varphi_k/\|\varphi_k\|\}_{k=1}^M$ *is scalable.*

Elementary properties

Proposition (G. Kutyniok, F. Philipp, K. O. (2014))

Let $M \geq N$, and $m \geq 1$ be integers.

(i) $\Phi \in \mathcal{SC}(M,N)$ *if and only if* $T(\Phi) \in \mathcal{SC}(M,N)$ *for one (and hence for all) orthogonal transformation(s)* T *on* $\mathbb{R}^N$.

(ii) *Let* $\Phi = \{\varphi_k\}_{k=1}^{N+1} \in \mathcal{F}(N+1,N) \setminus \{0\}$ *with* $\varphi_k \neq \pm\varphi_\ell$ *for* $k \neq \ell$. *If* $\Phi \in \mathcal{SC}_+(N+1,N)$, *then* $\Phi \notin \mathcal{SC}_+(N+1,N+1)$.

The scaling problem

$$\Phi = \{\varphi_i\}_{i=1}^M \text{ is } \mathit{scalable} \iff \exists \{c_i\}_{i=1}^M \subset [0, \infty) : \Phi C \Phi^T = I,$$

where $C = \text{diag}(c_i)$.

A reformulation

Fact

Φ *is* $(m$*-*$)$ *scalable* $\iff \exists \{x_k\}_{k\in I} \subset [0,\infty)$ *with* $\#I = m \geq N$ *such that* $\widetilde{\Phi} = \Phi X$ *satisfies*

$$\widetilde{\Phi}\widetilde{\Phi}^T = \Phi X^2 \Phi^T = \tilde{A} I_N = \frac{\sum_{k\in I} x_k^2 \|\varphi_k\|^2}{N} I_N \qquad (3)$$

where $X = \operatorname{diag}(x_k)$.
(3) *is equivalent to solving*

$$\Phi Y \Phi^T = I_N \qquad (4)$$

for $Y = \frac{1}{\tilde{A}} X^2$.

Scalable frame in $\mathbb{R}^2$

Question

When is $\Phi = \{\varphi_k\}_{k=1}^M \subset S^1$ *is a scalable frame in* $\mathbb{R}^2$*?*

Solution

Assume that $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R} \times \mathbb{R}_{+,0}$, $\|\varphi_k\| = 1$, *and* $\varphi_\ell \neq \varphi_k$ *for* $\ell \neq k$. *Let* $0 = \theta_1 < \theta_2 < \theta_3 < \ldots < \theta_M < \pi$, *then*

$$\varphi_k = \begin{pmatrix} \cos\theta_k \\ \sin\theta_k \end{pmatrix} \in S^1.$$

Describing $SC(3, 2)$

Example

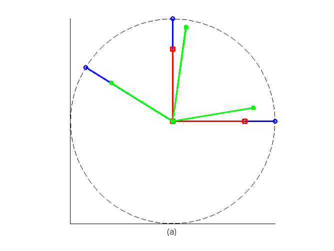


Figure : Blue=original frame; Red=the frames obtained by scaling; Green=associated canonical tight frame.

Describing $SC(3,2)$

Example

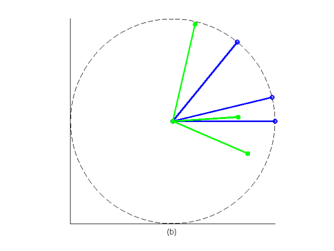


Figure : Blue=original frame; Red=the frames obtained by scaling; Green=associated canonical tight frame.

Describing $SC(3,2)$

Example

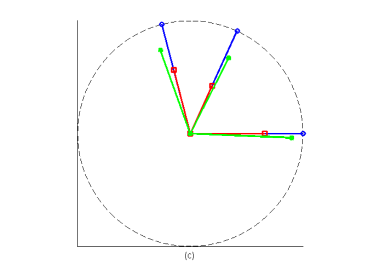


Figure : Blue=original frame; Red=the frames obtained by scaling; Green=associated canonical tight frame.

Describing $SC(4,2)$

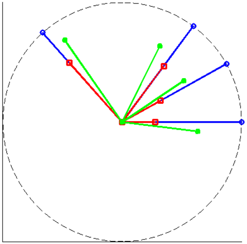


Figure : Blue=original frame; Red=the frames obtained by scaling; Green=associated canonical tight frame.

Describing $SC(4, 2)$

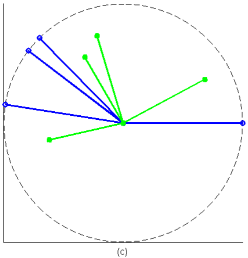


Figure : Blue=original frame; Red=the frames obtained by scaling; Green=associated canonical tight frame.

A more general reformulation

Setting

Let $F : \mathbb{R}^N \to \mathbb{R}^d$, $d := (N-1)(N+2)/2$, *defined by*

$$F(x) = \begin{pmatrix} F_0(x) & F_1(x) & \dots & F_{N-1}(x) \end{pmatrix}^T$$

$$F_0(x) = \begin{pmatrix} x_1^2 - x_2^2 \\ x_1^2 - x_3^2 \\ \vdots \\ x_1^2 - x_N^2 \end{pmatrix}, \dots, F_k(x) = \begin{pmatrix} x_k x_{k+1} \\ x_k x_{k+2} \\ \vdots \\ x_k x_N \end{pmatrix}$$

and $F_0(x) \in \mathbb{R}^{N-1}$, $F_k(x) \in \mathbb{R}^{N-k}$, $k = 1, 2, \dots, N-1$.

Remark

Remark

The map F is related to the diagram vector used by Copenhaver, Kim, Logan, Mayfield, Narayan, Petro, and Sheperd in their characterization of scalable frame

The map F when $N = 2$

Example

When $N = 2$ the map F reduces to

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ xy \end{pmatrix}.$$

Note that in the examples given above we consider

$$\widetilde{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}.$$

When is a frame scalable: A generic solution

Question

When is $\Phi = \{\varphi_k\}_{k=1}^{M} \subset \mathbb{R}^N$ scalable?

Proposition (G. Kutyniok, F. Philipp, K. O. (2014))

A frame Φ for $\mathbb{R}^N$ is m-scalable, respectively, strictly m-scalable, if and only if there exists a nonnegative $u \in \ker F(\Phi) \setminus \{0\}$ with $\|u\|_0 \leq m$, respectively, $\|u\|_0 = m$, and where $F(\Phi)$ is the $d \times M$ matrix whose k^{th} column is $F(\varphi_k)$.

A key tool: The Farkas Lemma

Lemma

For every real $N \times M$-matrix A exactly one of the following cases occurs:

(i) *The system of linear equations $Ax = 0$ has a nontrivial nonnegative solution $x \in \mathbb{R}^M$, i.e., all components of x are nonnegative and at least one of them is strictly positive.*

(ii) *There exists $y \in \mathbb{R}^N$ such that $y^T A$ is a vector with all entries strictly positive.*

Farkas lemma with $N = 2$, $M = 4$

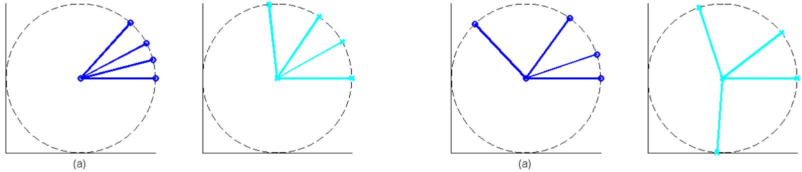


Figure : Bleu=original frame; Green=image by the map F.

Some convex geometry notions

Fact

Let $X = \{x_i\}_{k=1}^M \subset \mathbb{R}^N$.

1. *The polytope generated by* X *is denoted by* P_X.
2. *The relative interior of the polytope* P_X *denoted by* riP_X, *is*

$$riP_X = \left\{ \sum_{k=1}^M \alpha_k x_k : \alpha_k > 0, \sum_{k=1}^M \alpha_k = 1 \right\},$$

Scalable frames and Farkas's lemma

Theorem (G. Kutyniok, F. Philipp, K. O. (2014))

Let $M \geq N \geq 2$, and let m be such that $N \leq m \leq M$. Assume that $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}^*(M,N)$ is such that $\varphi_k \neq \pm\varphi_\ell$ when $k \neq \ell$. Then the following statements are equivalent:

(i) *Φ is $m-$scalable, respectively, strictly $m-$scalable,*

(ii) *There exists a subset $I \subset \{1, 2, \ldots, M\}$ with $\#I = m$ such that $0 \in P_{F(\Phi_I)}$, respectively, $0 \in riP_{F(\Phi_I)}$.*

(iii) *There exists a subset $I \subset \{1, 2, \ldots, M\}$ with $\#I = m$ for which there is no $h \in \mathbb{R}^d$ with $\langle F(\varphi_k), h\rangle > 0$ for all $k \in I$, respectively, with $\langle F(\varphi_k), h\rangle \geq 0$ for all $k \in I$, with at least one of the inequalities being strict.*

A useful property of F

For $x = (x_k)_{k=1}^N \in \mathbb{R}^N$ and $h = (h_k)_{k=1}^d \in \mathbb{R}^d$, we have that

$$\langle F(x), h\rangle = \sum_{\ell=2}^{N} h_{\ell-1}(x_1^2 - x_\ell^2) + \sum_{k=1}^{N-1}\sum_{\ell=k+1}^{N} h_{k(N-1-(k-1)/2)+\ell-1} x_k x_\ell.$$

Remark

$\langle F(x), h\rangle = \langle Q_h x, x\rangle = 0$ *defines a quadratic surface in* $\mathbb{R}^N$.

A geometric characterization of scalable frames

Theorem (G. Kutyniok, F. Philipp, K. Tuley, K.O. (2012))

Let $\Phi = \{\varphi_k\}_{k=1}^{M} \subset \mathbb{R}^N \setminus \{0\}$ be a frame for $\mathbb{R}^N$. Then the following statements are equivalent.

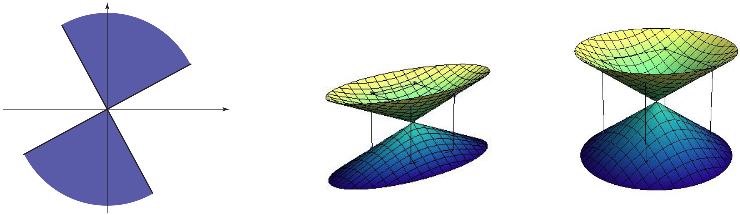
(i) Φ *is not scalable.*

(ii) *There exists a symmetric $M \times M$ matrix Y with $trace(Y) < 0$ such that $\langle \varphi_j, Y\varphi_j \rangle \geq 0$ for all $j = 1, \ldots, M$.*

(iii) *There exists a symmetric $M \times M$ matrix Y with $trace(Y) = 0$ such that $\langle \varphi_j, Y\varphi_j \rangle > 0$ for all $j = 1, \ldots, M$.*

Scalable frames in $\mathbb{R}^2$ and $\mathbb{R}^3$

Figures show sample regions of vectors of a non-scalable frame in $\mathbb{R}^2$ and $\mathbb{R}^3$.



(a) (b) (c)

Figure : (a) shows a sample region of vectors of a non-scalable frame in $\mathbb{R}^2$. (b) and (c) show examples of sets in $\mathcal{C}_3$ which determine sample regions in $\mathbb{R}^3$.

Fritz John's Theorem

Theorem (F. John (1948))

Let $K \subset B = B(0,1)$ be a convex body with nonempty interior. There exits a unique ellipsoid $\mathcal{E}_{min}$ of minimal volume containing K.
Moreover, $\mathcal{E}_{min} = B$ if and only if there exist $\{\lambda_k\}_{k=1}^m \subset [0, \infty)$ and $\{u_k\}_{k=1}^m \subset \partial K \cap S^{N-1}$, $m \geq N+1$ such that

(i) $\sum_{k=1}^m \lambda_k u_k = 0$

(ii) $x = \sum_{k=1}^m \lambda_k \langle x, u_k \rangle u_k, \forall x \in \mathbb{R}^N$.

Frame interpretation of F. John Theorem

Remark

Let $\{u_k\} \subset \partial K \cap S^{N-1}$ be the contact points of K and S^{N-1}. The second part of John's theorem can be written:

$$I_d = \sum_{k=1}^{m} \lambda_k \langle \cdot, u_k \rangle u_k = \sum_{k=1}^{m} \langle \cdot, \sqrt{\lambda_k} u_k \rangle \sqrt{\lambda_k} u_k.$$

F. John's characterization of scalable frames

Theorem (Chen, Kutyniok, Philipp, Wang, K.O. (2014))

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ be a frame. Set $P_\Phi = conv(\{\pm\varphi_k\}_{k=1}^M)$ and V_Φ the volume of P_Φ. Then Φ is scalable if and only if $V_\Phi = 1$. That is, the ellipsoid $\mathcal{E}_\Phi$ of minimal volume containing P_Φ is the euclidean unit ball B.

A quadratic programing approach to optimally conditioning frames

Setting

$\Phi = \{\varphi_i\}_{i=1}^M$ *is scalable* $\iff \Phi C \Phi^T = I.$
Let $C_\Phi = \{\Phi C \Phi^T = \sum_{i=1}^M c_i \varphi_i \varphi_i^T : c_i \geq 0\}$ *be the cone generated by* $\{\varphi_i \varphi_i^T\}_{i=1}^M$.
$\Phi = \{\varphi_i\}_{i=1}^M$ *is scalable* $\iff I \in C_\Phi.$

$$D_\Phi := \min_{C \geq 0 \text{ diagonal}} \left\| \Phi C \Phi^T - I \right\|_F$$

Comparing the measures of scalability

Values of V_Φ and D_Φ for randomly generated frames of M vectors in $\mathbb{R}^4$.

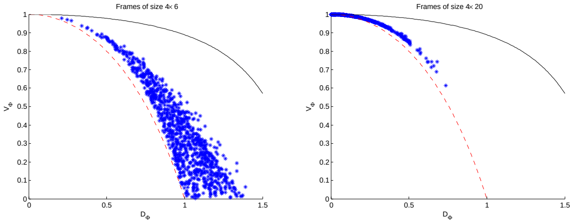


Figure : Relation between V_Φ and D_Φ with $M = 6, 20$. The black line indicates the upper bound in the last theorem, while the red dash line indicates the lower bound.

Wavelets and filter banks

Setting

1. *A function* $\psi \in L^2(\mathbb{R})$ *such that* $\{2^{k/2}\psi(2^k \cdot -\ell) : k, \ell \in \mathbb{Z}\}$ *is an ONB for* L^2 *is called a wavelet.*
2. *Wavelets usually arise from MRA through a scaling function* $\phi \in L^2$*:* $\phi(x) = \sum_\ell c_\ell \phi(2x - \ell)$.

Wavelets and filter banks

Setting

1. *Let $h : \mathbb{Z}^N \to \mathbb{R}$ be a FIR lowpass filter. Its z-transform is $H(z) := \sum_{k\in\mathbb{Z}^N} h(k)z^{-k}$.*
2. *A polyphase representation of h is a Laurent polynomial column vector $H(z) \in \mathcal{M}_q(z)$ such that*

$$H(z) = [H_{\nu_0}(z), H_{\nu_1}(z), \ldots, H_{\nu_{q-1}}(z)]^T,$$

where $H_\nu(z)$ is the z-transform of the filter h_ν defined as $h_\nu(k) = h(2k+\nu)$, $k \in \mathbb{Z}$.

LP^2 matrices

Setting

1. *Let*

$$\Phi_H(z) := \begin{bmatrix} H(z) & I - H(z)H^*(z) \end{bmatrix} \in \mathcal{M}_{q\times(q+1)}(z).$$

2. *We shall refer to the matrix $\Phi_H(z)$ as the LP^2 matrix (of order q) associated with $H(z)$.*

3. $$\Phi_H(z) \begin{bmatrix} H^*(z) \\ I \end{bmatrix} = I. \quad (5)$$

Properties of LP^2 matrices

Remark

1. *The LP^2 matrix $\Phi_H(z)$ is paraunitary, if*

$$\Phi_H(z)\Phi_H^*(z) = \mathrm{I}. \tag{6}$$

2. *The class of paraunitary LP^2 matrices is fundamentally related to the theory of tight filter banks.*
3. *The design of tight filter bank from a paraunitary LP^2 matrix $\Phi_H(z)$ is equivalent to the existence of a column vector $H(z)$ such that $H^*(z)H(z) = 1$.*

Example

Example

Let $\mathsf{H}(z) = [1, (1+z^{-1})/2]^T/\sqrt{2}$. Then $\mathsf{H}^*(z)\mathsf{H}(z) \neq 1$.

Question

Can one find matrices $M(z)$ whose entries are Laurent polynomials such that $\Phi_H(z)M(z)$ is paraunitary, i.e.

$$[\Phi_H(z)M(z)][M^*(z)\Phi_H^*(z)] = I.$$

Scaling LP^2 matrices

Theorem (Y. Hur, K. O. (2014))

Let $\Phi_H(z)$ be an LP^2 matrix associated with $H(z) \in \mathcal{M}_q(z)$. Then we have

$$\Phi_H(z)\mathrm{diag}([2 - H^*(z)H(z), 1, \ldots, 1])\Phi_H^*(z) = I.$$

Reducing the problem

Fact

Let h be a lowpass filter and $H(z) \in \mathcal{M}_q(z)$ be its polyphase representation. Suppose that there exists a Laurent polynomial $m_H(z)$ such that $2 - H^*(z)H(z) = |m_H(z)|^2$. Then

$$\Phi_H(z)\mathrm{diag}([m_H(z), 1, \dots, 1]) = \begin{bmatrix} m_H(z)H(z) & \mathtt{I} - H(z)H^*(z) \end{bmatrix}$$

is paraunitary, i.e. $\Phi_H(z)$ is scalable.

1-d tight wavelet frames

Lemma (Fejér-Riesz Lemma)

Suppose $P(z) = \sum_{k=-r}^{r} p(k) z^{-k} \geq 0$, *for all* $z \in \mathbb{T}$. *Then there exists a* 1*-D Laurent polynomial* $Q(z) = \sum_{k=0}^{r} q(k) z^{-k}$ *such that* $P(z) = |Q(z)|^2, \forall z \in \mathbb{T}$.

1-d tight wavelet frames

Theorem (Y. Hur, K. O. (2014))

Let h be a 1-D lowpass filter with positive accuracy and dilation $\lambda \geq 2$, and let $H(z)$ be its polyphase representation. Suppose $2 - H^*(z)H(z) > 0$, $\forall z \in \mathbb{T}$. Then there is a polynomial $m_H(z)$ such that $[m_H(z)H(z), I - H(z)H^*(z)]$ gives rise to a tight wavelet filter bank whose lowpass filter $\widetilde{h}$ is associated with $m_H(z)H(z)$ and has the same accuracy as h. Furthermore, if the support of h is contained in $\{0, 1, \ldots, s\}$, then the support of $\widetilde{h}$ is contained in $\{0, 1, \ldots, 2s\}$.

Example

Example

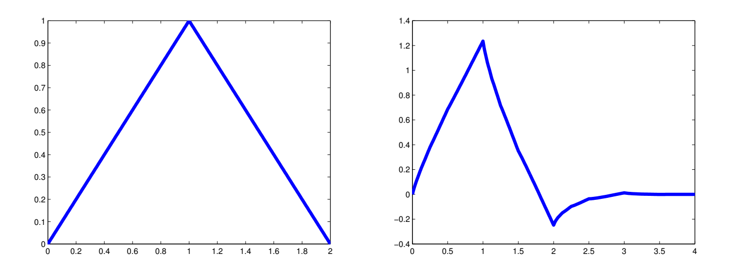


Figure : The original (ϕ, left) and the new ($\widetilde{\phi}$, right) refinable functions.

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Thank You!

http://www2.math.umd.edu/ okoudjou