Generic Results in Phaseless Reconstruction

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February 26, 2015

Outline

Analysis of speech

- Speech signal enhancement
- Speech recognition

Let $\{x(t), t = 1, 2, \cdots, T\}$ be the samples of a speech signal. The samples are transformed into the time-frequency domain by

$$X(k, \omega) = \sum_{t=0}^{M-1} g(t)x(t + kN)e^{-2\pi i\omega t/M}, \quad k = 0, 1, \cdots, \frac{T - M}{N}$$

In most algorithms, we only use the modulus of the transformed signal $|X(k, \omega)|$. If we do not use the phase, can we reconstruct the signal?

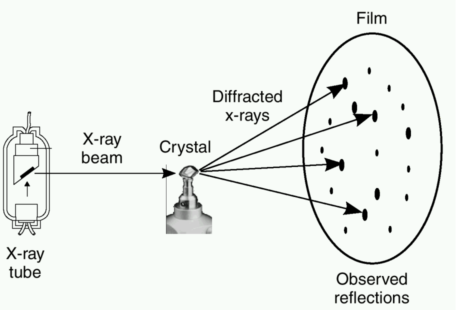

Figure : X-ray Crystallography
http://imgkid.com/x-ray-diffraction-pattern.shtml

In X-ray crystallography, the diffraction data contains only the amplitude of the transformed electron density.
To determine the structure of the crystall, it is important to retrieve the phase information.

Outline

- $\mathcal{H}$: Hilbert space with inner product $\langle \cdot, \cdot \rangle$
- $\mathcal{F}$: $\mathcal{F} = \{f_i : i \in \mathcal{I}\}$ is a frame in $\mathcal{H}$

Definition

$\mathcal{F} = \{f_i : i \in \mathcal{I}\}$ is a frame in $\mathcal{H}$ if there exist two constants A, $B > 0$ such that for every $x \in \mathcal{H}$,

$$A \|x\|^2 \leq \sum_{i \in \mathcal{I}} |\langle x, f_i \rangle|^2 \leq B \|x\|^2$$

- $\hat{\mathcal{H}} = \mathcal{H}/\sim$
  - $x \sim y$ if and only if there is a scalar $|c| = 1$ such that $y = cx$
- Consider the nonlinear map

$$\alpha : \hat{\mathcal{H}} \to l^2(\mathcal{I}), \qquad \alpha(\hat{x}) = \{|\langle x, f_i \rangle|\}_{i \in \mathcal{I}}, \quad x \in \hat{x}$$

Definition

$\mathcal{F}$ is phase retrievable if the map α is injective.

Problem setting

Questions of interest

- When is a frame phase retrievable?
- Is phaseless reconstruction stable under small perturbation?
- Is there an algorithm for phase retrieval with good performance?

Today we introduce some results on the first question.

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- Is phaseless reconstruction stable under small perturbation?
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Problem setting

Finite dimensional case

Case to consider: $\mathcal{H}$ is finite dimensional

- In this case, $\mathcal{F}$ is a frame for $\mathcal{H}$ if and only if $\mathcal{F}$ spans $\mathcal{H}$

Suppose $\mathcal{F} = \{f_1, \cdots, f_m\}$ for some $m \in \mathbb{Z}$, the map we consider reads

$$\alpha : \hat{\mathcal{H}} \to \mathbb{R}^m, \qquad \alpha(\hat{x}) = \{|\langle x, f_i \rangle|\}_{i=1}^m, \quad x \in \hat{x}$$

In some cases, it is useful to look at the map that is the square of α, explicitly,

$$\beta : \hat{\mathcal{H}} \to \mathbb{R}^m, \qquad \beta(\hat{x}) = \{|\langle x, f_i\rangle|^2\}_{i=1}^m, \quad x \in \hat{x}$$

where in the case that $\mathcal{H} = \mathbb{R}^n$ or $\mathbb{C}^n$,

$$|\langle x, f_i\rangle|^2 = f_i^* x x^* f_i = tr(f_i f_i^* x x^*)$$

In this case the map α induces a linear map $\mathcal{A}$ from $Sym(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H}, \quad T = T^*\}$ to $\mathbb{R}^m$:

$$\mathcal{A} : Sym(\mathcal{H}) \to \mathbb{R}^m, \qquad \mathcal{A}(T) = (\langle Tf_i, f_i\rangle)_{i=1}^m$$

Outline

Injectivity results

Real case

$\mathcal{H} = \mathbb{R}^n$

Theorem (R. Balan, P. Casazza, D. Edidin, 2005)

For $\mathcal{H} = \mathbb{R}^n$, the nonlinear map α is injective if and only if for any disjoint partition $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, either $\mathcal{F}_1$ spans $\mathcal{H}$ or $\mathcal{F}_2$ spans $\mathcal{H}$

Theorem (R. Balan, P. Casazza, D. Edidin, 2005)

1. *If α is injective, then $m \geq 2n - 1$;*
2. *If $m \leq 2n - 2$ then α is not injective;*
3. *If $m = 2n - 1$ then α is injective if and only if $\mathcal{F}$ is full spark (any subset of n elements is linearly independent);*
4. *If $m \geq 2n - 1$ and $\mathcal{F}$ is full spark then α is injective.*

Injectivity results
Complex case

$\mathcal{H} = \mathbb{C}^n$

Theorem (B. G. Bodmann, 2007)

$\forall n \in \mathbb{N}$, there is a frame of $m = 4n - 4$ elements such that α is injective;

Theorem (T. Heinosaari, L. Mazzarella, M. Wolf, 2013)

If α is injective then

$$m \geq 4n - 2 - 2\beta + \begin{cases} 2, & \text{if } n \text{ odd and } \beta = 3 \text{ mod } 4 \\ 1, & \text{if } n \text{ odd and } \beta = 2 \text{ mod } 4 \\ 0, & \text{elsewhere} \end{cases}$$

where β is the number of 1's in the binary expansion of $n - 1$.

Conjecture

If α is injective, then $m \geq 4n - 4$.

Definition

Let $\mathbb{K}$ be a field. A set $X \subset \mathbb{K}^n$ is called an algebraic variety if there are polynomials $p_1, \cdots, p_m$ such that
$X = \{x \in \mathbb{K}^n : p_1(x) = \cdots = p_m(x) = 0\}$.

Definition

The Zarisky topology is the induced topology in which algebraic varieties are closed.

Definition

We say that a generic point of $\mathbb{K}^d$ for a field $\mathbb{K}$ has a certain property if there is a non-empty Zarisky open set of points with that property.

- For $\mathcal{H} = \mathbb{K}^n$, a frame $\mathcal{F} = \{f_1, \cdots, f_m\}$ can be identified as an n-by-m matrix $[f_1, \cdots, f_m]$ of full rank. Let $\mathcal{F}(n, m)$ denote the set of all such matrices.
- $\mathcal{F}(n, m)$ is a Zarisky open set in $\mathbb{K}^{n \times m}$.
- A nonempty Zarisky open set is open and dense in Euclidean topology. Therefore, a generic frame being phase retrievable implies that with probability 1, a randomly chosen frame will be phase retrievable.

Theorem (R. Balan, P. Casazza, D. Edidin, 2005)

For $\mathcal{H} = \mathbb{R}^n$, if $m \geq 2n - 1$, then a generic frame $\mathcal{F}$ is phase retrievable.

What about the complex case? That is the "4n-4 conjecture".

Outline

4n-4 Conjecture

Conjecture

For $\mathcal{H} = \mathbb{C}^n$,
(a) If $m < 4n - 4$, *then* α *cannot be injective.*
(b) If $m \geq 4n - 4$, *then* α *is injective for generic* $\mathcal{F}$.

In *An Algebraic Characterization of Injectivity in Phase Retrieval* by A. Conca, D. Edidin, M. Hering, C. Vinzant, the authors proved (b) and a special case of (a).

4n-4 Conjecture

From now on, we fix $\mathcal{H} = \mathbb{C}^n$.
The following lemma is used to translate the injectivity problem into a question in algebraic geometry.

Lemma (A. Bandeira, J. Cahill, D. Mixon, A. Nelson, 2013)

The map α is not injective if and only if there is a nonzero Hermitian matrix $Q \in \mathbb{C}^{n \times n}$ for which

$$rank(Q) \leq 2 \quad and \quad f_k^* Q f_k = 0, \; \forall \; 1 \leq k \leq m$$

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Proof.

($\Rightarrow$) Suppose $\alpha(x) = \alpha(y)$ with $\hat{x} \neq \hat{y}$, take

$$Q = xx^* - yy^*$$

Then $\forall$ $1 \leq k \leq m$, we have $f_k^* Q f_k = (\alpha(x) - \alpha(y))_k = 0$.

4n-4 Conjecture

Lemma (A. Bandeira, J. Cahill, D. Mixon, A. Nelson, 2013)

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$$rank(Q) \leq 2 \quad and \quad f_k^* Q f_k = 0,\ \forall\ 1 \leq k \leq m$$

Proof (Cont'd).

($\Leftarrow$) (Q rank 1) $Q = xx^*$ for some $x \neq 0$. $\alpha(x) = \alpha(0) = 0$.
(Q rank 2) $Q = \lambda_1 xx^* + \lambda_2 yy^*$ where $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$. Then

$$0 = f_k^* Q f_k = \lambda_1 (\alpha(x))_k + \lambda_2 (\alpha(y))_k$$

Take $x' = |\lambda_1|^{1/2} x$, $y' = |\lambda_2|^{1/2} y$. $\alpha(x') - \alpha(y') = 0$. □

4n-4 Conjecture

Definition

Let $\mathcal{B}_{n,m}$ denote the subset (and in fact a subvariety) of $\mathbb{P}(\mathbb{C}^{n\times m} \times \mathbb{C}^{n\times m}) \times \mathbb{P}\left(\mathbb{C}^{n\times n}_{\text{sym}} \times \mathbb{C}^{n\times n}_{\text{skew}}\right)$ consisting of quadruples of matrices $([U, V], [X, Y])$ for which

$$\text{rank}(X + iY) \leq 2$$

and

$$u_k^T X u_k + v_k^T X v_k - 2u_k^T Y v_k = 0 \quad \forall\, 1 \leq k \leq m$$

where u_k and v_k are the k-th column of U and V, respectively.

4n-4 Conjecture

Proposition (♠)

Let $\mathcal{F} = \{f_k\}_{k=1}^m \subset \mathbb{C}^m$ be a complex frame. Write $f_k = u_k + iv_k$. Let U (resp. V) be the real matrix with columns u_k (resp. v_k). Then the map α is injective if and only if $[U, V]$ does not belong to the projection $\pi_1\left((\mathcal{B}_{n,m})_{\mathbb{R}}\right)$

This is based on the above lemma and the relation

$$(u_k + iv_k)^*(X + iY)(u_k + iv_k) = u_k^T X u_k + v_k^T X v_k - 2u_k^T Y v_k$$

4n-4 Conjecture

By Proprosition (♠), all "bad frames" are contained in $\pi_1\left((\mathcal{B}_{n,m})_{\mathbb{R}}\right)$.
We want to prove that the set $\pi_1\left((\mathcal{B}_{n,m})_{\mathbb{R}}\right)$ is small.
We want to bound the dimension of $\pi_1(\mathcal{B}_{n,m})$.
The following theorem gives the dimension of $\mathcal{B}_{n,m}$ itself.

Theorem

The projective complex variety $\mathcal{B}_{n,m}$ has dimension $2nm + 4n - m - 6$.

Proof.

STEP 1: Define $\mathcal{B}'_{n,m}$ with the same dimension as $\mathcal{B}_{n,m}$.

Definition

Let $\mathcal{B}'_{n,m}$ denote the subvariety of $\mathbb{P}(\mathbb{C}^{n\times m}\times\mathbb{C}^{n\times m})\times\mathbb{P}(\mathbb{C}^{n\times n})$ consisting of triples of matrices $([U,V],[Q])$ for which

$$\text{rank}(Q)\leq 2 \quad \text{and} \quad (u_k+iv_k)^*Q(u_k+iv_k)=0 \quad \forall\, 1\leq k\leq m$$

$\mathcal{B}_{n,m}$ and $\mathcal{B}'_{n,m}$ are linearly isomorphic. In fact, we can identify $\mathbb{C}^{n\times n}_{\text{sym}}\times\mathbb{C}^{n\times n}_{\text{skew}}$ with $\mathbb{C}^{n\times n}$ by the map

$$(X,Y)\mapsto Q=X+iY$$

4n-4 Conjecture

Proof (Cont'd).

On the other hand, any complex matrix Q can be uniquely written as $Q = X + iY$ where $X \in \mathbb{C}^{n\times n}_{\text{sym}}$ and $Y \in \mathbb{C}^{n\times n}_{\text{skew}}$. Explicitly, that is given by

$$X = \frac{Q + Q^T}{2}, \qquad Y = \frac{Q - Q^T}{2i}$$

Hence it suffices to prove that $\mathcal{B}'_{n,m}$ has dimension $2nm - m + 4n - 6$. We determine the dimension of $\mathcal{B}'_{n,m}$ by finding the dimension of $\pi_2(\mathcal{B}'_{n,m})$ and $\pi_2^{-1}(Q)$ for $Q \in \mathbb{C}^{n\times n}$.

Proof (Cont'd).

STEP 2: Find the dimension of $\pi_2(\mathcal{B}'_{n,m})$.

Claim: $\pi_2(\mathcal{B}'_{n,m}) = \{Q \in \mathbb{P}(\mathbb{C}^{n\times n}) : \text{rank}(Q) \leq 2\}$.
Proof: It suffices to show "$\supset$":
Take any $(u, v) \in \mathbb{C}^n \times \mathbb{C}^n$ for which

$$(u - iv)^T Q(u + iv) = 0$$

Let U, V be matrices with columns $u_k = u$, $v_k = v$ for $1 \leq k \leq m$.
Then $([U, V], [Q]) \in \mathcal{B}'_{n,m}$ and $Q \in \pi_2(\mathcal{B}'_{n,m})$.

4n-4 Conjecture

Proof (Cont'd).

Proposition (J. Harris, Proposition 12.2)

The variety $M_k \subset M$ of $m \times n$ matrices of rank $\leq k$ is irreducible of codimension $(m-k)(n-k)$ in M.

In our case, the set of matrices of rank at most 2 in $\mathbb{C}^{n \times n}$ has codimension $(n-2)^2$, and thus dimension $n^2 - (n-2)^2 = 4n - 4$.
Therefore, its projectivization in $\mathbb{P}(\mathbb{C}^{n \times n})$ have dimension $4n - 5$.

Proof (Cont'd).

STEP 3: Fix Q in $\pi_2(\mathcal{B}'_{n,m})$. Find the dimension of $\pi_2^{-1}(Q)$.

Lemma

For a nonzero matrix $Q = (q_{lk}) \in \mathbb{C}^{n\times n}$, the polynomial

$$q(u, v) = (u - iv)^T Q(u + iv) \in \mathbb{C}[u_1, \cdots, u_n, v_1, \cdots, v_n]$$

where $u = (u_1, \cdots, u_n)^T$, and $v = (v_1, \cdots, v_n)^T$, is not identically zero.

4n-4 Conjecture

Proof of Lemma.

$$q(u,v) = \sum_{1\le k\le n} q_{kk}(u_k^2 + v_k^2) + \sum_{1\le l\le k\le n} (q_{lk} + q_{kl})(u_l u_k + v_l v_k) + i(q_{lk} - q_{kl})(u_l v_k - v_l u_k)$$

If $q(u,v)$ is identically zero, we would have

$$\begin{aligned} q_{kk} &= 0 \quad \forall\, 1 \le k \le n \\ q_{lk} + q_{kl} &= 0 \ \forall\, 1 \le l \le k \le n \\ q_{lk} - q_{kl} &= 0 \ \forall\, 1 \le l \le k \le n \end{aligned}$$

It follows that Q is the zero matrix.

4n-4 Conjecture

Proof (Cont'd).

By the lemma above, Q defines a nonzero equation

$$(u_k - iv_k)^T Q(u_k + iv_k) = 0$$

For each pair of columns (u_k, v_k), this defines a hypersurface of dimension $2n - 1$ in $(\mathbb{C}^n)^2$.
Thus $\pi_2^{-1}(Q)$ is a product of m copies of this hypersurface in $\left((\mathbb{C}^n)^2\right)^m$ and thus is of dimension $m(2n-1)$.
After projectivization, $\pi_2^{-1}(Q)$ has dimension $m(2n-1) - 1$.

Proof (Cont'd).

STEP 4: Put together.

Following (J. Harris, Proposition 11.13), the dimension of the projective variety $\mathcal{B}'_{n,m}$ is the sum of the dimension of $\pi_2(\mathcal{B}'_{n,m})$ and the dimension of $\pi_2^{-1}(Q)$.
Therefore,

$$\dim(\mathcal{B}'_{n,m}) = 4n - 5 + m(2n-1) - 1 = 2nm + 4n - m - 6$$

□

4n-4 Conjecture

Theorem (4n-4 Conjecture (b))

If $m \geq 4n-4$, then α is injective for a generic frame $\mathcal{F}$.

Proof.

$$dim(\pi_1(\mathcal{B}_{n,m})) \leq \dim(\mathcal{B}_{n,m}) = 2nm + 4n - m - 6$$

When $m \geq 4n-4$, $2nm + 4n - m - 6 \leq (2n-1)(4n-4) + 4n - 6 = 8n^2 - 12n - 10 < 8n^2 - 8n - 4 = 2nm - 1$.
The dimension of $\mathbb{P}((\mathbb{C}^{n\times m})^2)$ is $2nm-1$. Thus $\pi_1(\mathcal{B}_{n,m})$ is contained in some hypersurface defined by the vanishing of some polynomial $p_{m,n} = p_{m,n}^{real} + i \cdot p_{m,n}^{imag}$.
Consider now $\pi_1\left((\mathcal{B}_{n,m})_{\mathbb{R}}\right)$. It is contained in the hypersurface defined by the vanishing of $p_{m,n}^{real}$ or $p_{m,n}^{imag}$, whichever is non-zero. □

4n-4 Conjecture

Example: $n = 2,\ m = 4$.

$$U = (u_{jk}),\ V = (v_{jk}),\ Q = \begin{pmatrix} x_{11} & x_{12} + iy_{12} \\ x_{12} - iy_{12} & y_{11} \end{pmatrix}$$

$\mathcal{B}_{2,4}$ is defined by $g_k = 0,\ k = 1, \cdots, 4$, where

$$\begin{aligned} g_k =& (u_{1k}^2 + v_{1k}^2)x_{11} + 2(u_{1k}u_{2k} + v_{1k}v_{2k})x_{12} + \\ & (u_{2k}^2 + v_{2k}^2)x_{22} + 2(u_{2k}v_{1k} - u_{1k}v_{2k})y_{12} \end{aligned}$$

4n-4 Conjecture

Example: $n = 2$, $m = 4$.

$\pi_1(\mathcal{B}_{2,4})$ is determined by the hypersurface

$$\begin{vmatrix} u_{11}^2 + v_{11}^2 & 2(u_{11}u_{21} + v_{11}v_{21}) & u_{21}^2 + v_{21}^2 & 2(u_{21}v_{11} - u_{11}v_{21}) \\ u_{12}^2 + v_{12}^2 & 2(u_{12}u_{22} + v_{12}v_{22}) & u_{22}^2 + v_{22}^2 & 2(u_{22}v_{12} - u_{12}v_{22}) \\ u_{13}^2 + v_{13}^2 & 2(u_{13}u_{23} + v_{13}v_{23}) & u_{23}^2 + v_{23}^2 & 2(u_{23}v_{13} - u_{13}v_{23}) \\ u_{14}^2 + v_{14}^2 & 2(u_{14}u_{24} + v_{14}v_{24}) & u_{24}^2 + v_{24}^2 & 2(u_{24}v_{14} - u_{14}v_{24}) \end{vmatrix} = 0$$

What about Part(a) of the "4n-4 Conjecture"?

Conjecture

For $\mathcal{H} = \mathbb{C}^n$,
(a) If $m < 4n - 4$, then α cannot be injective.
(b) If $m \geq 4n - 4$, then α is injective for generic $\mathcal{F}$.

It is proved for some special values of n, Part(a) is true.

4n-4 Conjecture

Proposition ($\heartsuit$)

If $m \leq 4n-5$, then for every $[U,V] \in \mathbb{P}(\mathbb{C}^{n\times m})^2$, the preimage under the first projection $\pi_1^{-1}([U,V])$ is a non-empty variety of degree

$$d_{n,2} = \prod_{j=0}^{n-3} \frac{\binom{n+j}{2}}{\binom{2+j}{2}}$$

In particular, the projection $\pi_1(\mathcal{B}_{n,m})$ is all of $\mathbb{P}((\mathbb{C}^{n\times m})^2)$.

4n-4 Conjecture

Proof.

Let

$$L_{\mathcal{F}} = \{Q \in \mathbb{P}(\mathbb{C}^{n\times n}) : (u_k - iv_k)^T Q(u_k + iv_k) = 0 \quad \forall\, 1 \leq k \leq m\}$$

$$H_2 = \{Q \in \mathbb{P}(\mathbb{C}^{n\times n}) : \mathrm{rank}(Q) \leq 2\}$$

We have $\dim(L_{\mathcal{F}}) \geq n^2 - 1 - m$. $\dim(H_2) = 4n - 5$.
When $m \leq 4n - 5$ we have $\dim(L_{\mathcal{F}}) + \dim(H_2) \geq n^2 - 1$. Therefore there is a point in the intersection $L_{\mathcal{F}} \cap H_2$. By (J. Harris, Example 19.10), we have the degree of $H_2 = d_{n,2}$. Therefore, the degree of $L_{\mathcal{F}} \cap H_2$ is also $d_{n,2}$ given above. □

4n-4 Conjecture

Using Proposition (♠), we can restate the 4n-4 Conjecture (a) as follows:

Conjecture (4n-4 Conjecture (a))

If $m \leq 4n-5$, then $\pi_1((\mathcal{B}_{n,m})_{\mathbb{R}}) = \mathbb{P}((\mathbb{R}^{n\times m})^2)$.

If we could show $(\pi_1(\mathcal{B}_{n,m}))_{\mathbb{R}} \subset \pi_1((\mathcal{B}_{n,m})_{\mathbb{R}})$, then by Proposition (♡) we would get the conjecture.

Unfortunately it is not an easy task.

In the paper the authors prove the case for $n = 2^l + 1$.

4n-4 Conjecture

Lemma

When $n = 2^l + 1$, $d_{n,2}$ *is odd.*

Proof.

Legendre's formula: Let $s_p(m)$ denote the sum of the digits in the base p expansion of m. The highest power of a prime p dividing $m!$ is given by

$$\nu_p(m!) = \frac{m - s_p(m)}{p-1}$$

Recall that

$$d_{n,2} = \prod_{j=0}^{n-3} \frac{\binom{n+j}{2}}{\binom{2+j}{2}}$$

Proof (Cont'd).

The highest power of 2 dividing $d_{n,2}$ is thus

$$\left(\sum_{j=0}^{n-3} s_2(n+j-2) - s_2(n+j)\right) - \left(\sum_{j=0}^{n-3} s_2(j) - s_2(j+2)\right)$$

For $n = 2^l + 1$ we have for $0 \leq m \leq n-2$ that $s_2(n-1+m) = s_2(m) + 1$. Thus (43) simplifies to

$$\begin{aligned} & s_2(n-2) - s_2(n) - s_2(n-3) + s_2(m-1) \\ = \ & l - 2 - (l-1) + 1 = 0 \end{aligned}$$

□

4n-4 Conjecture

Theorem

If $n \leq 4m - 5$ and $n = 2^l + 1$, then α is not injective.

Proof.

It follows from the above lemma and the fact that any projective variety defined over $\mathbb{R}$ with odd degree has real point. Now for $[U, V]$ real, $\pi_1^{-1}([U, V])$ contains a real point. Therefore $\pi_1((\mathcal{B}_{n,m})_{\mathbb{R}}) = \mathbb{P}((\mathbb{R}^{n \times m})^2)$ □

4n-4 Conjecture

What about for a general n? It is false!
Recently, in *A Small Frame and a Certificate of its Injectivity*, C. Vinzant proved that for the case $n = 4$ and $m = 11$, the conjecture is false.
The author found a polynomial that contains all the "non-injectivity" points.

Thank you!

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