Frame theory
from signal processing and back again
– a sampling

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Outline

1. The narrow-band ambiguity function (with Robert L. Benedetto and Joseph Woodworth)
   - NWC Applications: Waveform design and radar
2. Ambiguity functions for vector-valued data (with Travis D. Andrews and Jeffrey J. Donatelli)
   - NWC Applications: Multi-sensor environments and MIMO
   - Set-up and problem
   - Group frame multiplications
3. Graph uncertainty principles (with Paul Koprowski)
   - NWC Applications: Non-linear spectral methods for dimension reduction and classification

Outline continued

4. Balayage and STFT frame inequalities (with E. Au-Yeung)
   - NWC Applications: Non-uniform sampling and super-resolution imaging
5. Quantum detection (with Andrew Kebo)
   - NWC Applications: Finite frames and probabilistic frames in terms of quantum detection and POVMs
6. Reactive sensing (with Michael Dellomo)
   - NWC Applications: Engine diagnosis with disabled sensors by mean of multiplicative frames, whose factors account for parameter intensity and sensor sensitivity

Frames

- Let H be a separable Hilbert space, e.g., $H = L^2(\mathbb{R}^d)$, $\mathbb{R}^d$, or $\mathbb{C}^d$.
- $F = \{x_n\} \subseteq H$ is a *frame* for H if

$$\exists\, A, B > 0 \text{ such that } \forall\, x \in H, \quad A\|x\|^2 \leq \sum |\langle x, x_n\rangle|^2 \leq B\|x\|^2.$$

Theorem

If $F = \{x_n\} \subseteq H$ is a frame for H then

$$\forall x \in H, \quad x = \sum \langle x, S^{-1}x_n\rangle x_n = \sum \langle x, x_n\rangle S^{-1}x_n,$$

where $S : H \to H$, $x \mapsto \sum \langle x, x_n\rangle x_n$ is well-defined.

- Frames are a natural tool for dealing with numerical stability, overcompleteness, noise reduction, and robust representation problems.

1

- THE NARROW BAND AMBIGUITY FUNCTION

Ambiguity function and STFT

- Woodward's (1953) *narrow band cross-correlation ambiguity function* of v, w defined on $\mathbb{R}^d$:

$$A(v,w)(t,\gamma) = \int v(s+t)\overline{w(s)}e^{-2\pi i s\cdot\gamma}ds.$$

- The *STFT* of v : $V_w v(t,\gamma) = \int v(x)\overline{w(x-t)}e^{-2\pi i x\cdot\gamma}dx.$
- $A(v,w)(t,\gamma) = e^{2\pi i t\cdot\gamma}V_w v(t,\gamma).$
- The *narrow band ambiguity function* $A(v)$ of v :

$$A(v)(t,\gamma) = A(v,v)(t,\gamma) = \int v(s+t)\overline{v(s)}e^{-2\pi i s\cdot\gamma}ds$$

The discrete periodic ambiguity function

- Given $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$.
- The *discrete periodic ambiguity function*,

$$A(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C},$$

of u is

$$A(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} u[m+k]\overline{u[k]} e^{-2\pi i k n/N}.$$

CAZAC sequences

- $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$ is
  Constant Amplitude Zero Autocorrelation (CAZAC) if

  $$\forall m \in \mathbb{Z}/N\mathbb{Z}, \quad |u[m]| = 1, \text{ (CA)}$$

  and

  $$\forall m \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}, \quad A(u)(m,0) = 0. \text{ (ZAC)}$$

- Are there only finitely many non-equivalent CAZAC sequences?
  - "Yes" for N prime and "No" for $N = MK^2$,
  - Generally unknown for N square free and not prime.

Björck CAZAC sequences

Let p be a prime number, and $(\frac{k}{p})$ the *Legendre symbol.*
A *Björck CAZAC sequence* of length p is the function $b_p : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ defined as

$$b_p[k] = e^{i\theta_p(k)}, \quad k = 0, 1, \ldots, p-1,$$

where, for $p = 1 \pmod 4$,

$$\theta_p(k) = \arccos\left(\frac{1}{1+\sqrt{p}}\right)\left(\frac{k}{p}\right),$$

and, for $p = 3 \pmod 4$,

$$\theta_p(k) = \frac{1}{2}\arccos\left(\frac{1-p}{1+p}\right)\left[(1-\delta_k)\left(\frac{k}{p}\right) + \delta_k\right].$$

δ_k is the Kronecker delta symbol.

Björck CAZAC discrete periodic ambiguity function

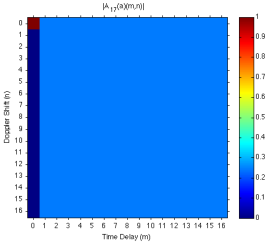
Let $A(b_p)$ be the Björck CAZAC discrete periodic ambiguity function defined on $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Theorem (J. and R. Benedetto and J. Woodworth)

$$|A(b_p)(m,n)| \leq \frac{2}{\sqrt{p}} + \frac{4}{p}$$

for all $(m,n) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \setminus (0,0)$.

- The proof is at the level of Weil's proof of the Riemann hypothesis for finite fields and depends on Weil's exponential sum bound.
- Elementary construction/coding and intricate combinatorial/geometrical patterns.



(a) (b)

Figure : Absolute value of the ambiguity functions of the Alltop and Björck sequences with $N = 17$.

Problems and remarks

- For given CAZACs u_p of prime length p, estimate minimal local behavior $|A(u_p)|$. For example, with b_p we know that the lower bounds of $|A(b_p)|$ can be much smaller than $1/\sqrt{p}$, making them more useful in a host of mathematical problems, cf. Welch bound.
- Even more, construct all CAZACs of prime length p.
- Optimally small coherence of b_p allows for computing sparse solutions of Gabor matrix equations by greedy algorithms such as OMP.

2

- AMBIGUITY FUNCTIONS FOR VECT0R-VALUED DATA

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Modeling for multi-sensor environments

- Multi-sensor environments and vector sensor and MIMO capabilities and modeling.
- Vector-valued DFTs
- Discrete time data vector $u(k)$ for a d-element array,

$$k \longmapsto u(k) = (u_0(k), \ldots, u_{d-1}(k)) \in \mathbb{C}^d.$$

We can have $\mathbb{R}^N \to GL(d, \mathbb{C})$, or even more general.

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Ambiguity functions for vector-valued data

- Given $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}^d$.
- For $d = 1$, $A(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$ is

$$A(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m + k)\overline{u(k)}e^{-2\pi ikn/N}.$$

Goal

Define the following in a meaningful, computable way:

- Generalized $\mathbb{C}$-valued periodic ambiguity function $A^1(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$
- $\mathbb{C}^d$-valued periodic ambiguity function $A^d(u)$.

The STFT is the *guide* and the *theory of frames* is the technology to obtain the goal.

Preliminary multiplication problem

- Given $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}^d$.
- If $d = 1$ and $e_n = e^{2\pi i n/N}$, then

$$A(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k)e_{nk}\rangle.$$

Preliminary multiplication problem

To characterize sequences $\{\varphi_k\} \subseteq \mathbb{C}^d$ and compatible multiplications $*$ and $\bullet$ so that

$$A^1(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * \varphi_{n\bullet k}\rangle \in \mathbb{C}$$

is a meaningful and well-defined *ambiguity function.* This formula is clearly motivated by the STFT.

$A^1(u)$ for DFT frames

- Given $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}^d, d \leq N$.
- Let $\{\varphi_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$, let $*$ be componentwise multiplication in $\mathbb{C}^d$ with a factor of $\sqrt{d}$, and let $\bullet = +$ in $\mathbb{Z}/N\mathbb{Z}$.

In this case $A^1(u)$ is well-defined by

$$A^1(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * \varphi_{n\bullet k}\rangle$$

$$= \frac{d}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\langle \varphi_j, u(k)\rangle\langle u(m+k), \varphi_{j+nk}\rangle.$$

$A^1(u)$ for cross product frames

- Take $* : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$ and let $\{i, j, k\}$ be the standard basis.
- $i * j = k, j * i = -k, k * i = j, i * k = -j, j * k = i, k * j = -i,$ $i * i = j * j = k * k = 0$. $\{0, i, j, k, -i, -j, -k, \}$ is a tight frame for $\mathbb{C}^3$ with frame constant 2. Let

$$\varphi_0 = 0, \varphi_1 = i, \varphi_2 = j, \varphi_3 = k, \varphi_4 = -i, \varphi_5 = -j, \varphi_6 = -k.$$

- The index operation corresponding to the frame multiplication is the non-abelian operation $\bullet : \mathbb{Z}_7 \times \mathbb{Z}_7 \longrightarrow \mathbb{Z}_7$, where $1 \bullet 2 = 3, 2 \bullet 1 = 6, 3 \bullet 1 = 2, 1 \bullet 3 = 5, 2 \bullet 3 = 1, 3 \bullet 2 = 4$, etc.
- We can write the cross product as

$$u \times v = u * v = \frac{1}{2^2} \sum_{s=1}^{6} \sum_{t=1}^{6} \langle u, \varphi_s \rangle \langle v, \varphi_t \rangle \varphi_{s \bullet t}.$$

- Consequently, $A^1(u)$ is well-defined.

Generalize to quaternion groups, order 8 and beyond.

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Frame multiplication

Definition (Frame multiplication)

Let $\mathcal{H}$ be a finite dimensional Hilbert space over $\mathbb{C}$, and let $\Phi = \{\varphi_j\}_{j \in J}$ be a frame for $\mathcal{H}$. Assume $\bullet : J \times J \to J$ is a binary operation. The mapping $\bullet$ is a *frame multiplication* for Φ if it extends to a bilinear product $*$ on all of $\mathcal{H}$.

- The mapping $\bullet$ is a frame multiplication for Φ if and only if there exists a bilinear product $* : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ such that

$$\forall j, k \in J, \quad \varphi_j * \varphi_k = \varphi_{j \bullet k}.$$

- There are frames with no frame multiplications.

Harmonic frames

- Slepian (1968) - *group codes*.
- Forney (1991) - *geometrically uniform* signal space codes.
- Bölcskei and Eldar (2003) - *geometrically uniform* frames.
- Han and Larson (2000) - *frame bases and group representations*.
- Zimmermann (1999), Pfander (1999), Casazza and Kovacević (2003), Strohmer and Heath (2003), Vale and Waldron (2005), Hirn (2010), Chien and Waldron (2011) - *harmonic frames*.
- Han (2007), Vale and Waldron (2010) - *group frames, symmetry groups*.

Harmonic frames

- $(\mathcal{G}, \bullet) = \{g_1, \ldots, g_N\}$ abelian group with $\widehat{G} = \{\gamma_1, \ldots, \gamma_N\}$.
- $N \times N$ matrix with (j, k) entry $\gamma_k(g_j)$ is *character table* of $\mathcal{G}$.
- $K \subseteq \{1, \ldots, N\}, |K| = d \leq N$, and columns $k_1, \ldots, k_d$.

Definition

Given $U \in \mathcal{U}(\mathbb{C}^d)$. The *harmonic frame* $\Phi = \Phi_{\mathcal{G},K,U}$ for $\mathbb{C}^d$ is

$$\Phi = \{U\left((\gamma_{k_1}(g_j), \ldots, \gamma_{k_d}(g_j))\right) : j = 1, \ldots, N\}.$$

Given $\mathcal{G}, K$, and $U = I$. Φ is the $DFT - FUNTF$ on $\mathcal{G}$ for $\mathbb{C}^d$. Take $\mathcal{G} = \mathbb{Z}/N\mathbb{Z}$ for usual $DFT - FUNTF$ for $\mathbb{C}^d$.

Group frames

Definition

Let $(\mathcal{G}, \bullet)$ be a finite group, and let $\mathcal{H}$ be a finite dimensional Hilbert space. A finite tight frame $\Phi = \{\varphi_g\}_{g \in \mathcal{G}}$ for $\mathcal{H}$ is a *group frame* if there exists

$$\pi : \mathcal{G} \to \mathcal{U}(\mathcal{H}),$$

a unitary representation of $\mathcal{G}$, such that

$$\forall g, h \in \mathcal{G}, \quad \pi(g)\varphi_h = \varphi_{g \bullet h}.$$

Harmonic frames are group frames.

Abelian results

Theorem (Abelian frame multiplications – 1)

Let $(\mathcal{G}, \bullet)$ be a finite abelian group, and let $\Phi = \{\varphi_g\}_{g \in \mathcal{G}}$ be a tight frame for $\mathcal{H}$. Then $\bullet$ defines a frame multiplication for Φ if and only if Φ is a group frame.

Abelian results

Theorem (Abelian frame multiplications – 2)

Let $(\mathcal{G}, \bullet)$ be a finite abelian group, and let $\Phi = \{\varphi_g\}_{g \in \mathcal{G}}$ be a tight frame for $\mathbb{C}^d$. If $\bullet$ defines a frame multiplication for Φ, then Φ is unitarily equivalent to a harmonic frame and there exists $U \in \mathcal{U}(\mathbb{C}^d)$ and $c > 0$ such that

$$cU\left(\varphi_g * \varphi_h\right) = cU\left(\varphi_g\right) cU\left(\varphi_h\right),$$

where the product on the right is vector pointwise multiplication and $*$ is defined by $(\mathcal{G}, \bullet)$, i.e., $\varphi_g * \varphi_h := \varphi_{g \bullet h}$.

Remarks

- There is an analogous characterization of frame multiplication for non-abelian groups (T. Andrews).
- Consequently, vector-valued ambiguity functions $A^d(u)$ exist for functions u on a finite dimensional Hilbert space $\mathcal{H}$ if frame multiplication is well-defined for a given tight frame for $\mathcal{H}$ and a given finite group $\mathcal{G}$.
- It remains to extend the theory to infinite Hilbert spaces and groups.
- It also remains to extend the theory to the non-group case, e.g., our cross product example.

3

- GRAPH UNCERTAINTY PRINCIPLES

Uncertainty principles – 1

The Heisenberg uncertainty principle inequality is

$$\forall f \in L^2(\mathbb{R}), \quad \|f\|^2_{L^2(\mathbb{R})} \leq 4\pi \, \|t\, f(t)\|_{L^2(\mathbb{R})} \left\|\gamma\, \hat{f}(\gamma)\right\|_{L^2(\widehat{\mathbb{R}})}.$$

Additively, we have

$$\forall f \in L^2(\mathbb{R}), \quad \|f\|^2_{L^2(\mathbb{R})} \leq 2\pi \, \left(\|t\, f(t)\|^2_{L^2(\mathbb{R})} + \left\|\gamma\, \hat{f}(\gamma)\right\|^2_{L^2(\widehat{\mathbb{R}})} \right).$$

Equivalently, for $f \in \mathcal{S}(\mathbb{R})$,

$$\|f\|^2_{L^2(\mathbb{R})} \leq \left\|\widehat{f'}\right\|^2_{L^2(\widehat{\mathbb{R}})} + \|f'\|^2_{L^2(\mathbb{R})}.$$

We shall extend this inequality to graphs.

Uncertainty principles – 2

- In signal processing, uncertainty principles dictate the trade off between high spectral and high temporal accuracy, establishing limits on the extent to which the “instantaneous frequency” of a signal can be measured (Gabor, 1946)
- Weighted, Euclidean, LCAG, non-L^2 uncertainty principles, proved by Fourier weighted norm inequalities, e.g., Plancherel, generalizations of Hardy’s inequality, e.g., integration by parts, and Hölder (alas).
- DFT: Chebatorov, Grünbaum, Donoho and Stark,Tao.
- Generalize the latter to graphs.

Graph theory – background

- Problem: propose, prove, and understand uncertainty principle inequalities for graphs, see A. Agaskar and Y. M. Lu on *A spectral graph uncertainty principle*
- Generally: There is no obvious solution because of the loss on general graphs of the cyclic structure associated with the DFT.
- Locally: Radar/Lidar data analysis at NWC uses non-linear spectral kernel methods, with *essential* graph theoretic components for dimension reduction and remote sensing.

Graph theory – definition

Definition

A *graph* is $G = \{V, \mathbf{E} \subseteq V \times V, w\}$ consisting of a set V called vertices, a set $\mathbf{E}$ called edges, and a weight function

$$w : V \times V \longrightarrow [0, \infty).$$

Write $V = \{v_j\}_{j=0}^{N-1}$ and keep the ordering fixed, but arbitrary.

Graph theory – assumptions

- For any $(v_i, v_j) \in V \times V$ we have

$$w(v_i, v_j) = \begin{cases} 0 & \text{if } (v_i, v_j) \in \mathbf{E}^c \\ c > 0 & \text{if } (v_i, v_j) \in \mathbf{E}. \end{cases}$$

- G is undirected, i.e., $w(v_i, v_j) = w(v_j, v_i)$.
- $w(v_i, v_i) = 0$, i.e., G has no loops.
- G is connected, i.e., given any v_i and v_j, there exists at most one edge between them, and there exists a sequence of vertices $\{v_k\}, k = 0, ..., d \leq |V| = N$, such that

$$(v_i, v_0), (v_0, v_1), ..., (v_d, v_j) \in \mathbf{E}.$$

- G is *unit weighted* if w takes only the values 0 and 1.

Graph Laplacian

- $N \times N$ symmetric *adjacency matrix*, A, for G :

$$A = (A_{ij}) = (w(v_i, v_j)).$$

- The *degree matrix*, D, is the $N \times N$ diagonal matrix,

$$D = diag\left(\sum_{j=0}^{N-1} A_{0j}, \sum_{j=0}^{N-1} A_{1j}, \cdots, \sum_{j=0}^{N-1} A_{(N-1)j}\right).$$

- The *graph Laplacian*,

$$L = D - A,$$

  is the $N \times N$ symmetric, positive semi-definite matrix, with real ordered eigenvalues $0 = \lambda_0 \leq \ldots \leq \lambda_{N-1}$ and orthonormal eigenbasis, $\{\chi_j\}_{j=0}^{N-1}$, for $\mathbb{R}^N$.

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Graph Fourier transform

- Formally, the Fourier transform $\hat{f}$ at γ of f defined on $\mathbb{R}$ is the inner product of f with the complex exponentials, that are the eigenfunctions of the Laplacian operator $\frac{d^2}{dt^2}$ on $\mathbb{R}$.
- Thus, define the *graph Fourier transform*, $\widehat{f}$, of $f \in \ell^2(G)$ in the graph Laplacian eigenbasis:

$$\widehat{f}[j] = \langle \chi_j, f \rangle, \quad j = 0, \dots, N-1.$$

If

$$\chi = [\chi_0, \chi_1, ..., \chi_{N-1}],$$

then $\widehat{f} = \chi^* f$, and, since χ is unitary, we have the *inversion formula*:

$$f = \chi\chi^* f = \chi\hat{f}.$$

Difference operator for graphs

The *difference operator*,

$$D_r : \ell^2(G) \longrightarrow \mathbb{R}^{|\mathbf{E}|},$$

with coordinate values representing the change in f over each edge, is defined by

$$(D_r f)[k] = (f[j] - f[i]) \, (w(e_k))^{1/2} \, ,$$

where $e_k = (v_j, v_i)$ and $j < i$.

- D_r can be defined by the *incidence matrix* of G.
- If G is a unit weighted circulant graph, then D_r is the intuitive difference operator of Lammers and Maeser.

Difference uncertainty principle for graphs

Theorem

Let G be a connected and undirected graph. Then,

$$\forall f \in \ell^2(G), \quad 0 < \tilde{\lambda}_0 \|f\|^2 \leq \|D_r f\|^2 + \left\|D_r \hat{f}\right\|^2 \leq \tilde{\lambda}_{N-1} \|f\|^2,$$

where

$$\Delta = \mathrm{diag}\{\lambda_0, \dots, \lambda_{N-1}\}$$

and where $0 < \tilde{\lambda}_0 \leq \tilde{\lambda}_1 \leq \dots \leq \tilde{\lambda}_{N-1}$ are the eigenvalues of $L + \Delta$. The bounds are sharp.

Frame difference uncertainty principle for graphs

$\{e_j\}_{j=0}^{N-1} \subseteq \mathbb{C}^d$ is a *frame* for $\mathbb{C}^d$ if

$$\exists\, 0 < A \leq B \text{ such that } \forall f \in \mathbb{C}^d, \quad 0 < A\,\|f\|^2 \leq \sum_{j=0}^{N-1} |\langle f, e_j\rangle|^2 \leq B\,\|f\|^2 .$$

- If $A = B = 1$ then the frame is a *Parseval frame*.
- Define the $d \times N$ matrix $E = [e_0, e_1, \ldots, e_{N-1}]$, where $\{e_j\}_{j=0}^{N-1}$ is a Parseval frame for $\mathbb{C}^d$. Then $EE^* = I_{d\times d}$.

Theorem

Let G be a connected and undirected graph. Then, for every $d \times N$ Parseval frame E,

$$\sum_{j=0}^{d-1} \tilde{\lambda}_j \leq \|D_r \chi^* E^*\|_{fr}^2 + \|D_r E^*\|_{fr}^2 \leq \sum_{j=N-d}^{N-1} \tilde{\lambda}_j .$$

The bounds are sharp.

Feasibility region

The difference operator *feasibility region FR* is

$$FR = \{(x, y) : \exists f \in \ell^2(G),\ \|f\| = 1,\ \text{such that } \|D_r f\|^2 = x \text{ and } \left\|D_r \widehat{f}\right\|^2 = y\}.$$

Theorem

a. FR is a closed subset subset of $[0, \lambda_{N-1}] \times [0, \lambda_{N-1}]$, where λ_{N-1} is the maximum eigenvalue of the Laplacian L.
b. $(\frac{1}{N}\sum_{j=0}^{N-1} \lambda_j, 0)$ and $(0, L_{0,0})$ are the only points of FR on the axes.
c. FR is in the half plane defined by $x + y \geq \tilde{\lambda}_0 > 0$ with equality if and only if $\hat{f}$ is in the eigenspace associated with $\tilde{\lambda}_0$.
d. If $N \geq 3$, then FR is a convex region.

Complete graph

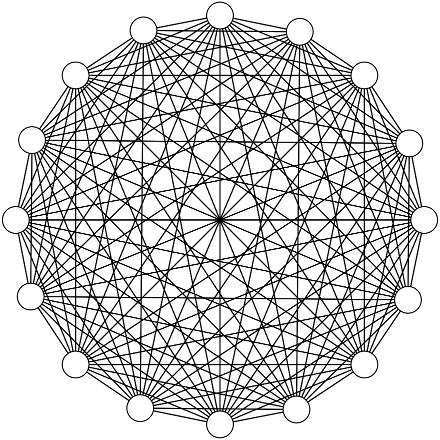
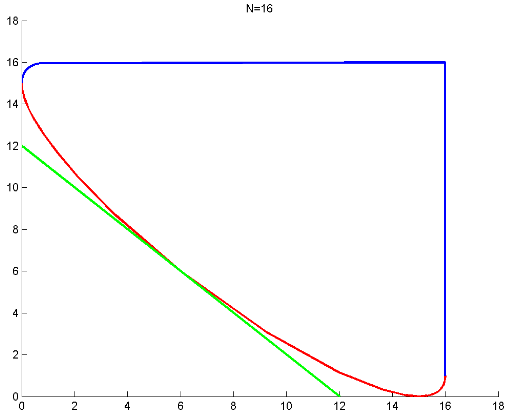


Figure : A unit weighted complete graph with 16 vertices.

Feasibility region



Difference uncertainty curve

The *difference uncertainty curve* ω is the lower boundary of FR defined as

$$\forall x \in [0, \lambda_{N-1}], \quad \omega(x) = \inf_{g \in \ell^2(G)} \langle g, Lg \rangle$$

$$\text{subject to } \langle g, \Delta g \rangle = x.$$

Given $x \in [0, \lambda_{N-1}]$. $g_x \in \ell^2(G)$ *attains the difference uncertainty curve* at x if, for all g for which $\langle g, \Delta g \rangle = x$, we have

$$\langle g_x, Lg_x \rangle \leq \langle g, Lg \rangle.$$

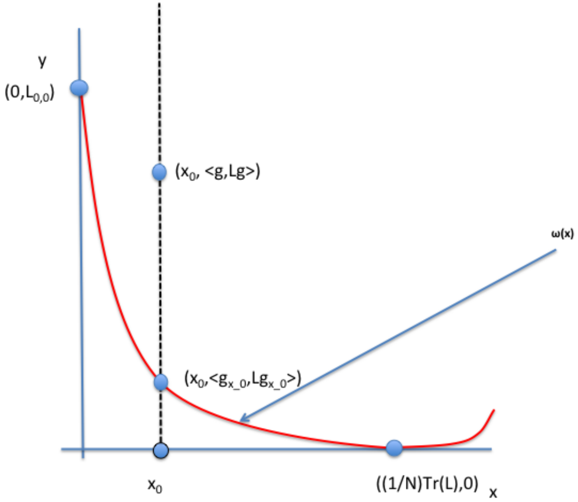


Figure : The difference uncertainty curve (red) for a connected graph G

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Uncertainty curve theorem

Theorem

A unit normed function $f \in \ell^2(G)$, with $\|D_r f\|^2 = x \in (0, \lambda_{N-1})$, achieves the uncertainty curve at x if and only if $\hat{f}$ is a nonzero eigenfunction for $K(\alpha) = L - \alpha\Delta$ associated with the minimal eigenvalue of $K(\alpha)$, where $\alpha \in (-\infty, \infty)$.

Uncertainty principle problem and comparison

- Lammers and Maeser, Grünbaum, Agaskar and Lu.
- The Agaskar and Lu problem.
- Critical comparison between the graph theoretical feasibility region and the comparable Bell Labs uncertainty principle region.

4

- BALAYAGE AND STFT FRAME INEQUALITIES

Balayage and spectral synthesis

Definition

(Balayage – Beurling) Let $E \subseteq \mathbb{R}^d$ and $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be closed sets. *Balayage* is possible for $(E, \Lambda) \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if

$$\forall \mu \in M_b(\mathbb{R}^d),\ \exists \nu \in M_b(E) \text{ such that } \widehat{\mu} = \widehat{\nu} \text{ on } \Lambda.$$

Define

$$\mathcal{C}(\Lambda) = \{f \in C_b(\mathbb{R}^d) : \text{supp}(\widehat{f}) \subseteq \Lambda\}.$$

Definition

(Spectral synthesis) A closed set $\Lambda \subseteq \widehat{\mathbb{R}}^d$ is a set of *spectral synthesis (S-set)* if

$$\forall f \in \mathcal{C}(\Lambda) \text{ and } \forall \mu \in M_b(\mathbb{R}^d), \quad \widehat{\mu} = 0 \text{ on } \Lambda \Rightarrow \int f\, d\mu = 0.$$

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Applications

Balayage and a non-uniform Gabor frame theorem

Let $\mathcal{S}_0(\mathbb{R}^d)$ be the *Feichtinger algebra*.

Theorem

Let $E = \{(s_n, \sigma_n)\} \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be a separated sequence; and let $\Lambda \subseteq \widehat{\mathbb{R}}^d \times \mathbb{R}^d$ be an S-set of strict multiplicity that is compact, convex, and symmetric about $0 \in \widehat{\mathbb{R}}^d \times \mathbb{R}^d$. Assume balayage is possible for (E, Λ). Given $g \in L^2(\mathbb{R}^d)$, such that $\|g\|_2 = 1$. Then

$$\exists\ A,\ B > 0,\ \text{such that}\ \forall f \in \mathcal{S}_0(\mathbb{R}^d),\ \text{for which}\ \mathrm{supp}(\widehat{V_g f}) \subseteq \Lambda,$$

$$A\,\|f\|_2^2 \leq \sum_{n=1}^{\infty} |V_g f(s_n, \sigma_n)|^2 \leq B\,\|f\|_2^2 .$$

Balayage and a non-uniform Gabor frame theorem (continued)

Theorem

Consequently, the frame operator, $S = S_{g,E}$, is invertible in $L^2(\mathbb{R}^d)$–norm on the subspace of $\mathcal{S}_0(\mathcal{R}^d)$, whose elements f have the property, *supp* $(\widehat{V_g f}) \subseteq \Lambda$. Further, if $f \in \mathcal{S}_0(\mathbb{R}^d)$ and $\mathrm{supp}(\widehat{V_g f}) \subseteq \Lambda$, then

$$f = \sum_{n=1}^{\infty} \langle f, \tau_{s_n} e_{\sigma_n} g \rangle S_{g,E}^{-1}(\tau_{s_n} e_{\sigma_n} g),$$

where the series converges unconditionally in $L^2(\mathbb{R}^d)$.

E does not depend on g.

Remarks

- There is a formulation of the non-uniform Gabor frame theorem in terms of the Woodward ambiguity function.
- The theory is also developed for pseudo-differential operators.
- Elementary examples satisfy hypotheses of non-uniform Gabor frame theorem.
- Analogous results, with give and take of hypotheses and conclusions:
  - Gröchenig's theorem involving an analysis of convolution operators on the Heisenberg group;
  - Meyer - Matei theory involving quasi-crystals.

Thats all folks!