Anisotropic Harmonic Analysis and Integration of Remotely Sensed Data

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Items with $*$ are presented in this talk.

Background on Harmonic Analysis: Classical and Anisotropic

Approach of Harmonic Analysis

- Harmonic analysis provides methods for the decomposition of functions into simpler constituent atoms.
- The nature of the decomposition is flexible, and the development of new methods is a major component of the field.
- Given a function with certain known properties, a particular method of decomposition might be especially convenient.
- Beyond being of theoretical interest, these decomposition can be used for applications.

Fourier Series

- The most classical decomposition system of the subject is Fourier series, with ideas dating back at least as far as Lagrange's study of vibrating strings in the eighteenth century.
- Suppose $f \in L^2([0,1]^d)$. Then f may be decomposed in the following manner, with convergence in the $L^2([0,1]^d)$ norm:

$$f(x) = \sum_{m \in \mathbb{Z}^d} \left(\int_{[0,1]^d} f(y) e^{-2\pi i \langle m,y \rangle} dy \right) e^{-2\pi i \langle m,x \rangle}.$$

- This method decomposes f with respect to its *frequency* content, as measured by the Fourier coefficients

$$\int_{[0,1]^d} f(y) e^{-2\pi i \langle m,y \rangle} dy.$$

Discrete Wavelet Decompositions

- A different type of decomposition, based on *scale* and *translation*, was pioneered in the 1980s and 1990s [1,2].
- Let $f \in L^2(\mathbb{R}^2)$, and let ψ be a *wavelet function*. Then f may be decomposed in the following manner, with convergence in the $L^2(\mathbb{R}^2)$ norm:

$$f = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}^2} \langle f, \psi_{m,n} \rangle \psi_{m,n},$$

where $\psi_{m,n}(x) := |\det A|^{\frac{m}{2}} \psi(A^m x - n),\ A \in GL_2(\mathbb{R})$. A typical choice for A is the dyadic isotropic matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

[1] I. Daubechies. “Orthonormal bases of compactly supported wavelets.” Communications on pure and applied mathematics 41.7 (1988): 909-996.

[2] S.G. Mallat. “Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$”. Transactions of the American Mathematical Society 315.1 (1989): 69-87.

Classical Methods are Good...

- Fourier methods proved fundamental in the early development of signal processing, and also in the study of physics.
- Wavelets revolutionized the fields of image compression[3], fusion, and registration.
- Both methods can be implemented with fast, efficient numerical algorithms in both low level (C) and high level (MATLAB) languages.

[3] S. Athanassios, C. Christopoulos, and T. Ebrahimi. "The JPEG 2000 still image compression standard." IEEE Signal Processing Magazine 18.5 (2001): 36-58.

...But Not Perfect

- These classical methods are known to be suboptimal for representing singularities in functions.
- That is, if a function is singular (one dimension) or singular in a given direction (higher dimensions), these transforms fail to efficiently represent the singularity in the coefficients they generate.
- For Fourier series, this is the *Gibbs phenomenon*: many Fourier coefficients are needed to accurately account for a discontinuity.

The Need for Anisotropy

- Wavelets are good for one dimensional jump discontinuities, but are poor in dimensions 2 or more[4].
- This is a major weakness, since one of the most widely-lauded applications of wavelet methods is image analysis, which is two-dimensional at its simplest.
- In higher dimensions, singularities have a *directional character*, but wavelets are fundamentally *isotropic*. This limits wavelets' effectiveness for resolving key aspects of images, such as edges.
- What is needed are decomposition systems that are *anisotropic*, taking directionality into account.

[4] F. Hartmut, L. Demanet, F. Friedrich. "Document and Image Compression". In *Beyond wavelets: New image representation paradigms*. 179-206. 2006.

Shearlets

- Starting in the early 2000s, several anisotropic systems were proposed: Curvelets[5], Contourlets[6], Ridgelets[7], Shearlets[8], and more.
- We note that ridgelets, curvelets, and shearlets fall into the overarching anisotropic paradigm of α-molecules.
- Let $f \in L^2(\mathbb{R}^2)$ and ψ be a *shearlet function*. Then f may be decomposed in the following manner, with convergence in the $L^2(\mathbb{R}^2)$ norm:

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \langle f, \psi_{j,k,m} \rangle \psi_{j,k,m}.$$

[5] E.J. Candès and D. L. Donoho. "New tight frames of curvelets and optimal representations of objects with piecewise $\mathcal{C}^2$ singularities." Communications on pure and applied mathematics 57.2 (2004): 219-266.

[6] M. Do and M. Vetterli. "Contourlets: a directional multiresolution image representation." Proceedings of IEEE International Conference on Image Processing. 2002.

[7] E.J. Candès. "Ridgelets: theory and applications". Diss. Stanford University, 1998.

[8] D. Labate, W.-Q. Lim, G. Kutyniok, and G. Weiss. "Sparse multidimensional representation using shearlets." Proceedings of SPIE Optics & Photonics. 2005.

Shearlets

- Here,
  - $\psi_{j,k,m}(x) := 2^{\frac{3j}{4}} \psi(S_k A_{2^j} x - m)$.
  - $A_a = \begin{pmatrix} a & 0 \\ 0 & a^{\frac{1}{2}} \end{pmatrix}$, $S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.
- Note that A has been replaced with A_a, which is no longer isotropic; this will allow our new analyzing functions to be more pronounced in a particular direction.
- The new matrix S_k, a shearing matrix, lets us select the direction.
- As a becomes larger, the direction selected by S_k will be emphasized to a proportionally greater degree.
- Shearlets have the benefit of fast numerical methods, so we focus on them.

Shearlet Optimality

- One of the theoretical benefits of shearlets is their optimality for representing a certain class of functions.

Definition

The set of *cartoon-like images* in $\mathbb{R}^2$ is

$$\mathcal{E} := \{f \mid f = f_0 + \chi_B f_1,\ f_i \in \mathcal{C}^2([0,1]^2), \|f_i\|_{\mathcal{C}^2} \leq 1,\ B \subset [0,1]^2,\ \partial B \in \mathcal{C}^2([0,1])\}.$$

- The space of cartoon-like images is a quantitative definition of signals that represent images. That is, although images are discrete, if we are to consider only continuous signals, then $\mathcal{E}$ models the class of signals corresponding to images.

Shearlet Optimality

- Shearlets are known to be optimal for $\mathcal{E}$ over *all reasonable representation systems* [9].
- That is, elements of $\mathcal{E}$ may be written with optimally few shearlet coefficients, when compared to the number of coefficients required by other representation systems.
- From a practical standpoint, this suggest shearlets should be superior to classical methods for the analysis of images.
- We shall investigate the efficacy of shearlets for image registration in the final third of this talk.

[9] K. Guo and D. Labate. "Optimally sparse multidimensional representation using shearlets." SIAM journal on mathematical analysis 39.1 (2007): 298-318.

Directional Gabor Theory

Gabor Systems

- Gabor theory was developed based on the work of Nobel laureate Dennis Gabor [10], who was interested in whether modulates and translates of Gaussians could form a basis for $L^2(\mathbb{R})$.

Definition

Let $g \in L^2(\mathbb{R}^d)$. Let $\alpha, \beta > 0$. The family of functions

$$\mathcal{G}(g, \alpha, \beta) := \{g(x - \alpha n)e^{-2\beta\pi i\langle m,x\rangle}\}_{m,n\in\mathbb{Z}^d}$$

is a *regular Gabor system.* Let $\Lambda \subset \mathbb{R}^{2d}$ be discrete. The family of functions

$$\mathcal{G}(g, \Lambda) := \{g(x - \lambda)e^{-2\pi i\langle\gamma,x\rangle}\}_{(\gamma,\lambda)\in\Lambda}$$

is an *(irregular) Gabor system.*

[10] D. Gabor. “Theory of communication. Part 1: The analysis of information.” Journal of the Institution of Electrical Engineers-Part III: Radio and Communication Engineering 93.26 (1946): 429-441.

Gabor Systems

- Gabor systems decompose a signal with respect to *translations* and *modulations*.
- Gabor theory is rich and subtle, and numerical applications of Gabor systems have proven useful, particularly for the analysis of auditory signals.
- However, like Fourier series and wavelets, Gabor systems are fundamentally isotropic.
- Grafakos and Sansing introduced a directional generalization of Gabor systems[11]. Their work is related to the ridgelets of Candès and Donoho.

[11] L. Grafakos and C. Sansing. "Gabor frames and directional time-frequency analysis." Applied and Computational Harmonic Analysis 25.1 (2008): 47-67.

Directional Gabor Systems

- For $g \in L^2(\mathbb{R})$, $g^{m,t} \in L^2(\mathbb{R})$ is the function defined by:

$$g^{m,t}(s) = e^{2\pi i m(s-t)} g(s-t) = T_t M_{-m} g(s), \ m, t \in \mathbb{R}.$$

Definition

Let $g \in \mathcal{S}(\mathbb{R})$ be a $\mathbb{R}$-valued, non-zero window function. For $d \geq 1$ an integer and $m, t \in \mathbb{R}$, we define:

$$G^{m,t}(s) := D_{\frac{d-1}{2}}(g^{m,t})(s) = (\widehat{g^{m,t}}(\gamma)|\gamma|^{\frac{d-1}{2}})^{\vee}(s), \ s \in \mathbb{R}.$$

The *weighted Gabor ridge functions* are

$$G_{m,t,u}(x) := G^{m,t}(\langle u, x\rangle) = (\widehat{g^{m,t}}(\gamma)|\gamma|^{\frac{d-1}{2}})^{\vee}(\langle u, x\rangle), \ x \in \mathbb{R}^d.$$

Directional Gabor Systems

- The weighted Gabor ridge functions incorporate directional information via the $\langle u, \cdot \rangle$ argument. This is similar to the approach of ridgelets.

Definition

Let $f \in \mathcal{S}(\mathbb{R}^d)$. The *Radon transform* of f is a function $R(f) : S^{d-1} \times \mathbb{R} \to \mathbb{C}$ given by the formula

$$R(f)(u,s) := \int_{\langle u,x\rangle = s} f(x)dx, \; u \in S^{d-1}, \; s \in \mathbb{R}.$$

Theorem

Let $f \in L^1(\mathbb{R}^d)$, and let $R_u(f)(s) := R(f)(u,s)$. For a fixed $u \in S^{d-1}$, the Fourier transform of f and $R_u(f)$ are related in the following way:

$$\widehat{R_u(f)}(\gamma) = \hat{f}(\gamma u).$$

Continuous Reproducing Formula

Theorem

Let $g, \psi \in \mathcal{S}(\mathbb{R})$ be two window functions such that $\langle g, \psi \rangle \neq 0$, with corresponding families of Gabor ridge functions $G_{m,t,u}, \Psi_{m,t,u}$. Suppose $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \in L^1(\widehat{\mathbb{R}}^d)$. Then:

$$f = \frac{1}{2\langle g, \psi \rangle} \int_{S^{d-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle f, G_{m,t,u} \rangle \Psi_{m,t,u} dmdtdu.$$

Semi-discrete Reproducing Formula

Theorem

There exist $g, \psi \in \mathcal{S}(\mathbb{R})$ *and* $\alpha, \beta > 0$ *such that for all* $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$*, we have have:*

$$A\|f\|_2^2 \leq \int_{S^{d-1}} \sum_{m\in\mathbb{Z}} \sum_{t\in\mathbb{Z}} |\langle f, G_{\alpha m,\beta t,u}\rangle|^2 du \leq B\|f\|_2^2, \tag{1}$$

where the constants A, B *depend only on* g, α, β *and* du *is the Lebesgue measure on* S^{d-1}*. Moreover, for this choice of* g, ψ*, we have:*

$$f = \frac{1}{2} \int_{S^{d-1}} \sum_{m\in\mathbb{Z}} \sum_{t\in\mathbb{Z}} \langle f, G_{\alpha m,\beta t,u}\rangle \Psi_{\alpha m,\beta t,u} du.$$

The Discretization Problem

- Grafakos and Sansing conjectured the above semi-discrete reproducing formula could be fully discretized, to produce a discrete frame.
- To this effect, we analyze systems of the form

$$\{g_{m,t,u}(x) := g^{m,t}(\langle u, x\rangle)\}_{(m,t,u)\in\Lambda}, \ \Lambda \subset S^1 \times \mathbb{R} \times \mathbb{R}.$$

- Note that the weighting in the Fourier domain is removed for these systems. This weighting is used to make the integration over S^1 work, so is not necessary in the discrete case.

Choosing the Correct Space of Functions

- Although stated here for dimension 2, the following result holds in higher dimensions.

Theorem

Let $\{g_{m,t,u}\}_{(m,t,u)\in\Lambda}$ be any discrete directional Gabor system. Then there exists a sequence of Schwartz functions $\{\phi_n\}_{n=1}^{\infty} \in \mathcal{S}(\mathbb{R}^2)$ such that

- $\lim_{n\to\infty} \|\phi_n\|_2 = 0$.
- $\sum_{(m,t,u)\in\Lambda} |\langle \phi_n, g_{m,t,u}\rangle|^2 \geq 1, \forall n$.

  Thus, $\{g_{m,t,u}\}_{(m,t,u)\in\Lambda}$ cannot be a discrete frame for any function space containing $\mathcal{S}(\mathbb{R}^2)$.

- So, we cannot hope to represent the entire class of Schwartz functions. We will have to aim for something smaller.

A Toy Example

Theorem

Let $g(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$*. Let* $\Gamma \subset S^1 \times \mathbb{R}$ *be such that the mapping* $\psi : \Gamma \to \mathbb{Z}^2$ *given by* $(u, m) \mapsto mu$ *is a bijection. Set* $\Lambda = \{(m, t, u) | (u, m) \in \Gamma, t \in \mathbb{Z}\}$*. Then we have:*

$$\sum_{(m,t,u)\in\Lambda} |\langle f, g_{m,t,u}\rangle|^2 = \|f\|_2^2,$$

for all $f \in L^2(\mathbb{R}^2)$ *supported on* $B_{\frac{1}{2}}(0)$*.*

Sufficient Condition

Definition

Let $\mathcal{U}$ be the subset of $\mathcal{S}(\mathbb{R}^2)$ of functions supported in $[-\frac{1}{2}, \frac{1}{2}]^2$.

Theorem

Let $g \in L^2(\mathbb{R})$ be such that supp$(\hat{g}) \subset K \subset (-\frac{1}{4}, \frac{1}{4})$, K compact. Then with Λ as above, $\{g_{m,t,u}\}_{(m,t,u)\in\Lambda}$ is a frame for $\mathcal{U}$.

- The condition that supp$(\hat{g}) \subset K \subset (-\frac{1}{4}, \frac{1}{4})$, K compact, is required to invoke Kadec's $\frac{1}{4}$ theorem [12].

[12] If $\Lambda = \{\lambda = (\lambda_1, ..., \lambda_d) \in \mathbb{R}^d\}$ and for each $\lambda \in \Lambda$, there is a unique $k = (k_1, ..., k_d) \in \mathbb{Z}^d$ such that $|\lambda_j - k_j| < \frac{1}{4}$ for all $j = 1, ..., d$, then $\{e^{-2\pi i \langle \lambda, \cdot \rangle}\}_{\lambda \in \Lambda}$ forms a Riesz basis for $L^2([-\frac{1}{2}, \frac{1}{2}]^2)$. This result may be found in: W. Sun and X. Zhou. "On Kadec's $\frac{1}{4}$-theorem and the stability of Gabor Frames." Applied and Computational Harmonic Analysis 7.2 (1999): 239-242.

Proof Sketch (1/2)

- The proof of this theorem is based on developing a theory analogous to the work of Hernández, Labate, and Weiss [13]. This theory studies almost periodic functions to develop necessary and sufficient conditions for a broad class of representing systems to be discrete frames, including Gabor and wavelet systems.
- Consider, for a fixed $f \in \mathcal{U}$, the function

$$\begin{aligned} w(x) &= \sum_{(u,m)} \sum_{n \in \mathbb{Z}} |\langle T_x f, g_{m,t,u} \rangle|^2 \\ &= \sum_{(u,m)} \sum_{k \in \mathbb{Z}} \hat{H}_{(u,m)}(k) e^{2\pi i \langle ku, x \rangle}. \end{aligned}$$

- We show this Fourier series is absolutely convergent and is continuous in x.

[13] E. Hernández, D. Labate, and G. Weiss. "A unified characterization of reproducing systems generated by a finite family, II." The Journal of Geometric Analysis 12.4 (2002): 615-662.

Proof Sketch (2/2)

- Then it suffices to show $A\|f\|_2^2 \leq w(0) \leq B\|f\|_2^2$.
- Directly computing the Fourier coefficients is possible, i.e. to get analytic expressions for $\hat{H}_{(u,m)}$ in terms of f, g.
- The sum of these coefficients can be shown to be bounded by $\|f\|_2^2$ by using Kadec's theorem.
- The frame bounds are determined by $\|g\|_2^2$ and the diameter of K.

Image Registration with Shearlets

Introduction to Image Registration

- The process of image registration seeks to align two or more images of approximately the same scene, acquired at different times or with different sensors.
- Images can differ in many ways:
  1. Geometrically: rotated, translated, warped, dilated.
  2. Modally: different sensors, different conditions at time of image capture.
- This problem is relevant to, among other fields, microscopy, biomedical imaging, remote sensing, and image fusion.
- Difficult images to register include those with few dominant features and images from very different sensors, i.e. different modalities.
- We are particularly interested in *multimodal registration*.
- Harmonic analytic techniques are well-suited for these types of problems, when compared to other methods.

Stages of Image Registration

Image registration may be viewed as the combination of four separate processes:

1. Selecting an appropriate **search space** of admissible transformations. This will depend on whether the images are at the same resolution, and what type of transformations will carry the input image to the reference image, i.e. rotation-scale-translation (RST), polynomial warping, etc.
2. Extracting relevant **features** to be used for matching. These could be individual pixels that are known to be in correspondence between the two images, or could be global structures in the images, such as roads, buildings, rivers, and textures.
3. Selecting a **similarity metric**, in order to decide if a transformed input image closely matches the reference image. This metric should make use of the features which are extracted from the image, be they specific pixels or global structures.
4. Selecting a **search strategy**, which is used to match the images based on maximizing or minimizing the similarity metric.

Wavelets and Shearlets for Registration

- Wavelet features are well-established for image registration; the separation of edges from textures is often useful for matching [14].
- However, the fundamentally isotropic nature of wavelets makes them suboptimal for registering images with strong edge features.
- To improve the registration of images with strong edges, we considered features generated from the anisotropic representation system of *shearlets*.
- This is good for registration algorithms, because sparse features increase the robustness of the optimization algorithm that computes the registration transformation.

[14] I. Zavorin and J. Le Moigne. "Use of multiresolution wavelet feature pyramids for automatic registration of multisensor imagery." IEEE Transactions on Image Processing, 14.6 (2005): 770-782.

A Scene from WA

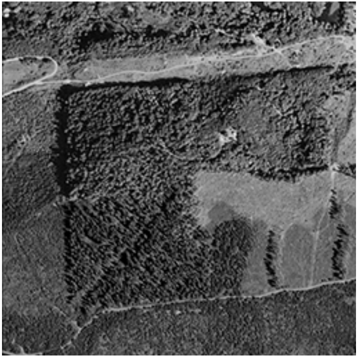

Figure: A grayscale optical image of a mixed land-cover area in Washington state containing both textural and edge-like features. The image is courtesy of Dr. David Harding at NASA GSFC.

WA Features

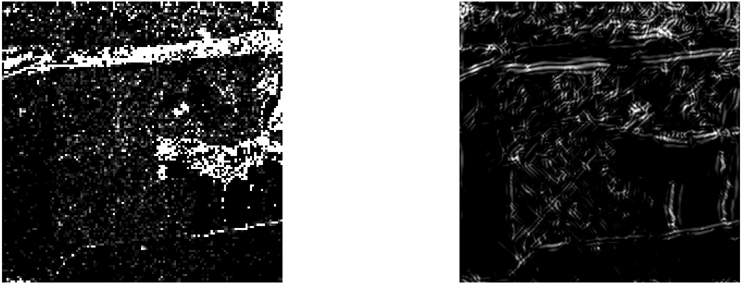

Figure: Wavelet (left) and shearlet (right) features extracted from previous image emphasizing textural and edge features, respectively.

Evaluating Registration Quality

- When registering images, there are two significant criteria of registration algorithm quality:
  1. **Accuracy** of computed registration when compared to the true registration.
  2. **Robustness** of algorithm to initial distance between the two images.
- The robustness of the algorithm is important because the initial closeness of the two images depends greatly on the GPS technology in the sensors and the distance of the sensing device to the location being imaged.
- If the images to be aligned start far apart, the registration algorithm could fail to converge.
- We expected using shearlet features would improve robustness.

Experimental Overview

- As we prototyped, we realized that using shearlets did increase robustness, but at a slight cost in accuracy, usually a few pixels.
- Consequently, we devised a two-stage registration algorithm: first, use shearlets to get an *approximate registration*, then refine this with another iteration of the algorithm, using wavelet features.
- We compared this algorithm to using wavelets alone.
- We performed experiments on *synthetic* images, as well as *multimodal* image pairs.

Basic Description of Algorithm

1. *Search Space:* RST. All of our examples feature images at the same scale, so effectively, our search space is the space of rotations and translations (RT).
2. *Features*: Wavelet features in one case and shearlet features coupled with wavelet features in another.
3. *Similarity Metric*: Unconstrained least squares. That is, if F_R and F_I are the reference and input features, N the number of relevant pixels, (x_i, y_i) the integer coordinates of each pixel, and T_p the transformation associated to parameters p, we seek to minimize the similarity metric given by

$$\chi^2(p) = \frac{1}{N}\sum_{i=1}^{N}(F_R(T_p(x_i, y_i)) - F_I(x_i, y_i))^2$$

4. *Search Strategy:* Modified Marquadt-Levenberg method of solving non-linear least squares problems.

Algorithm Details (1/2)

1. Input a reference image, I^r, and an input image I^i. These will be the images to be registered.
2. Input an initial registration guess $(\theta_0, T_{x_0}, T_{y_0})$.
3. Apply shearlet feature algorithm and wavelet feature algorithm to I^r and I^i. This produces a set of shearlet features for both, denoted $S_1^r, ..., S_n^r$ and $S_1^i, ..., S_n^i$, respectively, as well as a set of wavelet features for both, denoted $W_1^r, ..., W_n^r$ and $W_1^i, ..., W_n^i$. Here, n refers to the level of decomposition chosen. In general, n is bounded by the resolution of the images as

$$n \leq \lfloor \frac{1}{2} \log_2(\max\{M, N\}) \rfloor,$$

where I_r, I_i are $M \times N$ pixels. All our experiments are for 256×256 images, so $n \leq 4$.

Algorithm Details (2/2)

4. Match S_1^r with S_1^i with a least-squares optimization algorithm and initial guess $(\theta_0, T_{x_0}, T_{y_0})$ to get a transformation T_1^S. Using T_1^S as an initial guess, match S_2^r with S_2^i, to acquire a transformation T_2^S. Iterate this process by matching S_j^r with S_j^i using T_{j-1}^S as an initial guess, for $j = 2, ..., n$. At the end of this iterative matching, we acquire our *final shearlet-based registration*, call it $T^S = (\theta^S, T_x^S, T_y^S)$.
5. Using T^S as our initial guess, match W_1^r with W_1^i with a least-squares optimization algorithm to acquire a transformation T_1^W. Using T_1^W as an initial guess, match W_2^r with W_2^i, to acquire a transformation T_2^W. Iterate this process by matching W_j^r with W_j^i using T_{j-1}^W as an initial guess, for $j = 2, ..., n$. At the end of this iterative matching, we acquire our *final hybrid registration*, call it T^H.
6. Output $T^H = (\theta^H, T_x^H, T_y^H)$.

Overview of Experiments

- For this talk, we consider one set of experiments, in which multimodal images are registered.
- We shall perform many iterations of our algorithm. Each iteration, we shall *move the initial guess farther apart*.
- The distance is parametrized by rotation and translation in the x and y directions. For convenience, these are coupled together as RT. So, $RT = 1.8$ means a counterclockwise rotation of 1.8 degrees and a translation of 1.8 pixels in both the x and y direction. Fraction translations and rotations are interpolated by splines.

Belgium Multimodal Images

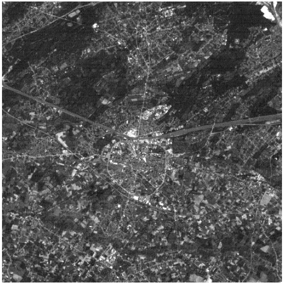

Figure: Multispectral band 1 (left) and panchromatic band 8 (right) images of Hasselt, Belguim acquired by Landsat ETM+. The images have been converted to grayscale. A subset is extracted from these images to ease computation. The images are courtesy of the IEEE Geoscience and Remote Sensing Society Data Fusion committee.

Panchromatic-to-Multispectral Results

Registration Technique	Number of Converged Experiments (out of 101)	Percentage of Converged Experiments	RMSE	Relative Improvement
Spline Wavelets	8	7.92%	.6376	-
Simoncelli Band-Pass	19	18.81%	.7534	-
Simoncelli Low-Pass	14	13.86%	.6034	-
Shearlet + Spline Wavelets	20	19.80%	.5185	150.00%
Shearlet + Simoncelli Band-Pass	27	26.73%	.6494	42.11%
Shearlet + Simoncelli Low-Pass	20	19.80%	.5513	42.86%

Table: Comparison of registration algorithms for panchromatic to multispectral experiment.

Conclusions and Moving Forward

Experiment conclusions

- For automatic image registration, using shearlets and wavelets together outperforms using only wavelets.
- The improvement is pronounced when there are substantial edges.
- When the image is texturally dominant, there is less noticeable improvement.

Future Directions

- Our theorem characterizing directional Gabor frames is sufficient, but not necessary. Indeed, our toy example does not fulfill the hypothesis of the theorem. In the spirit of the work of Hernández, Labate, and Weiss, we would like to develop necessary and sufficient conditions.

- Given the similarity between how directional Gabor frames and ridgelets incorporate directionality, it is of interest to explore how the theory of ridgelets relates to the work presented in the middle portion of this presentation.

- A fast, efficient numerical implementation of discrete directional Gabor frames is also desirable, for use in applications.

- In particular, could directional Gabor frames be incorporated into a registration algorithm? This would require a theoretical understanding what types of functions are optimally represented by directional Gabor frames.

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