Preconditioning techniques in frame theory and probabilistic frames

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Outline

Definition

Definition

$\Phi = \{\varphi_k\}_{k=1}^M \subseteq \mathbb{R}^N$ is a *frame* for $\mathbb{R}^N$ if $\exists\, A, B > 0$ such that $\forall x \in \mathbb{R}^N$,

$$A\|x\|^2 \leq \sum_{k=1}^{M} |\langle x, \varphi_k \rangle|^2 \leq B\|x\|^2.$$

If, in addition, $\|\varphi_k\| = 1$ for each k, we say that Φ is a *unit-norm frame*. The set of frames for $\mathbb{R}^N$ with M elements will be denoted by $\mathcal{F}$. In addition, we let $\mathcal{F}_u$ the the subset of unit-norm frames.

Analysis and Synthesis with frame

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$.

1. The *analysis operator*, is defined by

$$\mathbb{R}^N \ni x \mapsto \Phi^T x = \{\langle x, \varphi_k \rangle\}_{k=1}^M \in \mathbb{R}^M.$$

2. The *synthesis operator* is defined by

$$\mathbb{R}^M \ni c = (c_k)_{k=1}^M \mapsto \Phi c = \sum_{k=1}^M c_k \varphi_k \in \mathbb{R}^N.$$

3. The *frame operator* $S = \Phi\Phi^T$ is given by

$$\mathbb{R}^N \ni x \mapsto Sx = \sum_{k=1}^M \langle x, \varphi_k \rangle \varphi_k \in \mathbb{R}^N.$$

4. The *Gramian (operator)* $G = \Phi^T \Phi$ of the frame is the $M \times M$ matrix whose $(i,j)^{th}$ entry is $\langle \varphi_j, \varphi_i \rangle$.

Tight frames and FUNTFs

1. A frame Φ is a *tight frame* if we can choose $A = B$.
2. If $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^M$ is a frame then

$$\{\varphi_k^\dagger\}_{k=1}^M = \{S^{-1/2}\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$$

is a tight frame and for every $x \in \mathbb{R}^N$,

$$x = \sum_{k=1}^M \langle x, \varphi_k^\dagger \rangle \varphi_k^\dagger. \tag{1}$$

3. If Φ is a tight frame of unit-norm vectors, we say that Φ is a *finite unit-norm tight frame (FUNTF)*. In this case, the reconstruction formula (**??**) reduces to

$$\forall x \in \mathbb{R}^N, \quad x = \frac{N}{M} \sum_{k=1}^M \langle x, \varphi_k \rangle \varphi_k. \tag{2}$$

The frame potential

Theorem (Benedetto and Fickus, 2003)

For each $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$, such that $\|\varphi_k\| = 1$ for each k, we have

$$FP(\Phi) = \sum_{j=1}^M \sum_{k=1}^M |\langle \varphi_j, \varphi_k \rangle|^2 \geq \tfrac{M}{N} \max(M, N). \tag{3}$$

Furthermore,
• If $M \leq N$, the minimum of FP is M and is achieved by orthonormal systems for $\mathbb{R}^N$ with M elements.
• If $M \geq N$, the minimum of FP is $\frac{M^2}{N}$ and is achieved by FUNTFs.
$FP(\Phi)$ is the frame potential.

Why frames and FUNTFs

Remark

1. *Geometry of FUNTFs: N. Strawn.*
2. *Constructing all FUNTFs: D. Mixon.*
3. *Applications of FUNTFs and frames: P. Casazza; R. Balan; G. Chen and D. Needell; A. Powell and O. Yilmaz.*

Optimally conditioned frames

Remark

1. *FUNTFs can be considered "optimally conditioned" frames. In particular the condition number of the frame operator is* 1.
2. *There are many preconditioning methods to improve the condition number of a matrix, e.g., Matrix Scaling.*
3. *A matrix* A *is (row/column) scalable if there exit diagonal matrices* D_1, D_2 *with positive diagonal entries such that* D_1A, AD_2*, or* D_1AD_2 *have constant row/column sum.*

Goals of this section

Remark

1. *How to transform a (non) tight frame into a tight one?*
2. *Give theoretical guarantees and algorithms.*
3. *What "transformations" are allowed?*
4. *For a given "transformation", what happens if a frame cannot be transformed exactly?*

In this part of the lecture we will only consider one "transform" and mostly answer the first two questions.

Main question

Question

Given a (non-tight) frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ can one transform Φ into a tight frame? If yes can this be done algorithmically and can the class of all frames that allow such transformations be described?

Solution

1. *If Φ denotes again the $N \times M$ synthesis matrix, a solution to the above problem is the associated canonical tight frame*

$$\{S^{-1/2}\varphi_k\}_{k=1}^M.$$

Involves the inverse frame operator.

2. *What "transformations" are allowed?*

Choosing a transformation

Question

Given a (non-tight) frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ *can one find nonnegative numbers* $\{c_k\}_{k=1}^M \subset [0, \infty)$ *such that* $\widetilde{\Phi} = \{c_k \varphi_k\}_{k=1}^M$ *becomes a tight frame?*

Definition

Definition

A frame $\Phi = \{\varphi_k\}_{k=1}^M$ in $\mathbb{R}^N$ is *scalable*, if $\exists \{c_k\}_{k=1}^M \subset [0, \infty)$ such that $\{c_k\varphi_k\}_{k=1}^M$ is a tight frame for $\mathbb{R}^N$.
The set of scalable frames is denoted by $\mathcal{SC}(M, N)$.
In addition, if $\{c_k\}_{k=1}^M \subset (0, \infty)$, the frame is called *strictly scalable* and the set of strictly scalable frames is denoted by $\mathcal{SC}_+(M, N)$.

A more general definition

Definition

Given, $N \leq m \leq M$, a frame $\Phi = \{\varphi_k\}_{k=1}^M$ is said to be m-*scalable*, respectively, *strictly* $m-$*scalable*, if $\exists\, \Phi_I = \{\varphi_k\}_{k\in I}$ with $I \subseteq \{1, 2, \ldots, M\}$, $\#I = m$, such that $\Phi_I = \{\varphi_k\}_{k\in I}$ is scalable, respectively, strictly scalable.
We denote the set of m-scalable frames, respectively, strictly m-scalable frames in $\mathcal{F}(M, N)$ by $\mathcal{SC}(M, N, m)$, respectively, $\mathcal{SC}_+(M, N, m)$.

Some basic examples

Example

1. When $M = N$, a frame $\Phi = \{\varphi_k\}_{k=1}^{N} \subset \mathbb{R}^N$ is scalable if and only if Φ is an orthogonal set.
2. When $M \geq N$, if Φ contains an orthogonal basis, then it is clearly $N-$scalable.
3. Thus, given $M \geq N$, the set $\mathcal{SC}(M, N, N)$ consists exactly of frames that contains an orthogonal basis for $\mathbb{R}^N$.

Useful remarks

Remark

We note that a frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ with $\varphi_k \neq 0$ for each $k = 1, \ldots, M$ is scalable if and only if $\Phi' = \{\frac{\varphi_k}{\|\varphi_k\|}\}_{k=1}^M$ is scalable.

Useful remarks

Remark

Given a frame $\Phi \subset \mathbb{R}^N$, *assume that* $\Phi = \Phi_1 \cup \Phi_2$ *where*

$$\Phi_1 = \{\varphi_k^{(1)} \in \Phi : \varphi_k^{(1)}(N) \geq 0\}$$

and

$$\Phi_2 = \{\varphi_k^{(2)} \in \Phi : \varphi_k^{(2)}(N) < 0\}.$$

Let

$$\Phi' = \Phi_1 \cup (-\Phi_2).$$

Φ *is scalable if and only if* Φ' *is scalable.*
We shall assume that all the frame vectors are in the upper-half space, i.e., $\Phi \subset \mathbb{R}^{N-1} \times \mathbb{R}_{+,0}$ *where* $\mathbb{R}_{+,0} = [0, \infty)$.

Elementary properties of scalable frames

Proposition

Let $M \geq N$, and $m \geq 1$ be integers.

(i) *If $\Phi \in \mathcal{F}$ is m-scalable then $m \geq N$.*

(ii) *For any integers m, m' such that $N \leq m \leq m' \leq M$ we have that*

$$\mathcal{SC}(M, N, m) \subset \mathcal{SC}(M, N, m'),$$

and

$$\mathcal{SC}(M, N) = \bigcup_{m=N}^{M} \mathcal{SC}(M, N, m).$$

(iii) *$\Phi \in \mathcal{SC}(M, N)$ if and only if $T(\Phi) \in \mathcal{SC}(M, N)$ for one (and hence for all) orthogonal transformation(s) T on $\mathbb{R}^N$.*

(iv) *Let $\Phi = \{\varphi_k\}_{k=1}^{N+1} \in \mathcal{F}(N+1, N) \setminus \{0\}$ with $\varphi_k \neq \pm\varphi_\ell$ for $k \neq \ell$. If $\Phi \in \mathcal{SC}_+(N+1, N)$, then $\Phi \notin \mathcal{SC}_+(N+1, N+1)$.*

Scalable frames: When and How?

Question

1. *When is a frame* $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ *scalable?*
2. *If* $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ *is scalable, how to find the coefficients?*
3. *If* Φ *is not scalable, how close to scalable is it?*
4. *What are the topological properties of* $\mathcal{SC}(M, N)$*?*

A reformulation

Fact

Φ is (m-) scalable $\iff \exists\{x_k\}_{k\in I} \subset [0,\infty)$ with $\#I = m \geq N$ such that $\widetilde{\Phi} = \Phi X$ satisfies

$$\widetilde{\Phi}\widetilde{\Phi}^T = \Phi X^2 \Phi^T = \tilde{A} I_N = \frac{\sum_{k\in I} x_k^2 \|\varphi_k\|^2}{N} I_N \tag{4}$$

where $X = \mathrm{diag}(x_k)$.
(4) *is equivalent to solving*

$$\Phi Y \Phi^T = I_N \tag{5}$$

for $Y = \frac{1}{\tilde{A}} X^2$.

Scalable frame in $\mathbb{R}^2$

Question

When is $\Phi = \{\varphi_k\}_{k=1}^M \subset S^1$ is a scalable frame in $\mathbb{R}^2$?

Solution

Assume that $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R} \times \mathbb{R}_{+,0}$, $\|\varphi_k\| = 1$, and $\varphi_\ell \neq \varphi_k$ for $\ell \neq k$. Let $0 = \theta_1 < \theta_2 < \theta_3 < \ldots < \theta_M < \pi$, then

$$\varphi_k = \begin{pmatrix} \cos\theta_k \\ \sin\theta_k \end{pmatrix} \in S^1.$$

Let $Y = (y_k)_{k=1}^M \subset [0,\infty)$, then (5) *becomes*

$$\begin{pmatrix} \sum_{k=1}^M y_k \cos^2\theta_k & \sum_{k=1}^M y_k \sin\theta_k \cos\theta_k \\ \sum_{k=1}^M y_k \sin\theta_k \cos\theta_k & \sum_{k=1}^M y_k \sin^2\theta_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (6)$$

Scalable frame in $\mathbb{R}^2$

Solution

(6) *is equivalent to*

$$\begin{cases} \sum_{k=1}^M y_k \sin^2\theta_k & = & 1 \\ \sum_{k=1}^M y_k \cos 2\theta_k & = & 0 \\ \sum_{k=1}^M y_k \sin 2\theta_k & = & 0. \end{cases}$$

Consequently, for Φ to be scalable we must find a nonnegative vector $Y = (y_k)_{k=1}^M$ in the kernel of the matrix whose k^{th} column is $\begin{pmatrix} \cos 2\theta_k \\ \sin 2\theta_k \end{pmatrix}$.

Scalable frame in $\mathbb{R}^2$

Solution

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Scalable frame in $\mathbb{R}^2$

Solution

The problem is equivalent to finding non-trivial nonnegative vectors in the nullspace of

$$\begin{pmatrix} 1 & \cos 2\theta_2 & \dots & \cos 2\theta_M \\ 0 & \sin 2\theta_2 & \dots & \sin 2\theta_M \end{pmatrix}. \tag{7}$$

Describing $SC(3,2)$

Example

We first consider the case $M = 3$. In this case, we have $0 = \theta_1 < \theta_2 < \theta_3 < \pi$, and the (7) becomes

$$\begin{pmatrix} 1 & \cos 2\theta_2 & \cos 2\theta_3 \\ 0 & \sin 2\theta_2 & \sin 2\theta_3 \end{pmatrix}. \tag{8}$$

Describing $\mathcal{SC}(3, 2)$

Example

If $\theta_{k_0} = \pi/2$ for $k_0 \in \{2, 3\}$, then the corresponding frame contains an ONB and, hence is scalable.

For example, when $k_0 = 2$, then $0 = \theta_1 < \theta_2 = \pi/2 < \theta_3 < \pi$. In this case, the fame is $2-$ scalable but not $3-$ scalable.

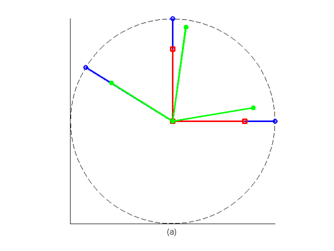

Figure : Blue=original frame; Red=the frames obtained by scaling; Green=associated canonical tight frame.

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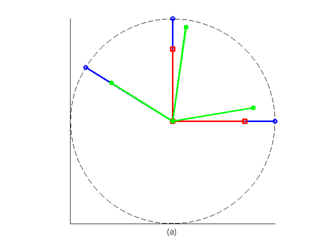

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Describing $SC(3,2)$

Example

Suppose $\theta_k \neq \pi/2$ for $k = 2, 3$. If $\theta_3 < \pi/2$, then the frame cannot be scalable. Indeed, $u = (z_1, z_2, z_3)$ belongs to the kernel of (8) if and only if

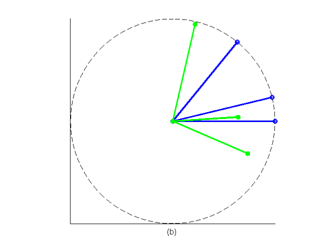
$$\begin{cases} z_1 = \frac{\sin 2(\theta_3 - \theta_2)}{\sin 2\theta_2} z_3, \\ z_2 = -\frac{\sin 2\theta_3}{\sin 2\theta_2} z_3, \end{cases} \tag{9}$$

where $z_3 \in \mathbb{R}$. The choice of the angles implies that $z_2 z_3 < 0$, unless $z_3 = 0$.

Describing $SC(3,2)$

Example

This is illustrated by



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Describing $\mathcal{SC}(3, 2)$

Example

Suppose that $0 = \theta_1 < \theta_2 < \pi/2 < \theta_3 < \pi$. From (9) $z_2 > 0$ for all $z_3 > 0$ and $z_1 > 0$ for all $z_3 > 0$ if and only if $\theta_3 - \theta_2 < \pi/2$. Consequently, when $0 = \theta_1 < \theta_2 < \pi/2 < \theta_3 < \pi$ the frame $\Phi \in \mathcal{SC}_+(3, 2, 3)$ if and only if $0 < \theta_3 - \theta_2 < \pi/2$.

Describing $SC(3,2)$

Example

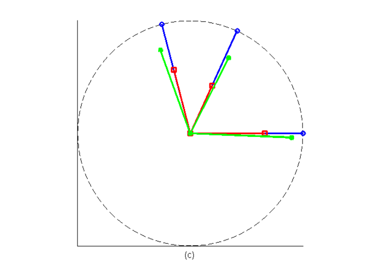


Figure : Blue=original frame; Red=the frames obtained by scaling; Green=associated canonical tight frame.

Describing $SC(4, 2)$

Example

When $M = 4$ we are lead to seek nonnegative non-trivial vectors in the null space of

$$\begin{pmatrix} 1 & \cos 2\theta_2 & \cos 2\theta_3 & \cos 2\theta_4 \\ 0 & \sin 2\theta_2 & \sin 2\theta_3 & \sin 2\theta_4 \end{pmatrix}.$$

Describing $SC(4,2)$

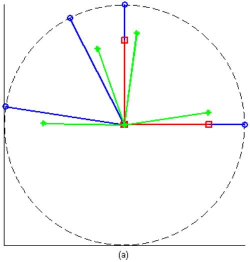


Figure : Blue=original frame; Red=the frames obtained by scaling; Green=associated canonical tight frame.

Describing $SC(4, 2)$

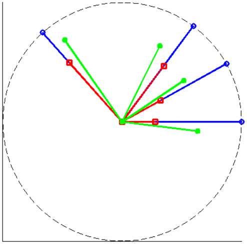


Figure : Blue=original frame; Red=the frames obtained by scaling; Green=associated canonical tight frame.

Describing $SC(4,2)$

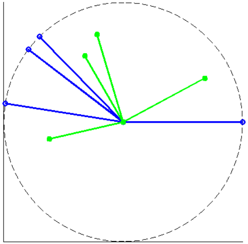


Figure : Blue=original frame; Red=the frames obtained by scaling; Green=associated canonical tight frame.

A more general reformulation

Setting

Let $F : \mathbb{R}^N \to \mathbb{R}^d$, $d := (N-1)(N+2)/2$, *defined by*

$$F(x) = \begin{pmatrix} F_0(x) \\ F_1(x) \\ \vdots \\ F_{N-1}(x) \end{pmatrix}$$

$$F_0(x) = \begin{pmatrix} x_1^2 - x_2^2 \\ x_1^2 - x_3^2 \\ \vdots \\ x_1^2 - x_N^2 \end{pmatrix}, \ldots, F_k(x) = \begin{pmatrix} x_k x_{k+1} \\ x_k x_{k+2} \\ \vdots \\ x_k x_N \end{pmatrix}$$

and $F_0(x) \in \mathbb{R}^{N-1}$, $F_k(x) \in \mathbb{R}^{N-k}$, $k = 1, 2, \ldots, N-1$.

The map F when $N = 2$

Example

When $N = 2$ the map F reduces to

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ xy \end{pmatrix}.$$

Note that in the examples given above we consider

$$\widetilde{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}.$$

When is a frame scalable: A generic solution

Question

When is $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ *scalable?*

Proposition

A frame Φ *for* $\mathbb{R}^N$ *is* m*-scalable, respectively, strictly* m*-scalable, if and only if there exists a nonnegative* $u \in \ker F(\Phi) \setminus \{0\}$ *with* $\|u\|_0 \leq m$*, respectively,* $\|u\|_0 = m$*, and where* $F(\Phi)$ *is the* $d \times M$ *matrix whose* k^{th} *column is* $F(\varphi_k)$.

A key tool: The Farkas Lemma

Lemma

For every real $N \times M$-matrix A exactly one of the following cases occurs:

(i) *The system of linear equations $Ax = 0$ has a nontrivial nonnegative solution $x \in \mathbb{R}^M$, i.e., all components of x are nonnegative and at least one of them is strictly positive.*

(ii) *There exists $y \in \mathbb{R}^N$ such that $y^T A$ is a vector with all entries strictly positive.*

Farkas lemma with $N = 2$, $M = 4$

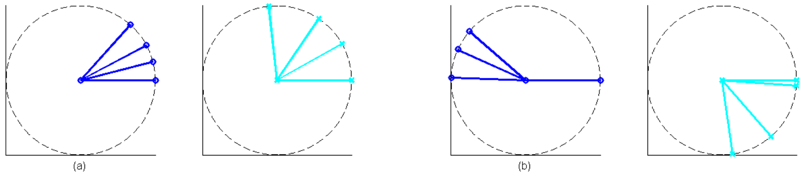


Figure : Bleu=original frame; Green=image by the map F. Both of these examples result in non scalable frames.

Farkas lemma with $N = 2$, $M = 4$

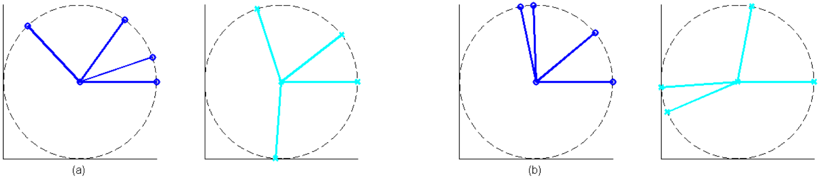


Figure : Bleu=original frame; Green=image by the map F. Both of these examples result in scalable frames.

Some convex geometry notions

Fact

Let $X = \{x_i\}_{k=1}^M \subset \mathbb{R}^N$.

1. *The polytope generated by* X *is the convex hull of* X, *denoted by* P_X *(or* $\mathrm{co}(X)$*).*
2. *The affine hull generated by* X *is denoted by* $\mathrm{aff}(X)$.
3. *The relative interior of the polytope* $\mathrm{co}(X)$ *denoted by* $ri\,\mathrm{co}(X)$, *is the interior of* $\mathrm{co}(X)$ *in the topology induced by* $\mathrm{aff}(X)$.
4. *It is true that* $ri\,\mathrm{co}(X) \neq \emptyset$ *whenever* $\#X \geq 2$, *and*

$$ri\,\mathrm{co}(X) = \left\{\sum_{k=1}^M \alpha_k x_k : \alpha_k > 0, \sum_{k=1}^M \alpha_k = 1\right\},$$

Scalable frames and Farkas's lemma

Theorem

Let $M \geq N \geq 2$, and let m be such that $N \leq m \leq M$. Assume that $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}^*(M, N)$ is such that $\varphi_k \neq \pm\varphi_\ell$ when $k \neq \ell$. Then the following statements are equivalent:

(i) *Φ is $m-$scalable, respectively, strictly $m-$scalable,*

(ii) *There exists a subset $I \subset \{1, 2, \ldots, M\}$ with $\#I = m$ such that $0 \in \text{co}(F(\Phi_I))$, respectively, $0 \in ri\,\text{co}(F(\Phi_I))$.*

(iii) *There exists a subset $I \subset \{1, 2, \ldots, M\}$ with $\#I = m$ for which there is no $h \in \mathbb{R}^d$ with $\langle F(\varphi_k), h\rangle > 0$ for all $k \in I$, respectively, with $\langle F(\varphi_k), h\rangle \geq 0$ for all $k \in I$, with at least one of the inequalities being strict.*

A useful property of F

For $x = (x_k)_{k=1}^N \in \mathbb{R}^N$ and $h = (h_k)_{k=1}^d \in \mathbb{R}^d$, we have that

$$\langle F(x), h \rangle = \sum_{\ell=2}^{N} h_{\ell-1}(x_1^2 - x_\ell^2) + \sum_{k=1}^{N-1} \sum_{\ell=k+1}^{N} h_{k(N-1-(k-1)/2)+\ell-1} x_k x_\ell. \tag{10}$$

Consequently, fixing $h \in \mathbb{R}^d$, $\langle F(x), h \rangle$ is a homogeneous polynomial of degree 2 in $x_1, x_2, \ldots, x_N$. The set of all polynomials of this form can be identified with the subspace of real symmetric $N \times N$ matrices whose trace is 0.

A useful property of F

Remark

$\langle F(x), h\rangle = \langle Q_h x, x\rangle = 0$ *defines a quadratic surface in* $\mathbb{R}^N$*, and condition (iii) in the last Theorem stipulates that for* Φ *to be scalable, one cannot find such a quadratic surface such that the frame vectors (with index in* I*) all lie on (only) "one side" of this surface.*

A geometric characterization of scalable frames

Theorem (G. Kutyniok, F. Philipp, K. Tuley, K.O. (2012))

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N \setminus \{0\}$ *be a frame for* $\mathbb{R}^N$. *Then the following statements are equivalent.*

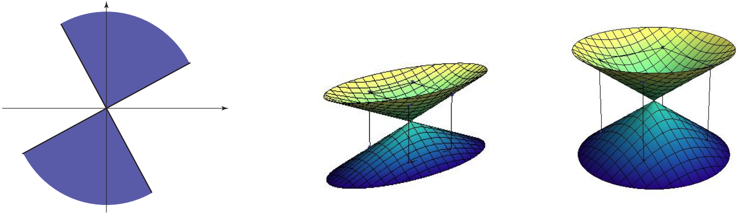
(i) Φ *is not scalable.*

(ii) *There exists a symmetric* $M \times M$ *matrix* Y *with* $trace(Y) < 0$ *such that* $\langle \varphi_j, Y\varphi_j \rangle \geq 0$ *for all* $j = 1, \ldots, M$.

(iii) *There exists a symmetric* $M \times M$ *matrix* Y *with* $trace(Y) = 0$ *such that* $\langle \varphi_j, Y\varphi_j \rangle > 0$ *for all* $j = 1, \ldots, M$.

Scalable frames in $\mathbb{R}^2$ and $\mathbb{R}^3$

Figures show sample regions of vectors of a non-scalable frame in $\mathbb{R}^2$ and $\mathbb{R}^3$.



(a) (b) (c)

Figure : (a) shows a sample region of vectors of a non-scalable frame in $\mathbb{R}^2$. (b) and (c) show examples of sets in $\mathcal{C}_3$ which determine sample regions in $\mathbb{R}^3$.

Fritz John's Theorem

Theorem (F. John (1948))

Let $K \subset B = B(0,1)$ *be a convex body with nonempty interior. There exits a unique ellipsoid* $\mathcal{E}_{min}$ *of minimal volume containing* K*.*
Moreover, $\mathcal{E}_{min} = B$ *if and only if there exist* $\{\lambda_k\}_{k=1}^m \subset (0, \infty)$ *and* $\{u_k\}_{k=1}^m \subset \partial K \cap S^{N-1}$, $m \geq N+1$ *such that*

(i) $\sum_{k=1}^m \lambda_k u_k = 0$

(ii) $x = \sum_{k=1}^m \lambda_k \langle x, u_k \rangle u_k, \forall x \in \mathbb{R}^N$

where ∂K *is the boundary of* K *and* S^{N-1} *is the unit sphere in* $\mathbb{R}^N$*.*

F. John's characterization of scalable frames

Setting

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ be a frame for $\mathbb{R}^N$. We apply F. John's theorem to the convex body $K = P_\Phi = conv(\{\pm\varphi_k\}_{k=1}^M)$. Let $\mathcal{E}_\Phi$ denote the ellipsoid of minimal volume containing P_Φ, and $V_\Phi =$ Vol$(\mathcal{E}_\Phi)/\omega_N$ where ω_N is the volume of the euclidean unit ball.

Theorem

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ be a frame. Then Φ is scalable if and only if $V_\Phi = 1$. In this case, the ellipsoid $\mathcal{E}_\Phi$ of minimal volume containing $P_\Phi = conv(\{\pm\varphi_k\}_{k=1}^M)$ is the euclidean unit ball B.

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A measure of scalability

Remark

Let $\Phi \subset S^{N-1}$ be a frame. Then V_Φ is a "measure of scalability": the closer it is to 1 the more scalable is the frame.

A quadratic programing approach to scalability

Setting

$$\Phi = \{\varphi_i\}_{i=1}^M \text{ is scalable } \iff \exists \{c_i\}_{i=1}^M \subset [0,\infty) : \Phi C \Phi^T = I,$$

where $C = \mathit{diag}(c_i)$.

$$C_\Phi = \{\Phi C \Phi^T = \sum_{i=1}^M c_i \varphi_i \varphi_i^T : c_i \geq 0\}$$

is the (closed) cone generated by $\{\varphi_i \varphi_i^T\}_{i=1}^M$.

$$\Phi = \{\varphi_i\}_{i=1}^M \text{ is scalable } \iff I \in C_\Phi.$$

$$D_\Phi := \min_{C \geq 0 \text{ diagonal}} \left\| \Phi C \Phi^T - I \right\|_F$$

A second measure of scalability

Remark

Let $\Phi \subset S^{N-1}$ be a frame. Then D_Φ is a "measure of scalability": the closer it is to 0 the more scalable is the frame.

Comparing the measures of scalability

Values of V_Φ and D_Φ for randomly generated frames of M vectors in $\mathbb{R}^4$.

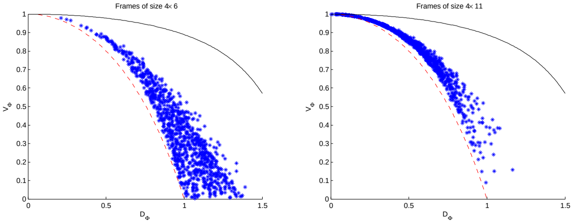


Figure : Relation between V_Φ and D_Φ with $M = 6, 11$. The black line indicates the upper bound in the last theorem, while the red dash line indicates the lower bound.

Comparing the measures of scalability

Values of V_Φ and D_Φ for randomly generated frames of M vectors in $\mathbb{R}^4$.

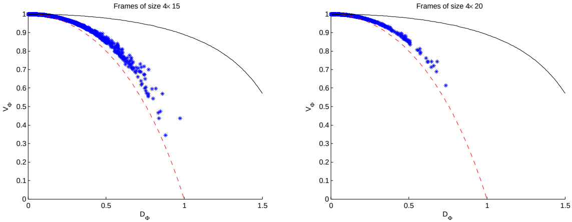


Figure : Relation between V_Φ and D_Φ with $M = 15, 20$. The black line indicates the upper bound in the last theorem, while the red dash line indicates the lower bound.

Concluding remarks on scalable frames

1. The problem can be reformulated as a linear programing one leading to numerical solutions.
2. When frame not scalable, one can define how close or far to being scalable it is: Notion of "almost scalable."
3. Role of redundancy.
4. Size of $\mathcal{SC}(M, N)$.
5. Other methods of frame preconditioning

Goals of this section

Remark

1. *Standard tools used in frame theory include: Functional and Harmonic Analysis, Operator Theory, Linear Algebra, Differential Geometry, Differential Equations.*
2. *Identifying frames with probability measures leads analyzing frames in the setting of the Wasserstein metric spaces.*
3. *For example, gradient flow methods from optimal transport theory can be used to minimize certain common potentials in frame theory.*

Motivation: The Welch bound

Theorem

For any frame $\Phi = \{\varphi_k\}_{k=1}^{M} \subset S^{N-1}$ we have

$$\max_{k\neq\ell} |\langle \varphi_k, \varphi_\ell \rangle| \geq \sqrt{\frac{M-N}{N(M-1)}}, \tag{11}$$

and equality hold if and only if Φ is an ETF.
Furthermore, equality can hold only when $M \leq \frac{N(N+1)}{2}$.

Definition of the p^{th} frame potential

Definition

Let M be a positive integer, and $0 < p < \infty$. Given a collection of unit vectors $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$, the p-*frame potential* is the functional

$$\mathrm{FP}_{p,M}(\Phi) = \sum_{k,\ell=1}^{M} |\langle \varphi_k, \varphi_\ell \rangle|^p. \tag{12}$$

When, $p = \infty$, the definition reduces to

$$\mathrm{FP}_{\infty,M}(\Phi) = \max_{k \neq \ell} |\langle \varphi_k, \varphi_\ell \rangle|.$$

Special cases

1. $p = 2$ corresponds to the frame potential whose minimizers are the FUNTFs
2. For $p = \infty$ and fixed M, the minimizers of $\mathrm{FP}_{\infty,M}$ are called Grassmanian frames.
3. The potential $\mathrm{FP}_{\infty,M}$ always has a minimum but constructing these minimizers is challenging.

Question

What are the minimizers of $\mathrm{FP}_{p,M}$*?*

Example: $M = 3, N = 2$

Question

Find the minimizers of

$$\mathrm{FP}_{p,3}(\Phi) = \sum_{k,\ell=1}^{3} |\langle \varphi_k, \varphi_\ell \rangle|^p$$

when $p \in (0, \infty]$ *and* $\Phi = \{\varphi_k\}_{k=1}^{3} \subset S^1$.

Solution for $p = 2$ and $p = \infty$

Solution

1. *When* $p = 2$,

$$\mathrm{FP}_{2,3}(\Phi) = \sum_{k,\ell=1}^{3} |\langle \varphi_k, \varphi_\ell \rangle|^2 \geq 9/2$$

with equality if and only if $\Phi = \{\varphi_k\}_{k=1}^{3} \subset S^1$ *is a FUNTF. A minimizer of* $\mathrm{FP}_{2,3}$ *is the MB-frame, see next slide.*

2. *When* $p = \infty$,

$$\mathrm{FP}_{\infty,3}(\Phi) = \max_{k \neq \ell} |\langle \varphi_k, \varphi_\ell \rangle| \geq 1/\sqrt{2}$$

with equality if and only if $\Phi = \{\varphi_k\}_{k=1}^{3} \subset S^1$ *is an ETF. Hence a solution is also given by the MB frame*

the MB-frame

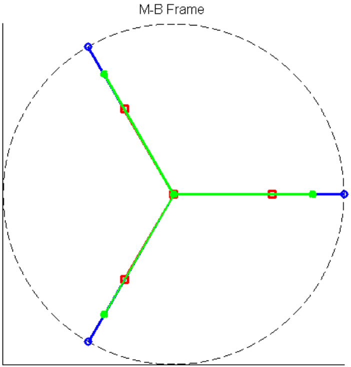


Figure : An example of Equiangular FUNTF: the MB-frame.

Minimizers of $\mathrm{FP}_{p,3}$ for $p \in (0, \infty]$

Proposition

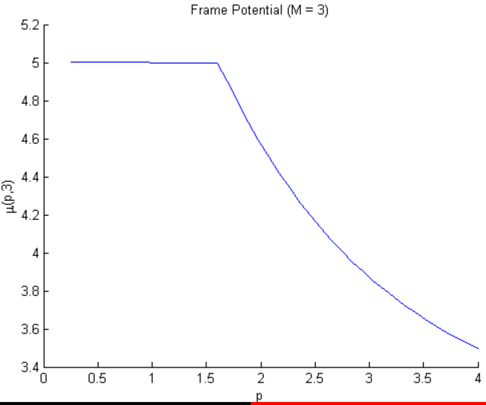
Let $p_0 = \frac{\log(3)}{\log(2)}$*. Then* $\mathrm{FP}_{p_0,3}(\Phi) \geq 5$*, with equality holding if and only if* $\Phi = \{\varphi_k\}_{k=1}^3$ *is an orthonormal basis plus one repeated vector or an ETF. Furthermore,*

1. *for* $0 < p < p_0$*, and* $\Phi = \{\varphi_k\}_{k=1}^3 \subset S^1$*, we have* $\mathrm{FP}_{p,3}(\Phi) \geq 5$*, and equality holds if and only if* $\Phi = \{\varphi_k\}_{k=1}^3$ *is an orthonormal basis plus one repeated vector,*
2. *for* $p > p_0$*, and* $\Phi = \{\varphi_k\}_{k=1}^3 \subset S^1$*, we have* $\mathrm{FP}_{p,3}(\Phi) \geq 2^{\frac{p}{p_0}} (6)^{1-\frac{p}{p_0}} + 3$*, and equality holds if and only if* $\Phi = \{\varphi_k\}_{k=1}^3$ *is an ETF.*

Minimizers of $\mathrm{FP}_{p,3}$ for $p \in (0, \infty)$

Remark

$$\mu_{p,3,2} = \min\{\mathrm{FP}_{p,2}(\Phi) : \Phi = \{\varphi_k\}_{k=1}^{3} \subset S^1\}$$



Minimizers of $\mathrm{FP}_{p,N+1}$ for $p \in (0, \infty)$

Theorem

Let $p \in (0, \infty]$, and N be positive integer. Let $\Phi = \{\varphi_k\}_{k=1}^{N+1} \subset S^{N-1}$. Set $p_0 = \frac{\log(\frac{N(N+1)}{2})}{\log(N)}$. Assume that $\mathrm{FP}_{p_0,N+1}(\Phi) \geq N + 3$, with equality holding if and only if $\Phi = \{\varphi_k\}_{k=1}^{N+1}$ is an orthonormal basis plus one repeated vector or an ETF. Then,

(1) *for $0 < p < p_0$, and $\Phi = \{\varphi_k\}_{k=1}^{N+1} \subset S^{N-1}$, we have $\mathrm{FP}_{p,N+1}(\Phi) \geq N + 3$, and equality holds if and only if $\Phi = \{\varphi_k\}_{k=1}^{N+1}$ is an orthonormal basis plus one repeated vector,*

(2) *for $p_0 < p < 2$, and $\Phi = \{\varphi_k\}_{k=1}^{N+1} \subset S^{N-1}$, we have $\mathrm{FP}_{p,N+1}(\Phi) \geq 2^{\frac{p}{p_0}} (N(N+1))^{1-\frac{p}{p_0}} + N + 1$, and equality holds if and only if $\Phi = \{\varphi_k\}_{k=1}^{N+1}$ is an ETF.*

Remarks on the Theorem

Remark

1. *The hypothesis of the last theorem can be verified when $N = 2$. But for $N \geq 3$ it is not known if this hypothesis is true.*
2. *There seems to be some "universality" of the minimizers of these potentials. With p_0 given above, any orthonormal basis plus one repeated vector minimizes $\mathrm{FP}_{p,N+1}$ for $0 < p \leq p_0$ and any ETF minimizes $\mathrm{FP}_{p,N+1}$ for $p_0 \leq p \leq \infty$.*

Partial results on minimizing $\mathrm{FP}_{p,M}$ for $p \in (0,\infty)$

Proposition

Let $p \in (0,\infty]$, M, N be positive integers. Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ we have:

(a) *If $M \geq N$ and $2 < p < \infty$, then*

$$\mathrm{FP}_{p,M}(\Phi) \geq M(M-1)\big(\tfrac{M-N}{N(M-1)}\big)^{p/2} + N,$$

and equality holds if and only if Φ is an ETF.

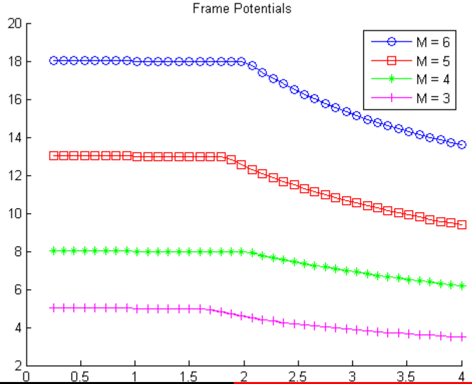
(b) *Let $0 < p < 2$ and assume that $M = kN$ for some positive integer k. Then the minimizers of the p-frame potential are exactly the k copies of any orthonormal basis modulo multiplications by ± 1. The minimum of (12) over all sets of $M = kN$ unit norm vectors is $k^2 N$.*

Numerical simulations for $N = 2$

Remark

We let

$$\mu_{p,M,2} = \min\{\mathrm{FP}_{p,2}(\Phi) : \Phi = \{\varphi_k\}_{k=1}^{M} \subset S^1\}$$



The p^{th} frame potential and t-design

Definition

Let t be a positive integer. A *spherical t-design* is a finite subset $\{x_i\}_{i=1}^M$ of the unit sphere S^{N-1} in $\mathbb{R}^N$, such that,

$$\frac{1}{M}\sum_{i=1}^{M} h(x_i) = \int_{S^{N-1}} h(x)d\sigma(x),$$

for all homogeneous polynomials h of total degree equals or less than t in N variables and where σ denotes the uniform surface measure on S^{N-1} normalized to have mass one.

FUNTFS and 2-design

Proposition

$\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ is a spherical 2-design if and only if Φ is a FUNTF and $\sum_{k=1}^M \varphi_k = 0$.

t-designs as minimizers of p^{th} frame potentials

Theorem

Let $p = 2k$ be an even integer and $\{x_i\}_{i=1}^M = \{-x_i\}_{i=1}^M \subset S^{N-1}$, then

$$\mathrm{FP}_{p,M}(\{x_i\}_{i=1}^M) \geq \frac{1 \cdot 3 \cdot 5 \cdots (p-1)}{N(N+2)\cdots(N+p-2)} M^2,$$

and equality holds if and only if $\{x_i\}_{i=1}^M$ is a spherical p-design.

Motivations

- Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a frame in $\mathbb{R}^N$ with bounds $0 < A \leq B < \infty$. Define

$$\mu_\Phi := \frac{1}{M}\sum_{i=1}^{M}\delta_{\varphi_i} \quad then \quad \int_{\mathbb{R}^N}|\langle x, y\rangle|^2 d\mu_\Phi(y) = \frac{1}{M}\sum_{k=1}^{M}|\langle x, \varphi_k\rangle|^2.$$

- For each $x \in \mathbb{R}^N$: $A/M\|x\|^2 \leq \int_{\mathbb{R}^N}|\langle x, y\rangle|^2 d\mu_\Phi(y) \leq B/M\|x\|^2$
- μ_Φ is an example of probabilistic frames.
- $\mathcal{P}$ is the set of probability measures on $\mathbb{R}^N$, and

$$\mathcal{P}_2 = \left\{\mu \in \mathcal{P} : M_2^2(\mu) = \int_{\mathbb{R}^N}\|y\|^2 d\mu(y) < \infty\right\}$$

Definition

Definition

A Borel probability measure $\mu \in \mathcal{P}$ is a *probabilistic frame* if there exist $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \int_{\mathbb{R}^N} |\langle x, y\rangle|^2 d\mu(y) \leq B\|x\|^2, \quad \text{for all } x \in \mathbb{R}^N. \tag{13}$$

When $A = B$, μ is called a *tight probabilistic frame*.

When is a probability measure a probabilistic frame?

Theorem

A Borel probability measure $\mu \in \mathcal{P}$ is a probabilistic frame if and only if $\mu \in \mathcal{P}_2$ and $E_\mu = \mathbb{R}^N$, where E_μ denotes the linear span of $\operatorname{supp}(\mu)$ in $\mathbb{R}^N$. Moreover, if μ is a tight probabilistic frame, then the frame bound is given by

$$A = \frac{1}{N} M_2^2(\mu) = \frac{1}{N} \int_{\mathbb{R}^N} \|y\|^2 d\mu(y).$$

Examples

Example

(a) Let $a = \{a_k\}_{k=1}^M \subset (0, \infty)$ with $\sum_{k=1}^M a_k = 1$. A set $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ is a frame if and only if the probability measure $\mu_{\Phi,a} = \sum_{k=1}^M a_k \delta_{\varphi_k}$ supported by the set Φ is a probabilistic frame.

(c) The uniform distribution on the unit sphere S^{N-1} in $\mathbb{R}^N$ is a tight probabilistic frame. That is, denoting the probability measure on S^{N-1} by $d\sigma$ we have that for all $x \in \mathbb{R}^N$,

$$\frac{\|x\|^2}{N} = \int_{\mathbb{R}^N} \langle x, y \rangle^2 d\sigma(y).$$

Probabilistic frame operator

Let $\mu \in \mathcal{P}$ be a probability measure.

1. The *probabilistic analysis operator* is given by

$$T_\mu : \mathbb{R}^N \to L^2(\mathbb{R}^N, \mu), \quad x \mapsto \langle x, \cdot \rangle.$$

2. The *probabilistic synthesis operator* is defined by

$$T_\mu^* : L^2(\mathbb{N}^d, \mu) \to \mathbb{R}^N, \quad f \mapsto \int_{\mathbb{R}^N} f(x) x d\mu(x).$$

3. The *probabilistic frame operator* of μ is

$$S_\mu = T_\mu^* T_\mu.$$

4. The *probabilistic Gram operator* of μ, is defined on $L^2(\mathbb{R}^N, \mu)$ by

$$G_\mu f(x) = T_\mu T_\mu^* f(x) = \int_{\mathbb{R}^N} \langle x, y \rangle f(y) d\mu(y).$$

Probabilistic frame operator

Fact

The probabilistic frame operator is given by

$$S_\mu : \mathbb{R}^d \to \mathbb{R}^d, \qquad S_\mu(x) = \int_{\mathbb{R}^N} \langle x, y \rangle y d\mu(y)$$

and is the matrix of second moments of μ:
If $\{e_j\}_{j=1}^N$ is the canonical orthonormal basis for $\mathbb{R}^N$, then

$$S_\mu e_i = \sum_{j=1}^N m_{i,j}(\mu) e_j,$$

where

$$m_{i,j}(\mu) = \int_{\mathbb{R}^N} y^{(i)} y^{(j)} d\mu(y).$$

Probabilistic frame operator

Proposition

Let $\mu \in \mathcal{P}$, then S_μ is well-defined (and hence bounded) if and only if

$$M_2(\mu) < \infty.$$

Furthermore, μ is a probabilistic frame if and only if S_μ is positive definite.

Duality

If μ is a probabilistic frame then S_μ is positive definite.

1. The *push-forward* of μ through S_μ^{-1} is given by

$$\tilde{\mu}(B) = \mu((S_\mu^{-1})^{-1}B) = \mu(S_\mu B).$$

2. $\tilde{\mu}$ is a probabilistic frame called the *probabilistic canonical dual frame* of μ.
3. The push-forward of μ through $S_\mu^{-1/2}$ is given by

$$\mu^\dagger(B) = \mu(S^{1/2}B).$$

Reconstruction formula

Proposition

Let $\mu \in \mathcal{P}$ be a probabilistic frame with bounds $0 < A \leq B < \infty$. Then:

(a) *$\tilde{\mu}$ is a probabilistic frame with frame bounds $1/B \leq 1/A$.*

(b) *$\mu^{\dagger}$ is a tight probabilistic frame.*

Consequently, for each $x \in \mathbb{R}^N$ we have:

$$\int_{\mathbb{R}^N} \langle x, y \rangle \, S_\mu y \, d\tilde{\mu}(y) = \int_{\mathbb{R}^N} \langle S_\mu^{-1} x, y \rangle \, y \, d\mu(y) = x, \tag{14}$$

and

$$\int_{\mathbb{R}^N} \langle x, y \rangle \, y \, d\mu^{\dagger}(y) = \int_{\mathbb{R}^N} \langle S_\mu^{-1/2} x, y \rangle \, S_\mu^{-1/2} y \, d\mu(y) = x. \tag{15}$$

Definition

Question

When is a probability measure μ a tight probabilistic frame?

Definition

The *probabilistic frame potential* is the nonnegative function defined on $\mathcal{P}$ and given by

$$\mathrm{PFP}(\mu) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} |\langle x, y \rangle|^2 \, d\mu(x) \, d\mu(y), \tag{16}$$

for each $\mu \in \mathcal{P}$.

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for each $\mu \in \mathcal{P}$.

The probabilistic frame potential and Gramian operator

Proposition

Let $\mu \in \mathcal{P}$, then $\mathrm{PFP}(\mu)$ is the Hilbert-Schmidt norm of the probabilistic Gramian operator G_μ, that is

$$\|G_\mu\|_{HS}^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \langle x, y\rangle^2 d\mu(x) d\mu(y).$$

Furthermore, if $\mu \in \mathcal{P}_2$, (which is the case when μ is a probabilistic frame) then we have

$$\mathrm{PFP}(\mu) \leq M_2^4(\mu) < \infty.$$

Probabilistic tight frames as minimizers of the PFP

Theorem

Let $\mu \in \mathcal{P}_2$ be such that $M_2(\mu) = 1$ and set $E_\mu = span(\mathrm{supp}(\mu))$, then the following estimate holds

$$\mathrm{PFP}(\mu) \geq 1/n \tag{17}$$

where n is the number of nonzero eigenvalues of S_μ. Moreover, equality holds if and only if μ is a tight probabilistic frame for E_μ.
In particular, given any probabilistic frame $\mu \in \mathcal{P}_2$ with $M_2(\mu) = 1$, we have

$$\mathrm{PFP}(\mu) \geq 1/N$$

and equality holds if and only if μ is a tight probabilistic frame.

Remark

When μ is a discrete measure, then $\mathrm{PFP}(\mu)$ is the frame potential.

Definition

For $p \in (0, \infty)$ set

$$\mathcal{P}_p = \{\mu \in \mathcal{P} : M_p^p(\mu) = \int_{\mathbb{R}^N} \|y\|^p d\mu(y) < \infty\}.$$

Definition

For each $p \in (0, \infty)$, the *probabilistic* $p-$*frame potential* is given by

$$\text{PFP}(\mu, p) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} |\langle x, y \rangle|^p \, d\mu(x) \, d\mu(y). \tag{18}$$

When $\text{supp}(\mu) = \Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$, $\text{PFP}(\mu, p)$ reduces to $\text{FP}_{p,M}$.

Minimizers of the probabilistic p^{th} frame potential

Theorem

Let $0 < p < 2$, then the minimizers of (18) *over all the probability measures supported on the unit sphere S^{N-1} are exactly those probability measures μ that satisfy*

(i) *there is an orthonormal basis $\{e_1, \ldots, e_N\}$ for $\mathbb{R}^N$ such that*

$$\{e_1, \ldots, e_N\} \subset \text{supp}(\mu) \subset \{\pm e_1, \ldots, \pm e_N\}$$

(ii) *there is $f : S^{N-1} \to \mathbb{R}$ such that $\mu(x) = f(x)\nu_{\pm x_1, \ldots, \pm x_N}(x)$ and*

$$f(x_i) + f(-x_i) = \frac{1}{N},$$

where the measure $\nu_{\pm x_1, \ldots, \pm x_N}(x)$ represent the counting measure of the set $\{\pm x_i : i = 1, \ldots, N\}$.

Probabilistic $p-$frame

Definition

For $0 < p < \infty$, we call $\mu \in \mathcal{M}(S^{N-1}, \mathcal{B})$ a *probabilistic p-frame* for $\mathbb{R}^N$ if and only if there are constants $A, B > 0$ such that

$$A\|y\|^p \leq \int_{S^{N-1}} |\langle x, y\rangle|^p d\mu(x) \leq B\|y\|^p, \quad \forall y \in \mathbb{R}^N. \tag{19}$$

We call μ a *tight probabilistic p-frame* if and only if we can choose $A = B$.

Examples

Example

By symmetry considerations, it is not difficult to show that the uniform surface measure σ on S^{N-1} is always a tight probabilistic p-frame, for each $0 < p < \infty$.

Lemma

If μ is probabilistic frame, then it is a probabilistic p-frame for all $1 \leq p < \infty$. Conversely, if μ is a probabilistic p-frame for some $1 \leq p < \infty$, then it is a probabilistic frame.

Tight probabilistic p-frames and spherical $t-$designs

Theorem

Let p be an even integer. For any probability measure μ on S^{N-1},

$$\mathrm{PFP}(\mu, p) \geq \frac{1 \cdot 3 \cdot 5 \cdots (p-1)}{N(N+2)\cdots(N+p-2)},$$

and equality holds if and only if μ is a probabilistic tight p-frame.

Tight probabilistic p-frames and spherical $t-$designs

Proposition

Let $p = 2k$ be an even positive integer. A set $\Phi = \{\varphi_k\}_{k=1}^{M} \subset S^{N-1}$ is a spherical $p-$design if and only if the probability measure $\mu_\Phi = \frac{1}{M}\sum_{k=1}^{N} \delta_{\varphi_k}$ is a probabilistic tight $p-$frame.

Concluding remarks on probabilistic frames

- The 2-Wasserstein metric given by

$$W_2^2(\mu,\nu) := \min\left\{\int_{\mathbb{R}^N\times\mathbb{R}^N} \|x-y\|^2 d\gamma(x,y), \gamma\in\Gamma(\mu,\nu)\right\}, \quad (20)$$

  where $\Gamma(\mu,\nu)$ is the set of all Borel probability measures γ on $\mathbb{R}^N\times\mathbb{R}^N$ whose marginals are μ and ν, respectively.
- $(\mathcal{P}_2, W_2)$ form a metric space.
- Construction of frame path with various constraint.
- Optimization of frame related functionals, e.g., the probabilistic p^{th} frame potentials, in the context of the Wasserstein metrics.

Concluding remarks on probabilistic frames

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Thank You!

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