ABSTRACT

Title of dissertation: FRAME MULTIPLICATION THEORY FOR VECTOR-VALUED HARMONIC ANALYSIS

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A tight frame Φ is a sequence in a separable Hilbert space $\mathcal{H}$ satisfying the frame inequality with equal upper and lower bounds and possessing a simple reconstruction formula. We define and study the theory of frame multiplication in finite dimensions. A frame multiplication for Φ is a binary operation on the frame elements that extends to a bilinear vector product on the entire Hilbert space. This is made possible, in part, by the reconstruction property of frames.

The motivation for this work is the desire to define meaningful vector-valued versions of the discrete Fourier transform and the discrete ambiguity function. We make these definitions and prove several familiar harmonic analysis results in this context. These definitions beget the questions we answer through developing frame multiplication theory.

For certain binary operations, those with the Latin square property, we give a characterization of the frames, in terms of their Gramians, that have these frame multiplications. Combining finite dimensional representation theory and Naimark's theorem, we show frames possessing a group frame multiplication are the projections of an orthonormal basis onto the isotypic components of the regular representations. In particular, for a finite group $\mathcal{G}$, we prove there are only finitely many inequivalent frames possessing the group operation of $\mathcal{G}$ as a frame multiplication, and we give an explicit formula for the dimensions in which these frames exist. Finally, we connect our theory to a recently studied class of frames; we prove that frames possessing a group frame multiplication are the central G-frames, a class of frames generated by groups of operators on a Hilbert space.

FRAME MULTIPLICATION THEORY FOR VECTOR-VALUED HARMONIC ANALYSIS

by

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In memory of Erline Frese.

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$\mathbb{R}$	the field of real numbers
$\mathbb{C}$	the field of complex numbers
$\mathbb{F}$	the field $\mathbb{R}$ or $\mathbb{C}$
$\mathcal{G}$	a finite group
$\widehat{\mathcal{G}}$	the dual group of $\mathcal{G}$
$\mathcal{H}, \mathcal{K}$	Hilbert spaces over a field $\mathbb{F}$
$L^1(\mathcal{G})$	the set of integrable functions on $\mathcal{G}$
$L^2(\mathcal{G})$	the set of square integrable functions on $\mathcal{G}$
$C(X)$	the set of continuous functions on a topological space X
$\mathcal{A}, \mathcal{B}$	Banach algebras
$\sigma(\mathcal{A})$	the spectrum of $\mathcal{A}$
$\sigma(x)$	the spectrum of $x \in \mathcal{A}$
$\mathcal{S}$	a set of operators on a Hilbert space
$\mathcal{B}(\mathcal{H})$	the set of all bounded operators on the Hilbert space $\mathcal{H}$
$\mathcal{S}'$	the commutant of $\mathcal{S}$
$(\mathcal{H}, \pi)$	a representation of a group $\mathcal{G}$
$\mathcal{A}_\pi$	the algebra of operators generated by the representation $(\mathcal{H}, \pi)$

Chapter 1

Introduction

Applied harmonic analysis is playing a central role in what is shaping up, to borrow a phrase, to be the golden age of Mathematical Engineering. Algorithms which take advantage of the sparsity of natural signals are promising, first steps towards advances in data transmission, imaging, and video to name a few. Dimension reduction techniques are improving classification methods and dealing with an influx of data, and the study of functions on graphs is finding applications in our connected world. Advancement of the theoretical underpinnings of applied harmonic analysis allows for these technological advancements, and in return the needs of the engineering and technology communities inform mathematicians posing abstract problems.

We shall study the meaningful extension of a classical harmonic analysis technique to the realm of vector-valued functions. There is rationale for attempting such an extension, e.g., to model vector sensing environments. Attempting such a definition leads us to questions in finite frame theory, which we answer by developing frame multiplication theory, a prime example of theory being informed by practice.

1.1 Fourier Analysis on Locally Compact Groups

In this section we present the necessary background material on Fourier analysis. For a more detailed account of Fourier analysis on locally compact groups see [19, 35], and for an extensive treatise on abstract harmonic analysis see [28, 29].

A *locally compact group* $\mathcal{G}$ is a topological group whose topology is locally compact and Hausdorff. A *left (resp. right) Haar measure* on $\mathcal{G}$ is a nonzero Radon measure μ on $\mathcal{G}$ that satisfies $\mu(xE) = \mu(E)$ (resp. $\mu(Ex) = \mu(E)$) for every Borel set $E \subset \mathcal{G}$ and every $x \in \mathcal{G}$. We focus our attention on locally compact groups because we can guarantee the existence of a left (therefore right) Haar measure.

Theorem 1.1.1 (Theorem 2.10 in [19])**.** *Every locally compact group $\mathcal{G}$ possesses a left Haar measure.*

Left Haar measures are unique up to a positive multiplicative constant. If $\mathcal{G}$ is abelian, then left Haar measures are both left and right translation invariant, and after choosing a normalization, we call one, simply, Haar measure on $\mathcal{G}$. If f is a function on $\mathcal{G}$, then we denote the integral of f with respect to Haar measure by $\int f(x)dx$. As with other measure and topological spaces, we have the usual function spaces: $L^p(\mathcal{G})$ $(1 \leq p \leq \infty)$, $C(\mathcal{G})$, and $C_c(\mathcal{G})$. For the remainder of this section we will assume $\mathcal{G}$ is abelian and write the group operation as addition; such groups are called locally compact abelian groups (LCAGs).

Definition 1.1.2 (Dual group)**.** Let $\mathcal{G}$ be a LCAG. Continuous homomorphisms from $\mathcal{G}$ into the circle group $\mathbb{T}$ (complex numbers of modulus 1 under multiplication) are called *characters*. The set of all characters of $\mathcal{G}$ is the *dual group* of $\mathcal{G}$, and we denote it by $\widehat{\mathcal{G}}$. As the name would suggest, endowed with the weak-$*$ topology and group operation pointwise multiplication, $\widehat{\mathcal{G}}$ is a locally compact abelian group. The Pontrjagin duality theorem tells us that $\widehat{\widehat{\mathcal{G}}} = \mathcal{G}$; i.e., $\mathcal{G}$ is the dual group of $\widehat{\mathcal{G}}$. The characters of $\widehat{\mathcal{G}}$ are evaluation at $x \in \mathcal{G}$. Because of this, we will use the symmetric notation

$$\langle x, \xi \rangle = \xi(x) \quad (x \in \mathcal{G}, \xi \in \widehat{\mathcal{G}}).$$

Definition 1.1.3 (Fourier transform)**.** Let $\mathcal{G}$ be a LCAG. The *Fourier transform* is the map from $L^1(\mathcal{G})$ to $C(\widehat{\mathcal{G}})$ defined by

$$\forall f \in L^1(\mathcal{G}), \quad \mathcal{F}f(\xi) = \widehat{f}(\xi) = \int f(x)\overline{\langle x, \xi \rangle}\, dx.$$

By the Pontrjagin duality theorem we can also define a Fourier transform on $\widehat{\mathcal{G}}$, and we can ask, what happens when we take $\widehat{\widehat{f}}$? It turns out, if Haar measure on $\widehat{\mathcal{G}}$ ($d\xi$) is suitably normalized with respect to Haar measure on $\mathcal{G}$, then we almost get back to where we started.

Theorem 1.1.4 (Fourier transform inversion)**.** *If $f \in L^1(\mathcal{G})$ and $\widehat{f} \in L^1(\widehat{\mathcal{G}})$, then $f(x) = \widehat{\widehat{f}}(-x)$ for almost every x; i.e.,*

$$f(x) = \int \widehat{f}(\xi) \, \langle x, \xi \rangle \; d\xi$$

for almost every x. If f is continuous, these relations hold for every x.

The situation for $L^2(\mathcal{G})$ and $L^2(\widehat{\mathcal{G}})$ is even nicer.

Theorem 1.1.5 (Plancherel theorem)**.** *The Fourier transform on $L^1(\mathcal{G}) \cap L^2(\mathcal{G})$ extends uniquely to a unitary isomorphism from $L^2(\mathcal{G})$ to $L^2(\widehat{\mathcal{G}})$.*

Definition 1.1.6 (Convolution)**.** Let $\mathcal{G}$ be a LCAG, and let $f, g \in L^1(\mathcal{G})$. The *convolution* of f and g in $L^1(\mathcal{G})$ is defined by the formula

$$(f * g)(x) = \int f(x - y) g(y)\, dy.$$

The convolution is well defined, $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$, and the resulting function is in $L^1(\mathcal{G})$.

The most important property of convolution that we will utilize is that it makes the Fourier transform into an algebra homomorphism from $L^1(\mathcal{G})$ to $C(\widehat{\mathcal{G}})$.

Proposition 1.1.7. *Let $\mathcal{G}$ be a LCAG. If $f, g \in L^1(\mathcal{G})$, then*

$$\widehat{f * g} = \widehat{f}\,\widehat{g}.$$

Example 1.1.8. The real numbers $\mathbb{R}$ under regular addition and with the usual topology are a LCAG. $\widehat{\mathbb{R}}$ is isomorphic to $\mathbb{R}$, Haar measure is Lebesgue measure, and we have the pairing $\langle x, \xi \rangle = e^{2\pi i x \xi}$. For $f \in L^1(\mathbb{R})$ the Fourier transform of f is

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx,$$

and if $f \in L^2(\mathbb{R})$, then we may take any sequence $\{f_n\}_{n=1}^{\infty} \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ converging to f and the Fourier transform of f is defined as the limit of the Fourier transforms of the f_n's. By the Plancherel theorem such a limit exists and is unique. In particular, we have

$$\widehat{f}(\xi) = \lim_{n \to \infty} \int_{-n}^{n} f(x) e^{-2\pi i x \xi}\, dx,$$

where the convergence is in $L^2(\mathbb{R})$.

1.2 Banach Algebras and the Gelfand Transform

The theory of Banach Algebras and the Gelfand transform play a supporting role in Fourier analysis on LCAGs and are the setting of Section 2.6. Banach algebras are studied in Functional analysis for their own sake under the wider umbrella of operator algebras; we will leverage some of these more general results in Chapter 4. As a reference for the role of Banach algebras in harmonic analysis we use [19]. For an introduction to operator algebras we recommend [44], and as references for C^* and von Neumann operator algebras we use [7, 15, 38].

Definition 1.2.1 (Banach algebra)**.** A *Banach algebra* is an algebra $\mathcal{A}$ over the field of complex numbers equipped with a norm $\|\cdot\|$ with respect to which it is a Banach space and which satisfies

$$\forall x, y \in \mathcal{A}, \quad \|xy\| \leq \|x\| \, \|y\| \, .$$

$\mathcal{A}$ is called *unital* if it possesses a multiplicative identity, which we denote by e. An *involution* on an algebra $\mathcal{A}$ is a map $x \mapsto x^*$ from $\mathcal{A}$ to $\mathcal{A}$ that satisfies

$$\forall x, y \in \mathcal{A},\, \lambda \in \mathbb{C}, \quad (x+y)^* = x^* + y^*, \quad (\lambda x)^* = \overline{\lambda} x^*, \quad (xy)^* = y^* x^*, \quad x^{**} = x.$$

An algebra with an involution is called a $*$-*algebra*.

If S is a subset of the Banach algebra $\mathcal{A}$, we say $\mathcal{A}$ is *generated* by S if the linear combinations of products of elements of S are dense in $\mathcal{A}$. Often we can reduce a problem in a Banach algebra to a problem which deals only with a set of generators.

Example 1.2.2. Let $\ell^1 = \ell^1(\mathbb{Z})$ be the space of all sequences $x = (x_n)_{n=-\infty}^{\infty}$ such that $\|x\|_1 = \sum_{n=-\infty}^{\infty} |x_n| < \infty$. ℓ^1 is a Banach space under coordinate-wise addition, and we may define a multiplication by convolution

$$(x * y)_n := \sum_{k=-\infty}^{\infty} x_k y_{n-k}$$

and an involution by $(x^*)_n = \overline{x_{-n}}$. With these operations ℓ^1 is a unital $*$-algebra with unit element δ defined by

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0. \end{cases}$$

For each $k \in \mathbb{Z}$, define $\delta^k \in \ell^1$ by $(\delta^k)_n = 1$ if $n = k$ and 0 otherwise. For $k \geq 1$, it can be shown $\delta^k = \delta^1 * \ldots * \delta^1$ and $\delta^{-k} = \delta^{-1} * \ldots * \delta^{-1}$, where the products on the right have k factors, and $\delta = \delta^1 * \delta^{-1}$. Also, for any $x \in \ell^1$, we have $x = \lim_{N\to\infty} \sum_{k=-N}^{N} x_k \delta^k$. Therefore, ℓ^1 is generated by the two element set $\{\delta^1, \delta^{-1}\}$.

Example 1.2.3. For our second example of a Banach $*$-algebra, let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. $\mathcal{B}(\mathcal{H})$ is a unital $*$-algebra with operations addition and composition of operators and with involution the adjoint operation. $\mathcal{B}(\mathcal{H})$ satisfies

$$\forall A \in \mathcal{B}(\mathcal{H}), \quad \|A^*A\| = \|A^*\| \, \|A\| \, ,$$

which is called the C^*-identity. Banach $*$-algebras satisfying the C^*-identity are called *C^*-algebras*. It turns out $\mathcal{B}(\mathcal{H})$ is essentially the only kind of C^*-algebra. The Gelfand-Naimark theorem states that any C^*-algebra is $*$-isomorphic to an algebra of bounded operators on some Hilbert space. It was not necessary for us to consider all of $\mathcal{B}(\mathcal{H})$; we could take a norm and adjoint closed subspace of $\mathcal{B}(\mathcal{H})$ and we would have a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

We have some additional terminology and a theorem involving the $*$-algebra described in the above example. We make use of the theorem in Section 4.3.2 where we apply it to the case of groups of operators generated by a representation.

Definition 1.2.4 (Commutant)**.** Let $\mathcal{S}$ be a set of bounded linear operators on a Hilbert space $\mathcal{H}$. The *commutant* of $\mathcal{S}$ is the algebra of operators that commute with every member of $\mathcal{S}$. We denote the commutant of $\mathcal{S}$ by $\mathcal{S}'$. In symbols,

$$\mathcal{S}' = \{A \in \mathcal{B}(\mathcal{H}) : \forall B \in \mathcal{S}, \quad AB = BA\}.$$

It is immediate from the definition of the commutant that

$$\mathcal{S} \subseteq \mathcal{S}'' \subseteq \mathcal{S}'''' \dots \quad \text{and} \quad \mathcal{S}' \subseteq \mathcal{S}''' \subseteq \mathcal{S}''''' \dots . \tag{1.1}$$

Also immediate is that any commutant is closed under the strong operator topology. The question of whether the sequences of subsets in (1.1) terminate is tied neatly to the strong closure of $\mathcal{S}$ by von Neumann's double commutant theorem.

Theorem 1.2.5 (von Neumann double commutant, Theorem 0.4.2 in [38])**.** *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ its algebra of bounded operators. Let $\mathcal{M}$ be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that contains the identity operator. The following are equivalent:*

1. *$\mathcal{M}'' = \mathcal{M}$, i.e., $\mathcal{M}$ equals its double commutant.*

2. *$\mathcal{M}$ is closed in the weak operator topology.*

3. *$\mathcal{M}$ is closed in the strong operator topology.*

Von Neumann's double commutant (or bicommutant) theorem is surprising because it connects a topological property, strong and weak closure, with a purely algebraic property, an algebra being equal to its double commutant.

Definition 1.2.6 (Spectrum)**.** Let $\mathcal{A}$ be a commutative unital Banach algebra. If $x \in \mathcal{A}$, the *spectrum* of x is

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda e - x \text{ is not invertible}\}.$$

Definition 1.2.7 (Multiplicative linear functionals)**.** Let $\mathcal{A}$ be a commutative unital Banach algebra. A *multiplicative linear functional* on $\mathcal{A}$ is a nonzero homomorphism from $\mathcal{A}$ to $\mathbb{C}$, i.e., a linear functional in the usual Banach space sense that additionally satisfies $h(xy) = h(x)h(y)$. The set of all multiplicative functionals on $\mathcal{A}$ is called the *spectrum* of $\mathcal{A}$. The spectrum of $\mathcal{A}$ is denoted the same as the spectrum of an element: $\sigma(\mathcal{A})$. $\sigma(\mathcal{A})$ is a subset of the closed unit ball B of the dual space $\mathcal{A}^*$, and when $\sigma(\mathcal{A})$ is endowed with the weak-$*$ topology, the topology of pointwise convergence on $\mathcal{A}$, it is a topological space. Because the pointwise limit of a multiplicative linear functional is multiplicative, $\sigma(\mathcal{A})$ is a closed subset of B, and so, by Alaoglu's theorem, $\sigma(\mathcal{A})$ is a compact Hausdorff space.

The spectrum of a commutative unital Banach algebra is sometimes referred to as the maximal ideal space, or simply, ideal space, for reasons which the next theorem makes clear.

Theorem 1.2.8 (Theorem 1.12 in [19])**.** *Let $\mathcal{A}$ be a commutative unital Banach algebra. The map $h \mapsto \ker(h)$ is a one-to-one correspondence between $\sigma(\mathcal{A})$ and the set of maximal ideals in $\mathcal{A}$.*

Definition 1.2.9 (Gelfand transform)**.** Let $\mathcal{A}$ be a commutative unital Banach algebra. For each $x \in \mathcal{A}$ define the function $\widehat{x}$ on $\sigma(\mathcal{A})$ by

$$\widehat{x}(h) = h(x).$$

$\widehat{x}$ is continuous on $\sigma(\mathcal{A})$ since a net $\{h_\lambda\}_\lambda$ in $\sigma(\mathcal{A})$ converges to h precisely when the net $h_\lambda(x)$ converges to $h(x)$ for every $x \in \mathcal{A}$. The map

$$\Gamma : \mathcal{A} \to C(\sigma(\mathcal{A})), \quad x \mapsto \widehat{x}$$

is called the *Gelfand transform* on $\mathcal{A}$. The Gelfand transform is a homomorphism from $\mathcal{A}$ to $C(\sigma(\mathcal{A}))$, and $\widehat{e}$ is the constant function 1 on $\sigma(\mathcal{A})$.

When $\mathcal{A}$ is not unital it can be embedded in a unital algebra $\widetilde{\mathcal{A}}$ such that it is identified with a maximal ideal. The spectrum of a nonunital commutative Banach algebra is defined in the same way as the unital case, and there is a one-to-one correspondence between $\sigma(\mathcal{A}) \cup \{0\}$ and $\sigma(\widetilde{\mathcal{A}})$. Utilizing this correspondence, it can be shown $\sigma(\mathcal{A}) \cup \{0\}$ is a weak-$*$ closed subset of the closed unit ball in $\mathcal{A}^*$, and therefore, it is a compact Hausdorff space. This means if $\{0\}$ is an isolated point of $\sigma(\mathcal{A}) \cup \{0\}$, then $\sigma(\mathcal{A})$ is compact, and if not, then $\sigma(\mathcal{A})$ is a locally compact Hausdorff space whose one-point compactification is $\sigma(\mathcal{A}) \cup \{0\}$. Lastly, the Gelfand transform on $\mathcal{A}$ is defined in the same way as above, and if we identify $\sigma(\widetilde{\mathcal{A}})$ with $\sigma(\mathcal{A}) \cup \{0\}$, then it is the Gelfand transform on $\widetilde{\mathcal{A}}$ (restricted to the maximal ideal $\mathcal{A}$) with the resulting functions $\widehat{x}$ restricted to $\sigma(\mathcal{A})$. The value of $\widehat{x}$ at the extra point $\{0\}$ is zero, and therefore, when $\sigma(\mathcal{A})$ is not compact $\widehat{x}$ vanishes at infinity. In this case, the Gelfand transform is a homomorphism from $\mathcal{A}$ to $C_0(\sigma(\mathcal{A}))$.

Example 1.2.10. Let $\ell^1 = \ell^1(\mathbb{Z})$ be as described in Example 1.2.2. The characters of $\mathbb{Z}$ are the functions $n \mapsto e^{2\pi i n \xi}$ for $\xi \in [0, 1)$; hence, the dual group $\widehat{\mathbb{Z}}$ can be identified with the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The Fourier transform on ℓ^1 is given by

$$\mathcal{F}x(\xi) = \sum_{n=-\infty}^{\infty} x_n e^{-2\pi i n \xi}.$$

Linearity and continuity of the Fourier transform implies the evaluation functionals $h_\xi(x) := \mathcal{F}x(\xi)$ ($\xi \in \mathbb{T}$) are in $(\ell^1)^*$, and Proposition 1.1.7 implies they belong to $\sigma(\ell^1)$. It can be shown for $\mathcal{G}$ a general LCAG, that any multiplicative linear functional on $L^1(\mathcal{G})$ is given by integration against a character, so the h_ξ constitute all the multiplicative linear functionals on ℓ^1. In short, $\sigma(\ell^1)$ can be identified with $\widehat{\mathbb{Z}}$ by $h_\xi \mapsto \xi$, and under this identification we have

$$\widehat{x}(\xi) = \widehat{x}(h_\xi) = h_\xi(x) = \mathcal{F}x(\xi),$$

i.e., the Fourier transform and Gelfand transform are the same.

The example above holds in general. For a locally compact abelian group $\mathcal{G}$ we can always identify $\widehat{\mathcal{G}}$ with $\sigma(L^1(\mathcal{G}))$ so that the Fourier transform on $L^1(\mathcal{G})$ is the same as the Gelfand transform on the convolution $*$-algebra $L^1(\mathcal{G})$. Because of this correspondence, whenever investigating a transform with Fourier transform like properties or attempting to define a Fourier transform, it is helpful to look for a Banach algebra structure lurking in the background. We take this route in Section 2.6.

1.3 Unitary Representations of Locally Compact Groups

Definition 1.3.1 (Unitary representation)**.** Let $\mathcal{G}$ be a locally compact group. A *unitary representation* of $\mathcal{G}$ is a Hilbert space $\mathcal{H}$ over $\mathbb{C}$ and a homomorphism π :

$\mathcal{G} \to \mathcal{U}(\mathcal{H})$ from $\mathcal{G}$ into the group of unitary operators on $\mathcal{H}$ that is continuous with respect to the strong operator topology. We enumerate these properties here for convenience:

1. $\forall g, h \in \mathcal{G}$, $\pi(gh) = \pi(g)\pi(h)$,

2. $\forall g \in \mathcal{G}$, $\pi(g^{-1}) = \pi(g)^{-1} = \pi(g)^*$,

3. for every net $\{g_\lambda\}_\lambda$ converging to g and every $x \in \mathcal{H}$, the net $\{\pi(g_\lambda)x\}_\lambda$ converges to $\pi(g)x$.

The dimension of $\mathcal{H}$ is called the *dimension* of π. When $\mathcal{G}$ is a finite group (as will be the case for us) $\mathcal{G}$ is given the discrete topology and continuity of π holds trivially. We denote a representation by $(\mathcal{H}, \pi)$ or, when $\mathcal{H}$ is understood, π.

Definition 1.3.2 (Equivalence of representations)**.** Let $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ be representations of $\mathcal{G}$. We say that a bounded linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ is an *intertwining operator* for π_1 and π_2 if

$$\forall g \in \mathcal{G}, \quad T\pi_1(g) = \pi_2(g)T.$$

π_1 and π_2 are said to be *unitarily equivalent* if there is a unitary intertwining operator U for π_1 and π_2.

More generally, we could consider non-unitary representations, where π is a homomorphism into the space of invertible operators on a Hilbert space. We do not do that here for two reasons. First, we will be mainly interested in the regular representations, which are unitary, and second, every finite dimensional representation of a finite group is unitarizable. That is, if $(\mathcal{H}, \pi)$ is a finite dimensional representation (not necessarily unitary) of $\mathcal{G}$ and $|\mathcal{G}| < \infty$, then there exists an inner product on $\mathcal{H}$ such that π is unitary. See Theorem 1.5 of [32] for a proof of this fact. Given that our focus will be on finite groups, we may as well consider only unitary representations. We will freely omit the word "unitary" and speak only of "representations", which we assume to be unitary unless stated otherwise. Similarly, we will omit the word "unitarily" when we speak of equivalence of representations.

Example 1.3.3. Let $\mathcal{G}$ be a finite group, and let $\ell^2 = \ell^2(\mathcal{G})$. The action of $\mathcal{G}$ on ℓ^2 by left translation is a unitary representation of $\mathcal{G}$. More concretely, let $\{e_h\}_{h\in\mathcal{G}}$ be the standard orthonormal basis for ℓ^2, and define $\lambda : \mathcal{G} \to \mathcal{U}(\ell^2)$ by

$$\forall g, h \in \mathcal{G}, \quad \lambda(g)e_h = e_{gh}.$$

λ is called the *left regular representation* of $\mathcal{G}$. The *right regular representation*, which we denote by ρ, is defined as translation on the right, i.e.,

$$\forall g, h \in \mathcal{G}, \quad \rho(g)e_h = e_{hg^{-1}}.$$

The construction is similar for general locally compact groups and takes place on $L^2(\mathcal{G})$.

1.3.1 Irreducible Representations

Definition 1.3.4 (Invariant subspace)**.** An *invariant subspace* of a unitary representation $(\mathcal{H}, \pi)$ is a closed subspace $X \subset \mathcal{H}$ such that $\pi(g)X \subset X$ for all $g \in \mathcal{G}$. The restriction of π to X is a unitary representation of $\mathcal{G}$ called a *subrepresentation*.

If π has a nontrivial subrepresentation, i.e., nonzero and not equal to π, or equivalently, has a nontrivial invariant subspace, then π is called *reducible*. If π has no nontrivial subrepresentations or, equivalently, has no nontrivial invariant subspaces, then π is called *irreducible*.

Definition 1.3.5 (Direct sum of representations)**.** Let $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ be representations of $\mathcal{G}$. Then

$$(\mathcal{H}_1 \oplus \mathcal{H}_2, \pi_1 \oplus \pi_2),$$

where $(\pi_1 \oplus \pi_2)(g)(x_1, x_2) := (\pi_1(g)(x_1), \pi_2(g)(x_2))$, for $g \in \mathcal{G}, x_1 \in \mathcal{H}_1, x_2 \in \mathcal{H}_2$, is a representation of $\mathcal{G}$ called the *direct sum* of the representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$.

More generally, for a positive integer m, we recursively define the direct sum of m representations $\pi_1 \oplus \ldots \oplus \pi_m$. If $(\mathcal{H}, \pi)$ is a representation of $\mathcal{G}$, we denote by $m\pi$ the representation that is the product of m copies of π, i.e., $(\mathcal{H} \oplus \ldots \oplus \mathcal{H}, \pi \oplus \ldots \oplus \pi)$, where each sum has m terms. Clearly, a direct sum of nontrivial representations cannot be irreducible, e.g., $(\mathcal{H}_1 \oplus \mathcal{H}_2, \pi_1 \oplus \pi_2)$ will have invariant subspaces $\mathcal{H}_1 \oplus \{0\}$ and $\{0\} \oplus \mathcal{H}_2$.

Definition 1.3.6 (Complete reducibility)**.** A representation $(\mathcal{H}, \pi)$ is called *completely reducible* if it is the direct sum of irreducible representations.

Two classical problems of harmonic analysis on a locally compact group $\mathcal{G}$ are to describe all the unitary representations of $\mathcal{G}$ and to describe how unitary representations can be built as direct sums of smaller representations. For finite groups the forthcoming Maschke's theorem tells us that the irreducible representations are the building blocks of representation theory that enable these descriptions.

Lemma 1.3.7. *Let $(\mathcal{H}, \pi)$ be a unitary representation of $\mathcal{G}$. If $X \subset \mathcal{H}$ is invariant under π, then $X^{\perp} = \{y \in \mathcal{H} : \forall x \in X, \langle x, y \rangle = 0\}$ is also invariant under π.*

Proof. Let $y \in X^{\perp}$. Then for any $x \in X$ and $g \in \mathcal{G}$, we have $\langle x, \pi(g)y \rangle = \langle \pi(g^{-1})x, y \rangle = 0$; therefore, $\pi(g)y \in X^{\perp}$. ■

We are ready to prove a version of Maschke's theorem for finite dimensional representations of finite groups.

Theorem 1.3.8 (Maschke's theorem)**.** *Every finite dimensional unitary representation of a finite group is completely reducible.*

Proof. Let $(\mathcal{H}, \pi)$ be a representation of a finite group $\mathcal{G}$ with dimension $n < \infty$. If π is irreducible we are done; otherwise, let X_1 be a nontrivial invariant subspace of π. By Lemma 1.3.7, $X_2 := X_1^{\perp}$ is also an invariant subspace of π. Letting π_1 and π_2 be the restrictions of π to X_1 and X_2 respectively, we have $\pi = \pi_1 \oplus \pi_2$, $\dim X_1 < n$, and $\dim X_2 < n$. Proceeding inductively we obtain a sequence of representations of strictly decreasing dimension which must terminate and yield a decomposition of π into a direct sum of irreducible representations. ■

If $(\mathcal{H}, \pi)$ is a representation, then we let $\mathcal{A}_\pi$ denote the algebra of operators on $\mathcal{H}$ generated by $\{\pi(g)\}_{g \in \mathcal{G}}$. When $\mathcal{G}$ is finite we have

$$\mathcal{A}_\pi = \left\{ \sum_g a_g \pi(g) : \{a_g\}_{g \in \mathcal{G}} \subset \mathbb{C} \right\}.$$

There is a useful lemma that describes the commutant of an irreducible representation.

Lemma 1.3.9 (Schur's lemma, Lemma 3.5 of [19]).

1. *A unitary representation* $(\mathcal{H}, \pi)$ *is irreducible if and only if* $\mathcal{A}'_\pi$ *contains only scalar multiples of the identity.*

2. *Suppose* T *is an intertwining operator for irreducible representations* $(\mathcal{H}_1, \pi_1)$ *and* $(\mathcal{H}_2, \pi_2)$ *of* $\mathcal{G}$*. If* π_1 *and* π_2 *are inequivalent, then* $T = 0$*.*

■

1.3.2 Character Theory

Throughout this section $\mathcal{G}$ is a finite group. It will be convenient for us to define a normalized scalar product on $L^2(\mathcal{G})$ by scaling Haar measure on $\mathcal{G}$ such that the measure of the entire group is 1. That is, on $L^2(\mathcal{G})$ define the scalar product $(\cdot|\cdot)$ by

$$(f_1|f_2) = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} f_1(g)\overline{f_2(g)}.$$

This scalar product defines a norm equivalent to the usual norm on L^2.

Definition 1.3.10 (Characters). Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{G}$. The *character* of π is the function χ_π on $\mathcal{G}$ taking complex values defined by

$$\forall g \in \mathcal{G}, \quad \chi_\pi(g) = \mathrm{Tr}(\pi(g)),$$

where $\mathrm{Tr}(\pi(g))$ is the trace of $\pi(g)$. The *irreducible characters* of $\mathcal{G}$ is the set of characters of inequivalent irreducible representations of $\mathcal{G}$.

The trace of a product of matrices is invariant under cyclic permutations. This implies equivalent representations have the same character and on each conjugacy class the function χ_π is constant.

Definition 1.3.11 (Class function)**.** A *class function* on $\mathcal{G}$ is a function constant on each conjugacy class of $\mathcal{G}$. The class functions form a subspace of $L^2(\mathcal{G})$.

The following theorems are standard facts about characters.

Theorem 1.3.12 (Theorem 3.6 of [32])**.** *The irreducible characters form an orthonormal basis of the vector space of class functions.* ■

That the irreducible characters form an orthogonal set in $L^2(\mathcal{G})$ implies there are at most $|\mathcal{G}|$ irreducible characters. Consequently, there are finitely many inequivalent irreducible representations for any finite group $\mathcal{G}$. Furthermore, the vector space of class functions clearly has dimension equal to the number of conjugacy classes of $\mathcal{G}$. Therefore, the number of irreducible characters, and equivalence classes of irreducible representations, of a finite group is equal to the number of conjugacy classes of that group.

Let N be the number of conjugacy classes of $\mathcal{G}$. We denote by $\pi_1, \pi_2, \ldots, \pi_N$ the inequivalent irreducible representations of $\mathcal{G}$. More precisely, each π_i is a choice of representative from an equivalence class of irreducible representations of $\mathcal{G}$.

Theorem 1.3.13 (Theorem 2.15 of [32])**.** *Let π be a representation of $\mathcal{G}$ and χ_π its character. Then*

$$\pi = \bigoplus_{i=1}^{N} m_i \pi_i,$$

where

$$m_i = (\chi_\pi | \chi_{\pi_i}).$$

■

Definition 1.3.14 (Isotypic components)**.** Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{G}$. If $(\mathcal{H}, \pi)$ admits the decomposition

$$\pi = m_1\pi_1 \oplus m_2\pi_2 \oplus \ldots \oplus m_N\pi_N,$$

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_N,$$

where $\mathcal{H}_i = m_i\mathcal{K}_i$ and $\pi|_{\mathcal{K}_i} = \pi_i$, i.e., $\mathcal{H}_i$ is the direct sum of m_i copies of $\mathcal{K}_i$ and π acts on $\mathcal{K}_i$ as π_i. Then the nonnegative integer m_i is the *multiplicity* of π_i in π, $m_i\pi_i$ is the *isotypic component of type* π_i of π or the *i-th isotypic component*, and $\mathcal{H}_i$ is the *support* of the i-th isotypic component. If we let $\mathcal{K}_{i,j} = \mathcal{K}_i$ for $1 \leq j \leq m_i$, then $\mathcal{H}$ has the decomposition

$$\mathcal{H} = \bigoplus_{i=1}^{N} \bigoplus_{j=1}^{m_i} \mathcal{K}_{i,j},$$

and the restriction of π to $\mathcal{K}_{i,j}$ is π_i.

As consequences of Theorem 1.3.13 we have that the decomposition of a representation into its isotypic components is unique up to order and two representations with the same character have the same isotypic decomposition and are equivalent.

1.4 Frames

The setting for our later chapters is frame theory. Frames are a generalization of orthonormal bases where we relax Parseval's identity to allow for overcompleteness. Frames were first introduced in 1952 by Duffin and Schaeffer [17] and have become the subject of intense study since the 1980s. Christensen's book [11] contains a succinct overview of elementary frame theory, and Heil's book [26] is an excellent introduction to basis theory and a foundation for studying Frames. We also recommend the introduction to finite frame theory [9].

Definition 1.4.1 (Frame)**.** Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\Phi = \{\varphi_j\}_{j\in J}$ (finite or countably infinite) of elements of $\mathcal{H}$ is a *frame* if there are positive constants A and B such that

$$\forall x \in \mathcal{H}, \quad A\left\|x\right\|^2 \leq \sum_{j\in J} \left|\langle x, \varphi_j\rangle\right|^2 \leq B\left\|x\right\|^2. \tag{1.2}$$

The optimal constants, the supremum over all such A and infimum over all such B, are called the *lower* and *upper frame bounds* respectively. A λ*-tight frame* (a λ-TF) is a frame with frame bounds $A = B = \lambda$, and if $A = B = 1$ the frame is *Parseval tight.* A frame is *equal-norm* (a ENF) if all the elements in the frame sequence have the same norm and *unit-norm* (a UNF) if all the elements have norm 1. A sequence of elements of $\mathcal{H}$ satisfying an upper frame bound is a *Bessel sequence.*

Remark 1.4.2. The series in (1.2) is an absolutely convergent series of positive numbers; and so, any reordering of the sequence of frame elements or reindexing by another set of the same cardinality will remain a frame. We allow for repetitions of vectors in a frame so that strictly speaking the set of vectors, which we also call Φ, is a multiset. We will index frames by either an arbitrary set (such as J in the definition) or the positive integers when it is convenient to do so.

Remark 1.4.3. Geometrically pleasing examples of finite equal-norm tight frames abound, e.g., the vertices of the Platonic solids. Finite ENTFs also have an interesting characterization as the minima of a potential energy function, see [6] for the details of this result.

Let $\Phi = \{\varphi_j\}_{j\in J}$ be a frame for $\mathcal{H}$. We define four operators associated with every frame; these operators are crucial to frame theory and will be used extensively. The *analysis operator* $L : \mathcal{H} \to \ell^2(J)$ is defined by

$$Lx = \{\langle x, \varphi_j\rangle\}_{j\in J}.$$

Inequality (1.2) ensures that the analysis operator L is bounded and $\|L\|_{op} \leq \sqrt{B}$. The adjoint of the analysis operator is the *synthesis operator* $L^* : \ell^2(J) \to \mathcal{H}$, and it is defined by

$$L^*a = \sum_{j\in J} a_j\varphi_j.$$

From Hilbert space theory, we know that any bounded linear operator T on $\mathcal{H}$ satisfies $\|T\|_{op} = \|T^*\|_{op}$; therefore, the synthesis operator L^* is bounded and $\|L^*\|_{op} \leq \sqrt{B}$.

The *frame operator* is the map $S : \mathcal{H} \to \mathcal{H}$ defined as $S = L^*L$, i.e.,

$$\forall x \in \mathcal{H}, \quad Sx = \sum_{j\in J} \langle x, \varphi_j\rangle \varphi_j.$$

We will go into detail in describing S; first,

$$\forall x \in \mathcal{H}, \quad \langle Sx, x\rangle = \sum_{j\in J} |\langle x, \varphi_j\rangle|^2 .$$

Thus, S is a positive and self-adjoint operator, and (1.2) can be rewritten as

$$\forall x \in \mathcal{H}, \quad A\|x\|^2 \leq \langle Sx, x\rangle \leq B\|x\|^2$$

or, more compactly,

$$AI \leq S \leq BI.$$

It follows that S is invertible (Lemma 3.2.2 in [13]), S is a multiple of the identity precisely when Φ is a tight frame, and

$$B^{-1}I \leq S^{-1} \leq A^{-1}I. \tag{1.3}$$

Hence, S^{-1} is a positive self-adjoint operator and has a square root $S^{-1/2}$ (Theorem 12.33 in [36]). This square root can be written as a power series in S^{-1}; consequently, it commutes with every operator that commutes with S^{-1} (in particular S). Utilizing these facts we can prove a theorem that tells us frames share an important property with orthonormal bases: a reconstruction formula.

Theorem 1.4.4 (Frame reconstruction formula). *Let $\mathcal{H}$ be a separable Hilbert space, and let $\Phi = \{\varphi_j\}_{j\in J}$ be a frame for $\mathcal{H}$ with frame operator S. Then*

$$\forall x \in \mathcal{H}, \quad x = \sum_{j\in J} \langle x, \varphi_j \rangle S^{-1}\varphi_j = \sum_{j\in J} \left\langle x, S^{-1}\varphi_j \right\rangle \varphi_j = \sum_{j\in J} \left\langle x, S^{-1/2}\varphi_j \right\rangle S^{-1/2}\varphi_j.$$

Proof. The proof is three computations. From $I = S^{-1}S$, we have

$$\forall x \in \mathcal{H}, \quad x = S^{-1}Sx = S^{-1}\sum_{j\in J} \langle x, \varphi_j \rangle \varphi_j = \sum_{j\in J} \langle x, \varphi_j \rangle S^{-1}\varphi_j;$$

from $I = SS^{-1}$, we have

$$\forall x \in \mathcal{H}, \quad x = SS^{-1}x = \sum_{j\in J} \left\langle S^{-1}x, \varphi_j \right\rangle \varphi_j = \sum_{j\in J} \left\langle x, S^{-1}\varphi_j \right\rangle \varphi_j;$$

and from $I = S^{-1/2}SS^{-1/2}$, it follows that

$$\forall x \in \mathcal{H},\ x = S^{-1/2}SS^{-1/2}x = S^{-1/2}\sum_{j\in J} \left\langle S^{-1/2}x, \varphi_j \right\rangle \varphi_j = \sum_{j\in J} \left\langle x, S^{-1/2}\varphi_j \right\rangle S^{-1/2}\varphi_j.$$

■

From the frame reconstruction formula and (1.3), it follows that $\{S^{-1}\varphi_j\}_{j\in J}$ is a frame with frame bounds B^{-1} and A^{-1} and $\{S^{-1/2}\varphi_j\}_{j\in J}$ is a Parseval tight frame.

Definition 1.4.5 (Canonical dual)**.** Let $\Phi = \{\varphi_j\}_{j\in J}$ be a frame for a separable Hilbert space $\mathcal{H}$ with frame operator S. The frame $S^{-1}\Phi = \{S^{-1}\varphi_j\}_{j\in J}$ is called the *canonical dual frame* of Φ. The frame $S^{-1/2}\Phi = \{S^{-1/2}\varphi_j\}_{j\in J}$ is called the *canonical tight frame* of Φ.

The *Gramian operator* is the map $G : \ell^2(J) \to \ell^2(J)$ defined as $G = LL^*$. If $\{e_j\}_{j\in J}$ is the standard orthonormal basis for $\ell^2(J)$, then

$$\forall a = \{a_j\}_{j\in J} \in \ell^2(J), \quad \langle Ga, e_k\rangle = \sum_{j\in J} a_j \langle \varphi_j, \varphi_k\rangle \,. \tag{1.4}$$

The following theorem, a weak variant of Naimark's dilation theorem, tells us every Parseval tight frame is the projection of an orthonormal basis in a larger space. The general form of Naimark's dilation theorem is a result for an uncountable family of increasing operators on a Hilbert space satisfying some additional conditions. It states that it is possible to construct an embedding into a larger space such that the dilation of the operators to this larger space commute and are a resolution of the identity. For an excellent description of this dilation problem and an independent geometric proof of a finite version of Naimark's dilation theorem we recommend an article by C. H. Davis, [14]. To see the connection of this general theorem with the one below, consider the finite sums of the rank one projections onto the subspaces spanned by elements of a Parseval frame.

Theorem 1.4.6 (Naimark's Theorem, Theorem 2.1 of Casazza and Kovačević [8], cf. Naimark [1] and Han and Larson [25]). *A set $\Phi = \{\varphi_j\}_{j\in J}$ in a Hilbert space $\mathcal{H}$ is a Parseval tight frame for $\mathcal{H}$ if and only if there is a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and an orthonormal basis $\{e_j\}_{j\in J}$ for $\mathcal{K}$ such that the orthogonal projection P of $\mathcal{K}$ onto $\mathcal{H}$ satisfies*

$$\forall j \in J, \quad Pe_j = \varphi_j.$$

Proof. Let $\Phi = \{\varphi_j\}_{j\in J}$ be a Parseval tight frame for $\mathcal{H}$, let $\mathcal{K} = \ell^2(J)$, and let L be the analysis operator of Φ. Since Φ is a Parseval tight frame for $\mathcal{H}$, we have

$$\|Lx\|^2 = \sum_{j\in J} |\langle x, \varphi_j\rangle|^2 = \|x\|^2 .$$

Thus, L is an isometry, and we can embed $\mathcal{H}$ into $\mathcal{K}$ by identifying $\mathcal{H}$ with $L(\mathcal{H})$. Let P be the orthogonal projection from $\mathcal{K}$ onto $L(\mathcal{H})$. Denote the standard orthonormal basis for $\mathcal{K}$ by $\{e_j\}_{j\in J}$. We claim that $Pe_n = L\varphi_n$. For any $m \in J$, we have

$$\begin{aligned} \langle L\varphi_m, Pe_n\rangle &= \langle PL\varphi_m, e_n\rangle = \langle L\varphi_m, e_n\rangle \\ &= \langle \varphi_m, \varphi_n\rangle = \langle L\varphi_m, L\varphi_n\rangle . \end{aligned} \tag{1.5}$$

In (1.5) we use the fact that P is an orthogonal projection for the first equality, that $L\varphi_m$ is in the range of P for the second, the definition of L and $\{e_j\}_{j\in J}$ for the third, and that L is an isometry for the last. Rearranging (1.5) yields

$$\langle L\varphi_m, Pe_n - L\varphi_n\rangle = 0.$$

Since the vectors $L\varphi_m$ span $L(\mathcal{H})$ it follows that $Pe_n - L\varphi_n \perp L(\mathcal{H})$, but $Pe_n - L\varphi_n \in L(\mathcal{H})$. Thus, $Pe_n - L\varphi_n = 0$ as claimed.

For the converse, assume that $\mathcal{H} \subset \mathcal{K}$, $\{e_j\}_{j \in J}$ is an orthonormal basis for $\mathcal{K}$, P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, and $Pe_j = \varphi_j$. We claim that $\Phi = \{\varphi_j\}_{j \in J}$ is a Parseval tight frame for $\mathcal{H}$. For any $x \in \mathcal{K}$, we have Parseval's identity

$$\|x\|^2 = \sum_{j \in J} |\langle x, e_j \rangle|^2 .$$

For $x \in \mathcal{H}$, we additionally have $Px = x$. Thus,

$$\forall x \in \mathcal{H}, \quad \|x\|^2 = \sum_{j \in J} |\langle x, e_j \rangle|^2 = \sum_{j \in J} |\langle Px, e_j \rangle|^2 = \sum_{j \in J} |\langle x, Pe_j \rangle|^2 ,$$

i.e., $\{\varphi_j\}_{j \in J} = \{Pe_j\}_{j \in J}$ is a Parseval tight frame for $\mathcal{H}$. ■

Remark 1.4.7. If Φ is a Parseval tight frame, then $L^*L = S = I$, so $G^2 = LL^*LL^* = LL^* = G$. Hence, G is a projection, and since it is self-adjoint it is an orthogonal projection. Furthermore, $Ge_j = LL^*e_j = L\varphi_j$. So the orthogonal projection P onto $L(\mathcal{H})$ from Naimark's theorem is precisely G.

When $\mathcal{H}$ is finite dimensional ($\mathbb{C}^d$ or $\mathbb{R}^d$) and $\Phi = \{\varphi_j\}_{j=0}^{N-1}$, each of the above operators can be realized as multiplication on the left by a matrix. The synthesis operator is the $d \times N$ matrix with the frame elements as its columns

$$L^* = \left(\begin{array}{c|c|c|c} \varphi_0 & \varphi_1 & \dots & \varphi_{N-1} \end{array} \right),$$

and the analysis operator is the $N \times d$ matrix with the conjugate transpose of the frame elements as its rows

$$L = \begin{pmatrix} \varphi_0^* \\ \varphi_1^* \\ \vdots \\ \varphi_{N-1}^* \end{pmatrix} .$$

The frame operator and Gramian are the proper products of these matrices. From direct multiplication of LL^* or (1.4) it is apparent that the Gramian or Gram matrix has entries

$$G_{jk} = \langle \varphi_k, \varphi_j \rangle .$$

1.5 Outline of Results

In Section 2.1, we define the vector-valued discrete Fourier transform (DFT), and in Section 2.2 we prove the invertibility of the vector-valued DFT and its modulation and translation properties. Leveraging this theory, in Section 2.3, we define the discrete vector-valued ambiguity function A_p^d, a goal we introduce at the beginning of Chapter 2. In Section 2.5.2, we expand on the vector-valued DFT theory by proving an uncertainty principle, and in Section 2.6 we describe the Banach algebra from which the vector-valued DFT arises.

In Chapter 3, we introduce the notion of frame multiplication, which is motivated by the vector-valued theory of Chapter 2. In Section 3.2, we define frame multiplication and prove necessary and sufficient conditions for an operation to define a frame multiplication. We also define the multiplications of a frame and prove equivalent tight frames share the same set of multiplications while the converse is false. In particular, there exists an uncountable family of inequivalent tight frames all of which share the same frame multiplications. In Section 3.3, we prove a necessary and sufficient condition on the Gramian of a frame for a quasigroup operation to be a frame multiplication.

In Chapter 4, we specialize to the case of group frame multiplications. In Section 4.2, we prove the equivalence of G-frames with the family of frames for which an abelian group $\mathcal{G}$ defines a frame multiplication. We then describe the equivalence classes of frames and the associated bilinear products for which an abelian group defines a frame multiplication. In Section 4.3, we generalize to the case of non-abelian groups. In Section 4.3.2, we characterize frames for which an arbitrary group defines a frame multiplication as those whose Gramian is in the commutant of the regular representations. We then prove an explicit formula for these Gramian operators as finite sums of certain projections. In particular, we show that for any group $\mathcal{G}$, there are only finitely many inequivalent tight frames for which $\mathcal{G}$ defines a frame multiplication. We also give a formula for the dimensions in which such frames exist. In Section 4.3.3, we prove the equivalence of frames for which $\mathcal{G}$ defines a frame multiplication with the family of central G-frames as defined by Vale and Waldron in [41].

Chapter 2

Vector-valued Harmonic Analysis

In 1953, P. M. Woodward [43] defined the narrow band radar ambiguity function or, simply, ambiguity function. The ambiguity function is a two-dimensional function of delay t and Doppler frequency γ that measures the correlation between a waveform w and its Doppler distorted version. The information given by the ambiguity function is important for practical purposes in radar. In fact, the *waveform design problem* is the problem of designing waveforms with "good" ambiguity functions for practical purposes. The *ambiguity function* $A(w)$ of $w \in L^2(\mathbb{R})$ is

$$A(w)(t,\gamma) = \int w(s+t)\overline{w(s)}e^{-2\pi i s\gamma}\,ds \tag{2.1}$$

for $(t,\gamma) \in \mathbb{R}^2$. We shall only be interested in the discrete version of (2.1). For an N-periodic function $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ the *discrete periodic ambiguity function* is

$$A_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} u(m+k)\overline{u(k)}e^{-2\pi i kn/N},$$

for $(m,n) \in \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$.

Remark 2.0.1. If $v, w \in L^2(\mathbb{R})$, the *narrow band cross-ambiguity function* $A(v,w)$ of v and w is

$$\begin{aligned} A(v,w)(t,\gamma) &= \int v(s+t)\overline{w(s)}e^{-2\pi i s\gamma}\,ds \\ &= e^{2\pi i t\gamma}\int v(s)\overline{w(s-t)}e^{-2\pi i s\gamma}\,ds. \end{aligned} \tag{2.2}$$

Evidently, $A(w) = A(w, w)$, so that the ambiguity function is a special case of the cross-ambiguity function. From (2.2) we see that the cross-ambiguity function is the short-time Fourier transform (STFT) of v with window $\overline{w}$ multiplied by $e^{2\pi i t\gamma}$, which has modulus 1; hence, $|A(v, w)|$ is the *spectrogram* of v.

For $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$, Benedetto and Donatelli [5] defined the discrete vector-valued ambiguity function $A_p^d(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$ by observing the following. Given $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$. If $d = 1$, then we can write $A_p(u)$ as

$$\begin{aligned} A_p(u)(m, n) &= \frac{1}{N} \sum_{k=0}^{N-1} \langle u(m + k), u(k) W_{kn} \rangle \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \left\langle \tau_{-m} u(k), F^{-1}(\tau_n \widehat{u})(k) \right\rangle, \end{aligned} \tag{2.3}$$

where $W_n = e^{2\pi i n/N}$, τ_k is the usual translation operator, and F^{-1} is the inverse Fourier transform on $\mathbb{Z}/N\mathbb{Z}$. Note that $\{W_n\}_{n=0}^{N-1}$ is a tight frame for $\mathbb{C}$. If instead u is vector-valued, i.e., $d > 1$, then we may define the discrete periodic ambiguity function of u in two ways: as a $\mathbb{C}$-valued function or as a $\mathbb{C}^d$-valued function.

First, we consider the case of a $\mathbb{C}$-valued ambiguity function. Inspired by (2.3), for $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$, we define $A_p^1(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ by

$$A_p^1(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} \langle u(m + k), u(k) * \varphi_{kn} \rangle \tag{2.4}$$

where $\{\varphi_n\}_{n=0}^{N-1} \subset \mathbb{C}^d$ and $*$ is a vector multiplication. Presented with this definition, a natural question is: how do we find a sequence of vectors $\{\varphi_n\}$ and a multiplication $*$ such that (2.4) makes sense and is a meaningful ambiguity function?

Motivated by the fact that $W_m W_n = W_{m+n}$, Benedetto and Donatelli make the following *ambiguity function assumptions*. They assume there is a sequence

$\{\varphi_n\}_{n=0}^{N-1} \subseteq \mathbb{C}^d$ and multiplication $* : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$ such that

$$\varphi_m * \varphi_n = \varphi_{m+n}, \tag{2.5}$$

for $m, n \in \mathbb{Z}/N\mathbb{Z}$. Next, in order to deal with the product $u(k) * \varphi_{kn}$ in (2.4), they assume that $\{\varphi_n\}_{n=0}^{N-1}$ is a tight frame for $\mathbb{C}^d$ and that the multiplication $*$ is bilinear. This allows $u(k) * \varphi_{kn}$ to be carried out using the frame expansion of $u(k)$ and (2.5). It was known that there exist tight frames satisfying these assumptions, e.g., DFT frames, but the question of classifying them was open. In Section 4.2, we characterize all such tight frames and multiplications.

Second, we define the vector-valued version of the ambiguity function A_p^d as follows: for $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$, define $A_p^d(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$ by

$$A_p^d(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m+k) * \overline{u(k)} * \overline{\varphi_{kn}},$$

where $\{\varphi_n\}_{n=0}^{N-1}$ and $*$ abide by the ambiguity function assumptions. In Section 2.3, we shall see that this definition is compatible with that of $A_p(u)$ in (2.3). Before we can make this connection we require an extension of the discrete Fourier transform to the vector-valued setting.

2.1 Extending the Discrete Fourier Transform

Consider the locally compact abelian group $\mathbb{Z}/N\mathbb{Z}$. The characters of $\mathbb{Z}/N\mathbb{Z}$ are the functions $\{\gamma_m\}_{m=0}^{N-1}$ defined by $n \mapsto e^{2\pi i mn/N}$, so that the dual $\widehat{(\mathbb{Z}/N\mathbb{Z})}$ is isomorphic to $\mathbb{Z}/N\mathbb{Z}$ under the identification $\gamma_m \mapsto m$. Hence, the Fourier transform on $\ell^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{C}^N$ is a linear map that can be expressed as

$$\widehat{x}(m) = \sum_{n=0}^{N-1} x(n) e^{-2\pi i m n/N}. \tag{2.6}$$

Apparently, the Fourier transform has matrix representation

$$D = (e^{-2\pi i m n/N})_{m,n=0}^{N-1}. \tag{2.7}$$

The Fourier transform on $\mathbb{C}^N$ is called the *discrete Fourier transform* (DFT), and the matrix D of this transform is commonly called the *DFT matrix.* The DFT has uses in digital signal processing and a plethora of numerical algorithms. Part of the reason why its use is so ubiquitous is that fast algorithms exist for its computation. The *Fast Fourier Transform* (FFT) allows the computation of the DFT to take place in $O(N \log N)$ operations; this is a significant boost over the $O(N^2)$ operations it would take to compute via (2.6). The fundamental paper on the FFT is due to Cooley and Tukey [12], in which they describe what is now referred to as the Cooley-Tukey FFT algorithm. The algorithm employs a divide and conquer method to break the N dimensional DFT into smaller DFTs that may then be further broken down, computed, and reassembled. For a more extensive description of the DFT, FFT, and their relationship to sampling and the Fourier transform on $\ell^1(\mathbb{Z})$ see [4].

Definition 2.1.1 (DFT frame)**.** Let $N \geq d$, and let $s : \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ be injective. The rows $\Phi = \{\varphi_m\}_{m=0}^{N-1}$ of the $N \times d$ matrix

$$\left(e^{2\pi i m s(n)/N}\right)_{m,n}$$

form an equal-norm tight frame for $\mathbb{C}^d$ called a *DFT frame.* The name comes from the fact that the elements of Φ are projections of the rows of the conjugate of the ordinary DFT matrix (2.7). That Φ is an equal-norm tight frame follows from Naimark's Theorem (Theorem 1.4.6) and the fact that the DFT matrix is an orthogonal matrix.

Definition 2.1.2 (Vector-valued discrete Fourier transform)**.** Let $\{\varphi_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$. Given $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$, we define the *vector-valued discrete Fourier transform* (vector-valued DFT) of u by

$$\forall p \in \mathbb{Z}/N\mathbb{Z}, \quad F(u)(p) = \widehat{u}(p) = \sum_{m=0}^{N-1} u(m)\varphi_{-mp}, \tag{2.8}$$

where the product $u(m)\varphi_{-mp}$ is pointwise multiplication. We have that

$$F : \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}) \to \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$$

is a linear operator.

We will carry the convention used in (2.8) through the rest of the chapter, i.e., the juxtaposition of vectors of equal dimension is the pointwise product of those vectors. We naturally extend this to functions whose values are vectors. For two functions $u, v \in \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$, we let uv be the coordinate-wise, where the coordinates are in $\mathbb{Z}/N\mathbb{Z}$, product of u and v, i.e.,

$$\forall m \in \mathbb{Z}/N\mathbb{Z}, \quad (uv)(m) := u(m)v(m),$$

where the product on the right is pointwise multiplication of vectors in $\ell^2(\mathbb{Z}/d\mathbb{Z})$.

Remark 2.1.3.

(1) We write $u \in \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$ as a function of two arguments so that $u(m)(n) \in \mathbb{C}$. With this notation we can think of u and $\widehat{u}$ as $N \times d$ matrices with entries $u(m)(n)$ and $\widehat{u}(p)(q)$ respectively.

(2) We have

$$\begin{aligned}\widehat{u}(p)(q) &= \left(\sum_{m=0}^{N-1} u(m)\varphi_{-pm}\right)(q)\\ &= \left(\sum_{m=0}^{N-1} u(m)(q)\varphi_{-pm}(q)\right).\end{aligned}$$

From this we see that $\widehat{u}(p)(q)$ depends only on $\{u(m)(q)\}_{m=0}^{N-1}$, i.e., when thought of as matrices the q-th column of $\widehat{u}$ depends only on the q-th column of u.

2.2 Inversion, Translation, and Modulation

The vector-valued DFT shares many properties with the regular DFT. Under certain conditions it can be shown to be invertible, this is the contents of Theorem 2.2.1, the expected relationship between translation on the "time" side and modulation on the "frequency" side holds, and in Section 2.5 we prove an uncertainty principle for the vector-valued DFT.

Theorem 2.2.1 (Andrews, Benedetto, Donatelli)**.** *The vector-valued discrete Fourier transform is invertible if and only if s, the function defining the DFT frame, has the property that*

$$\forall n \in \mathbb{Z}/d\mathbb{Z}, \quad (s(n), N) = 1.$$

The inverse is given by

$$\forall\ m \in \mathbb{Z}/N\mathbb{Z}, \quad u(m) = (F^{-1}\widehat{u})(m) = \frac{1}{N}\sum_{p=0}^{N-1} \widehat{u}(p)\varphi_{mp}.$$

In this case, we also have that $F^*F = FF^* = NI$, where I is the identity operator.

Proof. We first show the forward direction. Suppose there is $n_0 \in \mathbb{Z}/d\mathbb{Z}$ such that $(s(n_0), N) \neq 1$. Then there exists $j, l, M \in \mathbb{N}$ such that $j > 1$, $s(n_0) = jl$, and $N = jM$. Define a matrix A by

$$A := (e^{2\pi imks(n_0)/N})_{m,k=0}^{N-1} = (e^{2\pi imkl/M})_{m,k=0}^{N-1}.$$

A has rank strictly less than N since the 0-th and M-th rows are all 1's. Therefore we may choose a vector $v \in \mathbb{C}^N$ orthogonal to the rows of A. Define $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$ by

$$u(m)(n) = \begin{cases} v(m) & \text{if } n = n_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\forall n \neq n_0, \quad \widehat{u}(m)(n) = \sum_{k=0}^{N-1} u(k)(n)\varphi_{-mk}(n) = \sum_{k=0}^{N-1} 0 \cdot \varphi_{-mk}(n) = 0.$$

While for $n = n_0$,

$$\begin{aligned} \widehat{u}(m)(n_0) &= \sum_{k=0}^{N-1} u(k)(n_0)\varphi_{-mk}(n_0) \\ &= \sum_{k=0}^{N-1} u(k)(n_0)e^{-2\pi imks(n_0)/N} \\ &= \sum_{k=0}^{N-1} u(k)(n_0)e^{-2\pi imkl/M} \\ &= \left\langle u(\cdot)(n_0), e^{2\pi im(\cdot)l/M} \right\rangle. \\ &= \left\langle v, e^{2\pi im(\cdot)l/M} \right\rangle. \\ &= 0. \end{aligned}$$

The final equality follows from the fact that v is orthogonal to the rows of A.

Hence, the vector-valued Fourier transform given by s has non-trivial kernel and is not invertible.

We prove the converse and the formula for the inverse with a direct calculation. We compute

$$\begin{aligned}\sum_{n=0}^{N-1} \widehat{u}(n)\varphi_{mn} &= \sum_{n=0}^{N-1}\left(\sum_{k=0}^{N-1} u(k)\varphi_{-kn}\right)\varphi_{mn} \\ &= \sum_{k=0}^{N-1}\left(u(k)\left(\sum_{n=0}^{N-1}\varphi_{n(m-k)}\right)\right).\end{aligned}$$

We have the r-th component of the last summation is

$$\begin{aligned}\sum_{n=0}^{N-1}\varphi_{n(m-k)}(r) &= \sum_{n=0}^{N-1} e^{2\pi i n(m-k)s(r)/N} \\ &= \begin{cases} N & \text{if } (m-k)s(r) \equiv 0 \bmod N \\ 0 & \text{if } (m-k)s(r) \not\equiv 0 \bmod N. \end{cases}\end{aligned}$$

Since $(s(r), N) = 1$, the first cases occurs if and only if $k = m$. Continuing with the previous calculation, we have

$$\sum_{k=0}^{N-1}\left(u(k)\left(\sum_{n=0}^{N-1}\varphi_{n(m-k)}\right)\right) = Nu(m).$$

Finally, we compute the adjoint of F.

$$\begin{aligned}
\langle Fu, v\rangle &= \sum_{m=0}^{N-1}\sum_{n=0}^{d-1} \widehat{u}(m)(n)\overline{v(m)(n)} \\
&= \sum_{m=0}^{N-1}\sum_{n=0}^{d-1} \left(\sum_{k=0}^{N-1} u(k)(n)\varphi_{-mk}(n)\right)\overline{v(m)(n)} \\
&= \sum_{m=0}^{N-1}\sum_{n=0}^{d-1} \left(\sum_{k=0}^{N-1} u(k)(n)e^{-2\pi imks(n)/N}\right)\overline{v(m)(n)} \\
&= \sum_{k=0}^{N-1}\sum_{n=0}^{d-1} \left(\sum_{m=0}^{N-1} \overline{v(m)(n)e^{2\pi imks(n)/N}}\right)u(k)(n) \\
&= \sum_{k=0}^{N-1}\sum_{n=0}^{d-1} u(k)(n)\overline{\left(\sum_{m=0}^{N-1} v(m)(n)\varphi_{mk}(n)\right)} \\
&= \langle u, F^* v\rangle\,.
\end{aligned}$$

Therefore, F^* is defined by

$$(F^*v)(k) = \sum_{m=0}^{N-1} v(m)\varphi_{mk},$$

and $F^* = NF^{-1}$. ■

Given the above theorem, we may define the *unitary vector-valued discrete Fourier transform* $\mathcal{F}$ by the formula

$$\mathcal{F} = \frac{1}{\sqrt{N}}F.$$

With this definition, we have

$$\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = I,$$

and $\mathcal{F}$ is unitary as described.

Definition 2.2.2 (Translation and modulation)**.** Let $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$, and let

$\{\varphi_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$. For each $j \in \mathbb{Z}/N\mathbb{Z}$, define the *translation operators* $\tau_j : \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}) \to \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$ by

$$\tau_j u(m) = u(m-j)$$

and the *modulations* $\varphi^j : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$ by

$$\varphi^j(k) = \varphi_{jk}.$$

The usual translation and modulation properties of the Fourier transform hold.

Proposition 2.2.3. *Let $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$, and let $\{\varphi_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$ with associated vector-valued Fourier transform F. The following translation and modulation properties hold:*

(a) $F(\tau_j u) = \varphi^{-j}\widehat{u}$,

(b) $F(\varphi^j u) = \tau_j \widehat{u}$.

Proof. (a) We compute

$$\begin{aligned}
\widehat{\tau_j u}(n) &= \sum_{m=0}^{N-1} \tau_j u(m)\varphi_{-mn} \\
&= \sum_{m=0}^{N-1} u(m-j)\varphi_{-mn} \\
&= \sum_{k=-j}^{N-1-j} u(k)\varphi_{-(k+j)n} \\
&= \sum_{k=0}^{N-1} u(k)\varphi_{-kn-jn} \\
&= \varphi_{-jn}\left(\sum_{k=0}^{N-1} u(k)\varphi_{-kn}\right) \\
&= \varphi_{-jn}\widehat{u}(n).
\end{aligned}$$

The third equality follows by setting $k = m - j$, the fourth by reordering the sum and noting that the index of summation is modulo N, and the fifth follows from $\varphi_{j+k} = \varphi_j \varphi_k$ and bilinearity of pointwise products.

(b) We compute

$$\begin{aligned}
\widehat{\varphi^j u}(n) &= \sum_{m=0}^{N-1} (\varphi^j u)(m) \varphi_{-mn} \\
&= \sum_{m=0}^{N-1} \varphi_{jm} u(m) \varphi_{-mn} \\
&= \sum_{m=0}^{N-1} u(m) \varphi_{-m(n-j)} \\
&= \widehat{u}(n-j).
\end{aligned}$$

The third equality follows from commutativity and that $\varphi_{j+k} = \varphi_j \varphi_k$. ■

2.3 The Vector-valued Ambiguity Function

We return to the problem of defining the vector-valued ambiguity function, A_p^d. Let $\{\varphi_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$, and let $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$. Recall our earlier definition:

$$A_p^d(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m+k) * \overline{u(k)} * \overline{\varphi_{kn}}. \tag{2.9}$$

Recognizing that for a DFT frame the multiplication $*$ such that $\varphi_m * \varphi_n = \varphi_{m+n}$ is pointwise multiplication and utilizing the modulation functions $\varphi^j(k)$, we can rewrite (2.9) as

$$A_p^d(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} \tau_{-m} u(k) \overline{u(k) \varphi^n(k)}.$$

Furthermore, the modulation and translation properties of the vector-valued DFT, and Proposition 2.2.3, imply

$$\begin{aligned} A_p^d(u)(m,n) &= \frac{1}{N}\sum_{k=0}^{N-1} \tau_{-m}u(k)\overline{F^{-1}\left(\tau_n\widehat{u}\right)(k)} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} \left\{\tau_{-m}u(k), F^{-1}\left(\tau_n\widehat{u}\right)(k)\right\}, \end{aligned}$$

where $\{\cdot,\cdot\}$ is the generalized inner product $\{u,v\} := u\overline{v}$ for $u,v \in \mathbb{C}^d$. Thus, utilizing DFT frames our definition (2.9) gives the expected relationship between the vector-valued ambiguity function and the vector-valued discrete Fourier transform. This is also compatible with our view of defining the ambiguity function in the context of the STFT.

2.4 An Alternative Description of the Vector-valued DFT

In this brief section we describe a different way of viewing the vector-valued discrete Fourier transform that makes some of the above properties more apparent. Given $N \in \mathbb{N}$, define matrices $\mathcal{D}_\ell$ by the formula

$$\mathcal{D}_\ell := (e^{-2\pi i mn\ell/N})_{m,n=0}^{N-1}.$$

By definition of the vector-valued discrete Fourier transform,

$$\begin{aligned} \widehat{u}(p)(q) &= \left(\sum_{m=0}^{N-1} u(m)(q)\varphi_{-pm}(q)\right) \\ &= \left(\sum_{m=0}^{N-1} u(m)(q)e^{-2\pi i pms(q)/N}\right) \\ &= \left(\mathcal{D}_{s(q)}u(\cdot)(q)\right)(p). \end{aligned}$$

That is, the vector $\widehat{u}(\cdot)(q)$ is equal to the vector $\mathcal{D}_{s(q)}u(\cdot)(q)$. In other words, we get $\widehat{u}$ by applying the matrix $\mathcal{D}_{s(q)}$ to the q–th column of u for each $0 \leq q \leq d-1$. Therefore, F is invertible if and only if each matrix $\mathcal{D}_{s(q)}$ is invertible.

The rows of $\mathcal{D}_\ell$ are a subset of the rows of the regular DFT matrix, and each row of the DFT matrix is a character of $\mathbb{Z}/N\mathbb{Z}$. Taken as a collection, the characters form the dual group $\widehat{(\mathbb{Z}/N\mathbb{Z})} \simeq \mathbb{Z}/N\mathbb{Z}$ under pointwise multiplication. With this group operation and the fact that

$$\forall m, n \in \mathbb{Z}/N\mathbb{Z}, \quad e^{-2\pi i m n \ell/N} = (e^{-2\pi i n\ell/N})^m,$$

we see the rows of $\mathcal{D}_\ell$ are the orbit of some element $\gamma \in \widehat{(\mathbb{Z}/N\mathbb{Z})}$ repeated $|\gamma|/N$ times. Hence, $\mathcal{D}_\ell$ is invertible if and only if γ generates the entire dual group. From the theory of cyclic groups, γ is a generator of $\widehat{(\mathbb{Z}/N\mathbb{Z})}$ if and only if $\gamma = (e^{-2\pi i n\ell/N})_{n=0}^{N-1}$ for some ℓ relatively prime to N. Therefore, F is invertible if and only if $s(q)$ is relatively prime to N for each q.

Example 2.4.1. Let $N = 4$ and $\omega = e^{-2\pi i/4}$. We compute some of the matrices $\mathcal{D}_\ell$.

$$\mathcal{D}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{pmatrix} \quad \mathcal{D}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & 1 & 1 & 1 \\ 1 & \omega^2 & 1 & \omega^2 \end{pmatrix} \quad \mathcal{D}_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^3 & \omega^2 & \omega \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^1 & \omega^2 & \omega^3 \end{pmatrix}$$

It is easy to see that $\mathcal{D}_1$ and $\mathcal{D}_3$ are invertible while $\mathcal{D}_2$ is not invertible. In each case the matrix $\mathcal{D}_i$ is generated by pointwise powers of its second row, which have orders 4, 2, and 4 respectively.

2.5 Uncertainty Principles

Loosely speaking, inequalities involving both a function f and its Fourier transform $\widehat{f}$ are called uncertainty principles. In their excellent survey [20], Folland and Sitaram sum up the family of theorems in a single meta theorem:

> *A non-zero function and its Fourier transform cannot both be sharply localized.*

The most famous of these theorems is the classical Heisenberg uncertainty principle for $L^2(\mathbb{R}^d)$. The Heisenberg uncertainty principle has an interpretation in quantum mechanics where it asserts a limit to the precision in which the position and momentum of a particle may be known simultaneously. The theorem is named for theoretical physicist Werner Heisenberg who first developed the associated physical ideas in 1927 [27]. Donoho and Stark proved in [16] a refinement of the classical uncertainty principle. They showed that a function and its Fourier transform cannot both be highly concentrated on any two "sets of concentration". The Donoho-Stark uncertainty principle has a natural discrete analog, and in [39] Tao proved a refinement of this for the group $\mathbb{Z}/p\mathbb{Z}$ when p is a prime. For an overview of the role of uncertainty principles in time-frequency analysis we recommend [22].

2.5.1 Self-adjoint Operators and the Classical Uncertainty Principle

We begin our background material on uncertainty principles with the classical Heisenberg uncertainty principle for dimension $d = 1$.

Theorem 2.5.1 (Heisenberg uncertainty principle)**.** *If $f \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$, then*

$$\left(\int (x-a)^2 |f(x)|^2 \, dx\right)^{1/2} \left(\int (\gamma - b)^2 |\widehat{f}(\gamma)|^2 \, d\gamma\right)^{1/2} \geq \frac{1}{4\pi} \|f\|_2^2 .$$

Equality holds if and only if $f(x) = z_0 e^{2\pi i b x} e^{-c(x-a)^2}$ for some $z_0 \in \mathbb{C}$ and $c > 0$.

The theorem can be proved using a combination of integration by parts, the Cauchy-Schwarz inequality, and Plancherel's theorem, but we shall take a higher level approach which generalizes to the vector-valued discrete Fourier transform.

For two linear operators A and B on a Hilbert space $\mathcal{H}$, we denote their commutator by

$$[A, B] = AB - BA.$$

The expected value of a self-adjoint operator A in a state a is defined by the expression

$$\langle A \rangle = \langle Aa, a \rangle,$$

and since A is self-adjoint we have

$$\langle A^2 \rangle = \langle Aa, Aa \rangle = \|Aa\|^2 .$$

Lemma 2.5.2. *Let A and B be self-adjoint operators on a Hilbert space $\mathcal{H}$. Define the self-adjoint operators $T = AB + BA$ (the anti-commutator) and $S = \frac{1}{i}[A, B]$. Then*

$$\langle A^2 \rangle \langle B^2 \rangle \geq \frac{1}{4} \left(\langle a, Ta \rangle^2 + \langle a, Sa \rangle^2 \right) . \tag{2.10}$$

Equality holds in (2.10) *if and only if there exists $z_0 \in \mathbb{C}$ such that $Aa = z_0 Ba$.*

Proof. Applying the Cauchy-Schwarz inequality and self-adjointness of A we obtain

$$\langle A^2\rangle\langle B^2\rangle = \|Aa\|^2 \|Ba\|^2 \geq |\langle Aa, Ba\rangle|^2 = |\langle a, ABa\rangle|^2 . \tag{2.11}$$

By definition of T and S, we have $AB = \frac{1}{2}T + \frac{i}{2}S$. Therefore,

$$\begin{aligned} |\langle a, ABa\rangle|^2 &= \frac{1}{4} |\langle a, (T + iS)a\rangle|^2 \\ &= \frac{1}{4} |\langle a, Ta\rangle - i \langle a, Sa\rangle|^2 \\ &= \frac{1}{4} \left(\langle a, Ta\rangle^2 + \langle a, Sa\rangle^2\right) . \end{aligned} \tag{2.12}$$

The final equality holds because $\langle a, Ta\rangle$ and $\langle a, Sa\rangle$ are real, and (2.10) follows from (2.11) and (2.12). Lastly, equality holds if and only if we have equality in the application of Cauchy-Schwarz, and this occurs when Aa and Ba are linearly dependent. ■

Lemma 2.5.2 implies the more frequently used inequality for self-adjoint operators A and B:

$$\|Aa\| \|Ba\| \geq \frac{1}{2} |\langle [A, B]a, a\rangle| . \tag{2.13}$$

Indeed, dropping the anti-commutator term from the right side of (2.10) leaves

$$\frac{1}{4} \langle a, Sa\rangle^2 = \frac{1}{4} |\langle [A, B]a, a\rangle|^2 .$$

We have equality in (2.13) when Aa and Ba are linearly dependent (as above) and $\langle a, Ta\rangle = 0$, i.e., when $\langle Aa, Ba\rangle$ is completely imaginary. This weaker form of (2.10) is enough to prove Theorem 2.5.1, and thus the original is usually neglected; we, however, will make use of it.

Define the *position* and *momentum* operators respectively by

$$Qf(x) = xf(x), \quad Pf(x) = \frac{1}{2\pi i}f'(x).$$

Q and P are densely defined linear operators on $L^2(\mathbb{R})$; we denote the domain of an operator A by $D(A)$. When employing Hilbert space operator inequalities, such as (2.10) and (2.13), they are valid only for $a \in \mathcal{H}$ in the domain of all the operators in question, i.e., A, B, AB, and BA. We do not run into a problem with this here, but in general it can yield the inequalities useless, see [20] for a discussion of this problem. We are now ready to prove Theorem 2.5.1.

Proof of Theorem 2.5.1. Let Q and P be as defined above. Then for $f, g \in D(Q)$,

$$\langle Qf, g\rangle = \int xf(x)\overline{g(x)}\,dx = \int f(x)\overline{xg(x)}\,dx = \langle f, Qg\rangle\,,$$

and for $f, g \in D(P)$,

$$\langle Pf, g\rangle = \frac{1}{2\pi i}\int f'(x)\overline{g(x)}\,dx = -\frac{1}{2\pi i}\int f(x)\overline{g'(x)}\,dx = \langle f, Pg\rangle\,.$$

Therefore Q and P are self-adjoint. The operators $Q - a$ and $P - b$ are also self-adjoint and $[Q - a, P - b] = [Q, P]$. Thus, (2.13) implies for every f in the domain of Q, P, QP, and PQ, e.g., f a Schwartz function,

$$\frac{1}{2}\left|\langle [Q, P]f, f\rangle\right| \leq \|(Q - a)f\|\,\|(P - b)f\|\,. \tag{2.14}$$

For the commutator term we have

$$[Q, P]f(x) = \frac{1}{2\pi i}(xf'(x) - (f'(x) + xf'(x))) = -\frac{1}{2\pi i}f(x). \tag{2.15}$$

Combining (2.14) and (2.15) yields

$$\frac{1}{4\pi}\|f\|_2^2 \leq \|(Q - a)f\|\,\|(P - b)f\|\,.$$

It is an elementary fact from Fourier analysis that $(\widehat{\frac{d}{dx}f})(\gamma) = 2\pi i\gamma\widehat{f}(\gamma)$; applying this and Plancherel's theorem to the second term yields

$$\|(P-b)f\| = \left(\int (\gamma - b)^2 |\widehat{f}(\gamma)|^2 \, d\gamma\right)^{1/2},$$

and Heisenberg's inequality follows. ■

2.5.2 An Uncertainty Principle for the Vector-valued DFT

The uncertainty principle we prove for the vector-valued discrete Fourier transform is an extension of an uncertainty principle proved by Grünbaum for the discrete Fourier transform in [23]. We begin by defining two operators meant to stand in for the position and momentum operators from Section 2.5.1. Define $P : \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}) \to \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$ by the formula

$$\forall m \in \mathbb{Z}/N\mathbb{Z}, \quad P(u)(m) = i(u(m+1) - u(m-1)),$$

and given a fixed real valued $q \in \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$, define $Q : \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}) \to \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$ by the formula

$$\forall m \in \mathbb{Z}/N\mathbb{Z}, \quad Q(u)(m) = q(m)u(m).$$

Proposition 2.5.3. *The operators P and Q defined above are linear and self-adjoint.*

Proof. Linearity of P and Q and self-adjointness of Q are obvious. To show P is self adjoint, let $u, v \in \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$. We compute

$$\begin{aligned}
\langle Pu, v\rangle &= \sum_{m=0}^{N-1} \langle P(u)(m), v(m)\rangle \\
&= \sum_{m=0}^{N-1} \langle i(u(m+1) - u(m-1)), v(m)\rangle \\
&= \sum_{m=0}^{N-1} i \langle u(m+1), v(m)\rangle - i \langle u(m-1), v(m)\rangle \\
&= \sum_{m=0}^{N-1} i \langle u(m), v(m-1)\rangle - i \langle u(m), v(m+1)\rangle \text{ (reordering terms)} \\
&= \sum_{m=0}^{N-1} \langle u(m), i(v(m+1) - v(m-1))\rangle \\
&= \langle u, Pv\rangle .
\end{aligned}$$

■

Define $T = QP + PQ$ and $S = \frac{1}{i}[Q, P]$. Because our Hilbert space is finite dimensional, T and S are linear self-adjoint operators defined everywhere. Applying Lemma 2.5.2 gives an uncertainty principle for the operators Q and P:

$$\forall u \in \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}), \quad \langle Q^2\rangle\langle P^2\rangle \geq \frac{1}{4}\left(\langle u, Tu\rangle^2 + \langle u, Su\rangle^2\right). \tag{2.16}$$

In this form, (2.16) does not appear to be related to the vector-valued discrete Fourier transform. We now endeavor to make this connection by finding convenient expressions for each of the terms above and yielding a form of the Heisenberg inequality for the vector-valued DFT.

The expected value of Q and P are

$$\begin{aligned}
\langle Q^2 \rangle &= \langle Qu, Qu \rangle \\
&= \sum_{m=0}^{N-1} \langle Q(u)(m), Q(u)(m) \rangle \\
&= \sum_{m=0}^{N-1} \langle q(m)u(m), q(m)u(m) \rangle \\
&= \sum_{m=0}^{N-1} \|q(m)u(m)\|^2_{\ell^2(\mathbb{Z}/d\mathbb{Z})} \\
&= \|qu\|^2
\end{aligned}$$

and

$$\begin{aligned}
\langle P^2 \rangle &= \langle Pu, Pu \rangle \\
&= \|Pu\|^2 \\
&= \|i(\tau_{-1}u - \tau_1 u)\|^2 \\
&= \|\mathcal{F}(\tau_{-1}u - \tau_1 u)\|^2 \quad (\mathcal{F} \text{ is unitary}) \\
&= \left\|\varphi^1 \widehat{u} - \varphi^{-1}\widehat{u}\right\|^2 \quad (\text{Proposition 2.2.3}) \\
&= \left\|(\varphi^1 - \varphi^{-1})\widehat{u}\right\|^2 .
\end{aligned}$$

φ^1 and φ^{-1} are the modulation functions $\varphi^j(m) := \varphi_{jm}$. We record these for future use.

$$\langle Q^2 \rangle = \|qu\|^2 \text{ and } \quad \langle P^2 \rangle = \left\|(\varphi^1 - \varphi^{-1})\widehat{u}\right\|^2 . \tag{2.17}$$

We seek expressions for the terms $\langle u, Tu \rangle^2$ and $\langle u, Su \rangle^2$. Computing the commutator and anticommutator of Q and P gives

$$[Q, P]u(m) = i(q(m) - q(m+1))u(m+1) - i(q(m) - q(m-1))u(m-1)$$

and

$$(QP+PQ)u(m) = i(q(m)+q(m+1))u(m+1) - i(q(m)+q(m-1))u(m-1).$$

Recall $T = QP + PQ$ and $S = \frac{1}{i}[Q, P]$; therefore,

$$\begin{aligned}
&\langle u, Tu \rangle = \\
&= \sum_{m=0}^{N-1} \langle u(m), T(u)(m) \rangle \\
&= \sum_{m=0}^{N-1} \langle u(m), i(q(m)+q(m+1))u(m+1) - i(q(m)+q(m-1))u(m-1) \rangle \\
&= i \sum_{m=0}^{N-1} \langle u(m), (q(m)+q(m-1))u(m-1) \rangle - \langle u(m), (q(m)+q(m+1))u(m+1) \rangle \\
&= i \sum_{m=0}^{N-1} \langle (q(m)+q(m-1))u(m), u(m-1) \rangle - \langle u(m), (q(m)+q(m+1))u(m+1) \rangle \\
&= i \sum_{m=0}^{N-1} \langle (q(m+1)+q(m))u(m+1), u(m) \rangle - \langle u(m), (q(m)+q(m+1))u(m+1) \rangle \\
&= 2 \sum_{m=0}^{N-1} \Im \langle u(m), (q(m)+q(m+1))u(m+1) \rangle, \qquad (2.18)
\end{aligned}$$

and

$$\begin{aligned}
&\langle u, Su \rangle = \\
&= \sum_{m=0}^{N-1} \langle u(m), S(u)(m) \rangle \\
&= \sum_{m=0}^{N-1} \langle u(m), (q(m)-q(m+1))u(m+1) - (q(m)-q(m-1))u(m-1) \rangle \\
&= \sum_{m=0}^{N-1} \langle u(m), (q(m)-q(m+1))u(m+1) \rangle - \langle u(m), (q(m)-q(m-1))u(m-1) \rangle \\
&= \sum_{m=0}^{N-1} \langle u(m), (q(m)-q(m+1))u(m+1) \rangle - \langle (q(m+1)-q(m))u(m+1), u(m) \rangle \\
&= 2 \sum_{m=0}^{N-1} \Re \langle u(m), (q(m)-q(m+1))u(m+1) \rangle. \qquad (2.19)
\end{aligned}$$

Combining (2.17), (2.18), and (2.19) with inequality (2.16) gives a general uncertainty principle for the vector-valued DFT:

$$\|qu\|^2 \left\|(\varphi^1 - \varphi^{-1})\widehat{u}\right\|^2 \geq \left(\sum_{m=0}^{N-1} \Im \langle u(m), (q(m) + q(m+1))u(m+1)\rangle\right)^2 + \left(\sum_{m=0}^{N-1} \Re \langle u(m), (q(m) - q(m+1))u(m+1)\rangle\right)^2 .$$

The above holds for any real valued q, but to complete the analogy to that of the classical uncertainty principle we desire the operators Q and P to be unitarily equivalent via the Fourier transform, in this case, the vector-valued discrete Fourier transform. Indeed, setting $q = i(\varphi^1 - \varphi^{-1})$, we have $q(m)(n) = -2\sin(2\pi m s(n)/N)$ (q is real-valued) and $\mathcal{F}P = Q\mathcal{F}$ as desired. With this choice of Q we have shown the following version of the classical uncertainty principle for the vector-valued discrete Fourier transform.

Theorem 2.5.4 (Uncertainty principle for the vector-valued DFT)**.** *Let $q = i(\varphi^1 - \varphi^{-1})$. For every u in $\ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$ we have*

$$\left\|(\varphi^1 - \varphi^{-1})u\right\|^2 \left\|(\varphi^1 - \varphi^{-1})\widehat{u}\right\|^2 \geq \left(\sum_{m=0}^{N-1} \Im \langle u(m), (q(m) + q(m+1))u(m+1)\rangle\right)^2 + \left(\sum_{m=0}^{N-1} \Re \langle u(m), (q(m) - q(m+1))u(m+1)\rangle\right)^2 .$$

■

2.6 The Banach Algebra of the Vector-valued DFT

In this section we define a Banach algebra structure on $\mathcal{A} = L^1(\mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/d\mathbb{Z})$, describe the spectrum of this Banach algebra, and show that the Gelfand transform on $\mathcal{A}$ is the vector-valued discrete Fourier transform.

As described in Section 1.2, for a LCAG $\mathcal{G}$, $L^1(\mathcal{G})$ under convolution is a $*$-algebra, and the spectrum of $L^1(\mathcal{G})$ can be identified with $\widehat{\mathcal{G}}$ in such a way that the Gelfand transform is the Fourier transform. Using the group structure on $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ we can define convolution of $u, v \in L^1(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$ by the formula

$$(u * v)(m)(n) = \sum_{k=0}^{N-1} \sum_{l=0}^{d-1} u(k)(l)v(m-k)(n-l). \tag{2.20}$$

This definition is not ideal for our purposes because it treats u and v as functions that take Nd values. Our desire is to view u and v as functions that take N values which are each d dimensional vectors. The convolution (2.20) can be rewritten as follows:

$$(u * v)(m)(n) = \sum_{k=0}^{N-1} (u(k) * v(m-k))(n),$$

where the $*$ on the right hand side is regular d-dimensional convolution. Replacing this d-dimensional convolution with pointwise multiplication, we arrive at a new definition for convolution on $L^1(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$.

Definition 2.6.1 (Vector-valued convolution)**.** Let $u, v \in L^1(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$. Define the *vector-valued convolution* of u and v by the formula

$$(u * v)(m) = \sum_{k=0}^{N-1} u(k)v(m-k).$$

For the remainder of this section $*$ will denote the vector-valued convolution. $L^1(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$ equipped with the vector-valued convolution is a commutative Banach algebra that we will call $\mathcal{A}$. The only non-obvious property to be checked is that $\|u * v\|_1 \leq \|u\|_1 \|v\|_1$ holds. Indeed,

$$\begin{aligned}
\|u * v\|_1 &= \sum_{m=0}^{N-1} \|u * v(m)\|_{L^1(\mathbb{Z}/d\mathbb{Z})} \text{ (def. of } L^1 \text{ norm)} \\
&= \sum_{m=0}^{N-1} \left\| \sum_{k=0}^{N-1} u(k)v(m-k) \right\|_{L^1(\mathbb{Z}/d\mathbb{Z})} \\
&\leq \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} \|u(k)v(m-k)\|_{L^1(\mathbb{Z}/d\mathbb{Z})} \\
&\leq \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} \|u(k)\|_{L^1(\mathbb{Z}/d\mathbb{Z})} \|v(m-k)\|_{L^1(\mathbb{Z}/d\mathbb{Z})} \\
&= \sum_{k=0}^{N-1} \|u(k)\|_{L^1(\mathbb{Z}/d\mathbb{Z})} \sum_{m=0}^{N-1} \|v(m-k)\|_{L^1(\mathbb{Z}/d\mathbb{Z})} \\
&= \sum_{k=0}^{N-1} \|u(k)\|_{L^1(\mathbb{Z}/d\mathbb{Z})} \|v\|_1 \\
&= \|u\|_1 \|v\|_1 .
\end{aligned}$$

$\mathcal{A}$ is a unital $*$-algebra with unit e given by

$$e(m) = \begin{cases} \vec{1} & m = 0 \\ \vec{0} & m \neq 0, \end{cases}$$

where $\vec{1}$ and $\vec{0}$ are the vectors of 1's and 0's respectively, and with involution $u^*(m) = \overline{u(-m)}$. Tying this together with our Fourier transform theory above, we have the desired theorem relating $\mathcal{A}$ to the vector-valued Fourier transform.

Theorem 2.6.2 (Convolution theorem)**.** *Let $u, v \in L^1(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$. The vector-valued Fourier transform of the convolution of u and v is the vector product of their Fourier transforms, i.e.,*

$$F(u * v) = F(u)F(v).$$

Proof.

$$\begin{aligned}
F(u*v)(p) &= \sum_{m=0}^{N-1}(u*v)(m)\varphi_{-mp} \\
&= \sum_{m=0}^{N-1}\left(\sum_{k=0}^{N-1}u(k)v(m-k)\right)\varphi_{-mp} \\
&= \sum_{k=0}^{N-1}u(k)\left(\sum_{m=0}^{N-1}v(m-k)\varphi_{-mp}\right) \\
&= \sum_{k=0}^{N-1}u(k)\left(\sum_{l=0}^{N-1}v(l)\varphi_{-(k+l)p}\right) \\
&= \left(\sum_{k=0}^{N-1}u(k)\varphi_{-kp}\right)\left(\sum_{l=0}^{N-1}v(l)\varphi_{-lp}\right) \\
&= F(u)(p)F(v)(p)
\end{aligned}$$

■

We shall now describe the spectrum of $\mathcal{A}$ and the Gelfand transform on $\mathcal{A}$.

Define functions $\delta_{(i,j)}$ in $L^1(\mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/d\mathbb{Z})$ by

$$\delta_{(i,j)}(m)(n) = \begin{cases} 1 & (m,n)=(i,j) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\delta^k_{(1,j)} = \delta_{(1,j)} * \ldots * \delta_{(1,j)}$ (k factors) $= \delta_{(k,j)}$ so that $\{\delta_{(1,j)}\}_{j=0}^{d-1}$ generate $\mathcal{A}$. We shall find the spectrum of the individual elements of our generating set $\{\delta_{(1,j)}\}_{j=0}^{d-1}$, and with this information describe the spectrum of $\mathcal{A}$.

To find the spectrum of $\delta_{(1,j)}$ we first find necessary conditions on λ for $(\lambda e - \delta_{(1,j)})^{-1}$ to exist, and when these conditions are met we compute $(\lambda e - \delta_{(1,j)})^{-1}$ and thereby show the conditions are sufficient as well. To that end, suppose $u = (\lambda e - \delta_{(1,j)})^{-1}$ exists, i.e., $(\lambda e - \delta_{(1,j)}) * u = e$. Expanding the definitions on the left hand side

$$(\lambda e - \delta_{(1,j)}) * u(m) = \sum_{k=0}^{N-1} (\lambda e - \delta_{(1,j)})(k) u(m-k)$$

$$= \lambda u(m) - \delta_{(1,j)}(1) u(m-1).$$

Setting the result equal to $e(m)$ and dividing into the cases $m = 0$ and $m \neq 0$ yields two equations

$$\forall n \in \mathbb{Z}/d\mathbb{Z}, \quad \lambda u(0)(n) - \delta_{(1,j)}(1)(n) u(N-1)(n) = 1 \tag{2.21}$$

and

$$\forall n \in \mathbb{Z}/d\mathbb{Z}, \forall m \neq 0 \in \mathbb{Z}/N\mathbb{Z}, \quad \lambda u(m)(n) - \delta_{(1,j)}(1)(n) u(m-1)(n) = 0. \tag{2.22}$$

Plugging in $n = j$ into (2.21) yields

$$\lambda u(0)(j) - u(N-1)(j) = 1, \tag{2.23}$$

while for $n \neq j$ we have

$$u(0)(n) = \frac{1}{\lambda}.$$

Therefore, we must have $\lambda \neq 0$. Similarly, plugging in $n = j$ in (2.22) gives

$$\forall m \neq 0, \quad \lambda u(m)(j) - u(m-1)(j) = 0, \tag{2.24}$$

while

$$\forall n \neq j, \forall m \neq 0, \quad u(m)(n) = 0.$$

At this point we have specified the values of u except for $u(m)(j)$. Iterate (2.24) $N - 1$ times to find

$$\lambda^{N-1} u(N-1)(j) - u(0)(j) = 0. \tag{2.25}$$

Finally, multiplying (2.25) by λ and adding it to equation (2.23) gives

$$(\lambda^N - 1)u(N-1)(j) = 1,$$

and hence $\lambda^N \neq 1$. Using (2.24) we can find the remaining values of $u(m)(j)$:

$$u(m)(j) = \frac{\lambda^{N-m-1}}{\lambda^N - 1}.$$

This completes the computation of u. We have shown that for $\lambda e - \delta_{(1,j)}$ to be invertible λ must satisfy $\lambda \neq 0$ and $\lambda^N \neq 1$. Given that λ meets these requirements we found an explicit inverse; therefore $\sigma(\delta_{(1,j)}) = \{0, \lambda : \lambda^N = 1\}$.

By the Riesz representation theorem a linear functional on $\mathcal{A}$ is given by integration against a function in $\gamma \in L^\infty(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$ (which we also view as a function in L^∞ or simply a $N \times d$ matrix). Recall a basic result in Gelfand theory: for a commutative unital Banach algebra we have $\widehat{x}(\sigma(\mathcal{A})) = \sigma(x)$ (Theorem 1.13 of [19]). Combining this with our above calculations it follows that for a multiplicative linear functional γ,

$$\overline{\gamma(1)(n)} = \int \delta_{(1,n)}\overline{\gamma} = \gamma(\delta_{(1,n)}) \in \sigma(\delta_{(1,n)}).$$

Since γ is multiplicative,

$$\overline{\gamma(m)(n)} = \int \delta_{(m,n)}\overline{\gamma} = \int \delta^m_{(1,n)}\overline{\gamma} = \gamma(\delta^m_{(1,n)}) = \gamma(\delta_{(1,n)})^m = 0 \text{ or } \lambda^m \text{ where } \lambda^N = 1.$$

Therefore $\gamma(0)(n) = 0$ or 1, and since $1 = \gamma(e) = \sum_{k=0}^{d-1} \overline{\gamma(0)(k)}$, we have $\gamma(0)(n) \neq 0$ (and thus $\gamma(1)(n) \neq 0$) for only one n. It follows that for this n, $\gamma(1)(n) = \overline{\lambda}$ where $\lambda^N = 1$.

We have everything we need to describe $\sigma(\mathcal{A})$. The multiplicative linear functionals on $\mathcal{A}$ are $N \times d$ matrices of the form

$$\gamma_{\lambda,k}(m)(n) = \begin{cases} \lambda^{-m} & \text{for } n = k, \\ \\ 0 & \text{otherwise} \end{cases} \quad \text{where } \lambda^N = 1, \quad 0 \leq k \leq d-1.$$

Set $\omega = e^{-2\pi i/N}$. If $\lambda^N = 1$, then $\lambda = \omega^j$ for some $0 \leq j \leq N-1$, and we can write $\gamma_{\lambda,k}$ as $\gamma_{j,k}$. Thus, we can list all the elements of $\sigma(\mathcal{A})$ as $\{\gamma_{j,k}\}$, $0 \leq j < N-1$, $0 \leq k < d-1$, and there are Nd of them.

Let $s : \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ be injective and have the property that for every $n \in \mathbb{Z}/d\mathbb{Z}$, $(s(n), N) = 1$, i.e., the vector-valued DFT defined by s is invertible. Using s we can reorder $\sigma(\mathcal{A})$ as follows. For $0 \leq p \leq N-1$ and $0 \leq q \leq d-1$ define $\gamma'_{p,q}$ by

$$\gamma'_{p,q}(m)(n) = \begin{cases} \omega^{-pms(q)} & \text{for } n = q, \\ \\ 0 & \text{otherwise.} \end{cases}$$

We claim $\{\gamma'_{p,q}\}_{p,q}$ is a reordering of $\{\gamma_{j,k}\}_{j,k}$. To show this, first note that clearly $\{\gamma'_{p,q}\}_{p,q} \subseteq \{\gamma_{j,k}\}_{j,k}$. To demonstrate the reverse inclusion, for each $q \in \mathbb{Z}/d\mathbb{Z}$ find a multiplicative inverse to $s(q)$ in $\mathbb{Z}/N\mathbb{Z}$. This may be done because $(s(q), N) = 1$ for every q. Writing this inverse as $s(q)^{-1}$ it follows that

$$\gamma'_{js(k)^{-1},k} = \gamma_{j,k},$$

and therefore $\{\gamma_{j,k}\}_{j,k} \subseteq \{\gamma'_{p,q}\}_{p,q}$. Hence, we may identify $\sigma(\mathcal{A})$ with $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ via $\gamma'_{p,q} \leftrightarrow (p, q)$. Under this identification, for $x \in \mathcal{A}$, we may write the Gelfand transform of x, which is in $C(\sigma(\mathcal{A}))$, as the $N \times d$ matrix

$$\begin{aligned}\widehat{x}(p)(q) := \widehat{x}(\gamma'_{p,q}) = \gamma'_{p,q}(x) &= \sum_{m=0}^{N-1} x(m)(q)\omega^{pms(q)} \\ &= \sum_{m=0}^{N-1} x(m)(q)e^{-2\pi i pms(q)/N}.\end{aligned}$$

From this, we see that under the identification $\gamma'_{p,q} \leftrightarrow (p,q)$ the Gelfand transform on $\mathcal{A}$ is the vector-valued discrete Fourier transform. While this shows the transform we have discovered is itself nothing "new", it also shows that a classical transform can be redefined in the context of frame theory.

Chapter 3

Frame Multiplication

In this chapter we study bilinear products on a finite dimensional Hilbert space $\mathcal{H}$ ($\mathbb{C}^d$ or $\mathbb{R}^d$) over $\mathbb{F}$ ($\mathbb{C}$ or $\mathbb{R}$) and what we call frame multiplication. Questions of interest include: for a given binary operation, can we classify the frames, in terms of their elements or operators, for which the binary operation defines a frame multiplication? What is the relationship between equivalences of frames and frame multiplication, and given a bilinear product on $\mathcal{H}$, can we show whether or not it arises from a frame multiplication?

3.1 Motivation

Let $\{\varphi_n\}_{n=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$. In Chapter 2 we leveraged the relationship between the bilinear product pointwise multiplication and the operation of addition on the indices of Φ, i.e., $\varphi_m\varphi_n = \varphi_{m+n}$, to define the discrete vector-valued ambiguity function. In this context, the DFT frame is acting as a high dimensional analog to the roots of unity $\{W_n = e^{2\pi i n/N}\}_{n=0}^{N-1}$, which appear in the definition of the regular discrete ambiguity function.

It is not pre-ordained that the operation on the indices of the frame, induced by the bilinear vector multiplication, be addition mod N, as in the case of the DFT frames. We are interested in finding tight frames that behave similarly and whose index sets are abelian groups, non-abelian groups, or more general non-group sets and operations. This would, for example, allow us to define the discrete vector-valued ambiguity function for a function defined on an arbitrary group $\mathcal{G}$.

Example 3.1.1 (Cross product frame multiplication)**.** Define $* : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$. Let $\{i, j, k\}$ be the standard basis for $\mathbb{C}^3$, e.g., $i = (1, 0, 0) \in \mathbb{C}^3$. We have that $i * j = k$, $j * i = -k$, $k * i = j$, $i * k = -j$, $j * k = i$, $k * j = -i$, and $i * i = j * j = k * k = 0$. The union of two tight frames and the zero vector is a tight frame, so if we let $\varphi_0 = 0$, $\varphi_1 = i$, $\varphi_2 = j$, $\varphi_3 = k$, $\varphi_4 = -i$, $\varphi_5 = -j$, and $\varphi_6 = -k$, then $\Phi = \{\varphi_n\}_{n=0}^{6}$ is a tight frame for $\mathbb{C}^3$ with frame bound 2. Also, Φ is closed under the cross product, and the index operation corresponding to $*$ is the non-group operation $\bullet : \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \to \mathbb{Z}/7\mathbb{Z}$, where we compute

$$
\begin{array}{lllll}
1 \bullet 2 = 3, & 1 \bullet 3 = 5 & 1 \bullet 4 = 0, & 1 \bullet 5 = 6, & 1 \bullet 6 = 2, \\
2 \bullet 1 = 6, & 2 \bullet 3 = 1, & 2 \bullet 4 = 3, & 2 \bullet 5 = 0, & 2 \bullet 6 = 4, \\
3 \bullet 1 = 2, & 3 \bullet 2 = 4, & 3 \bullet 4 = 5, & 3 \bullet 5 = 1, & 3 \bullet 6 = 0,
\end{array}
$$

$$
n \bullet n = 0, \qquad n \bullet 0 = 0 \bullet n = 0, \qquad \text{etc.}
$$

We can now obtain the following formula:

$$
\begin{aligned}
\forall u, v \in \mathbb{C}^3, \quad u * v &= \frac{1}{4} \sum_{j=1}^{6} \sum_{k=1}^{6} \langle u, \varphi_j \rangle \langle v, \varphi_k \rangle \varphi_{j \bullet k} \\
&= \frac{1}{4} \sum_{n=1}^{6} \left(\sum_{j \bullet k = n} \langle u, \varphi_j \rangle \langle v, \varphi_k \rangle \right) \varphi_n. \qquad (3.1)
\end{aligned}
$$

One possible application of the above is that given frame representations for $u, v \in \mathbb{C}^3$, (3.1) allows us to compute the frame representation of $u * v$ without the cumbersome process of going back and forth between the frame representations and their standard orthogonal representations. While the utility of this small case is questionable, it is not hard to imagine more complicated examples where frame multiplication might have applicability.

3.2 Definition and Properties

Definition 3.2.1 (Frame multiplication)**.** Let $\Phi = \{\varphi_j\}_{j\in J}$ be a frame for a finite dimensional Hilbert space $\mathcal{H}$, and let $\bullet : J \times J \to J$ be a binary operation. We say $\bullet$ is a *frame multiplication* for Φ or, by abuse of language, a frame multiplication for $\mathcal{H}$, if it extends to a bilinear product $*$ on all of $\mathcal{H}$. That is, if there exists a bilinear product $* : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ such that

$$\forall j, k \in J, \quad \varphi_j * \varphi_k = \varphi_{j\bullet k}.$$

Fix a frame $\Phi = \{\varphi_j\}_{j\in J}$. By definition, a binary operation $\bullet : J \times J \to J$ is a frame multiplication for Φ when it extends by linearity to the entire space $\mathcal{H}$. Conversely, if there is a bilinear product $* : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ which agrees with $\bullet$ on Φ ($\varphi_j * \varphi_k = \varphi_{j\bullet k}$), then it must be the unique extension given by linearity (since Φ spans $\mathcal{H}$). Therefore, $\bullet$ defines a frame multiplication for Φ if and only if for every $x = \sum a_i\varphi_i$ and $y = \sum b_i\varphi_i \in \mathcal{H}$,

$$x * y := \sum_i \sum_j a_i b_j \varphi_{i\bullet j} \tag{3.2}$$

is defined and independent of the frame representations used for x and y.

Remark 3.2.2. Whether or not a particular binary operation is a frame multiplication depends on not just the elements of the frame but on the indexing of the frame.

For clarity of definitions and later theorems we make no attempt to define a notion of frame multiplication for multi-sets of vectors that is independent of the index set. This does not hinder the theory; a multiset of vectors, Φ, has some number of bilinear products on $\mathcal{H}$ for which it is closed, and all of these may be realized by fixing a single index, $\Phi = \{\varphi_j\}_{j\in J}$.

A distinction to keep in mind is that $\bullet$ is a set operation on the indices of a frame while $*$ is a bilinear vector product defined on all of $\mathcal{H}$. When $\bullet$ is a frame multiplication and $*$ is the corresponding bilinear product we have

$$\varphi_i * \varphi_j = \varphi_{i\bullet j}.$$

We shall investigate the interplay between bilinear vector products on $\mathcal{H}$, frames for $\mathcal{H}$ indexed by J, and binary operations on J. For example, if we fix a binary operation $\bullet$ on J, then what sort of frames indexed by J is $\bullet$ a frame multiplication for? Conversely, if we fix a frame $\Phi = \{\varphi_j\}_{j\in J}$, then what sort of binary operations on J are frame multiplications for $\mathcal{H}$?

Proposition 3.2.3. *Let $\Phi = \{\varphi_j\}_{j\in J}$ be a frame for a Hilbert space $\mathcal{H}$, and let $\bullet : J \times J \to J$ be a binary operation. Then $\bullet$ is a frame multiplication for Φ if and only if*

$$\forall \{a_i\}_{i\in J} \subset \mathbb{F}, \forall j \in J, \quad \sum_{i\in J} a_i\varphi_i = 0 \Rightarrow \sum_{i\in J} a_i\varphi_{i\bullet j} = 0 \text{ and } \sum_{i\in J} a_i\varphi_{j\bullet i} = 0 \qquad (3.3)$$

Proof. Suppose $*$ is the bilinear product defined by $\bullet$ and $\{a_i\}_{i\in J}$ is a sequence of scalars. If $\sum_{i\in J} a_i\varphi_i = 0$, then

$$\sum_{i\in J} a_i\varphi_{i\bullet j} = \sum_{i\in J} a_i\left(\varphi_i * \varphi_j\right) = \left(\sum_{i\in J} a_i\varphi_i\right) * \varphi_j = 0 * \varphi_j = 0.$$

Similarly, by multiplying on the left by φ_j, we can show $\sum_{i\in J} a_i\varphi_{j\bullet i} = 0$. For the converse suppose that statement (3.3) holds and $x = \sum a_i\varphi_i = \sum c_i\varphi_i$, $y = \sum b_j\varphi_j = \sum d_j\varphi_j \in \mathcal{H}$. By (3.3),

$$\forall j \in J, \quad \sum_i (a_i - c_i)\varphi_{i\bullet j} = 0, \text{ and} \tag{3.4}$$

$$\forall i \in J, \quad \sum_j (b_j - d_j)\varphi_{i\bullet j} = 0. \tag{3.5}$$

Therefore,

$$\begin{aligned}\sum_i\sum_j a_ib_j\varphi_{i\bullet j} &= \sum_i a_i \sum_j b_j\varphi_{i\bullet j} = \sum_i a_i \sum_j d_j\varphi_{i\bullet j} \quad \text{by (3.5)}\\ &= \sum_j d_j \sum_i a_i\varphi_{i\bullet j} = \sum_j d_j \sum_i c_i\varphi_{i\bullet j} \quad \text{by (3.4)}\\ &= \sum_i\sum_j c_id_j\varphi_{i\bullet j},\end{aligned}$$

and $*$ is well-defined by (3.2). ■

Definition 3.2.4 (Similarity)**.** Frames $\Phi = \{\varphi_j\}_{j\in J}$ and $\Psi = \{\psi_j\}_{j\in J}$ for a Hilbert space $\mathcal{H}$ are *similar* if there exists an invertible linear operator $A \in \mathcal{B}(\mathcal{H})$ such that

$$\forall j \in J, \quad A\varphi_j = \psi_j.$$

Lemma 3.2.5. *Suppose $\Phi = \{\varphi_j\}_{j\in J}$ and $\Psi = \{\psi_j\}_{j\in J}$ are frames for $\mathcal{H}$ and Φ is similar to Ψ. Then, a binary operation $\bullet : J \times J \to J$ is a frame multiplication for Φ if and only if it is a frame multiplication for Ψ.*

Proof. We remark that because $A^{-1}\psi_j = \varphi_j$ and A^{-1} is also an invertible operator, we need only prove one direction of the lemma. Suppose $\bullet$ is a frame multiplication for Φ and that $\sum_i a_i\psi_i = 0$. We have

$$0 = \sum_i a_i\psi_i = \sum_i a_iA\varphi_i = A\left(\sum_i a_i\varphi_i\right),$$

and since A is invertible it follows that $\sum_i a_i \varphi_i = 0$. By Proposition 3.2.3, since $\bullet$ is a frame multiplication for Φ,

$$\forall j, \quad \sum_{i \in J} a_i \varphi_{i \bullet j} = 0 \text{ and } \sum_{i \in J} a_i \varphi_{j \bullet i} = 0.$$

Applying A to both of these equations yields:

$$\forall j, \quad \sum_{i \in J} a_i \psi_{i \bullet j} = 0 \text{ and } \sum_{i \in J} a_i \psi_{j \bullet i} = 0.$$

Therefore, by Proposition 3.2.3, $\bullet$ is a frame multiplication for Ψ. ∎

Definition 3.2.6 (Multiplications of a frame)**.** Let $\Phi = \{\varphi_j\}_{j \in J}$ be a frame for a finite dimensional Hilbert space $\mathcal{H}$. The *multiplications* of Φ are defined and denoted by

$$mult\,(\Phi) := \{\bullet : J \times J \to J : \bullet \text{ is a frame multiplication on } \Phi\}.$$

$mult\,(\Phi)$ can be all functions (for example when Φ is a basis), empty, or somewhere between.

Example 3.2.7. Let $\alpha, \beta > 0$, $\alpha \neq \beta$, and $\alpha + \beta < 1$. Define $\Phi_{\alpha,\beta} = \{\varphi_1 = (1,0)^t, \varphi_2 = (0,1)^t, \varphi_3 = (\alpha, \beta)^t\}$. Then $\Phi_{\alpha,\beta}$ is a frame for $\mathbb{C}^2$ and $mult\,(\Phi_{\alpha,\beta}) = \emptyset$. An easy way to prove $mult\,(\Phi_{\alpha,\beta}) = \emptyset$ is to show that there are no bilinear operations on $\mathbb{C}^2$ which leave $\Phi_{\alpha,\beta}$ invariant. Suppose $*$ were such a bilinear operation. We have the linear relation $\varphi_3 = \alpha\varphi_1 + \beta\varphi_2$; hence, by bilinearity of $*$,

$$\varphi_1 * \varphi_3 = \alpha\varphi_1 * \varphi_1 + \beta\varphi_1 * \varphi_2. \tag{3.6}$$

Since $\|\varphi_i\|_2 \leq 1$ for $i = 1,2,3$, it must be that $\|\varphi_1 * \varphi_3\|_2 \leq \alpha + \beta < 1$, and since $*$ leaves $\Phi_{\alpha,\beta}$ invariant, $\varphi_1 * \varphi_3 = \varphi_3$. Furthermore, substituting φ_3 for $\varphi_1 * \varphi_3$ in equation (3.6) yields $\varphi_1 * \varphi_1 = \varphi_1$ and $\varphi_1 * \varphi_2 = \varphi_2$. Performing the same analysis on $\varphi_3 * \varphi_2$ (in place of $\varphi_1 * \varphi_3$ above) shows $\varphi_2 * \varphi_2 = \varphi_2$ and $\varphi_1 * \varphi_2 = \varphi_1$, a contradiction.

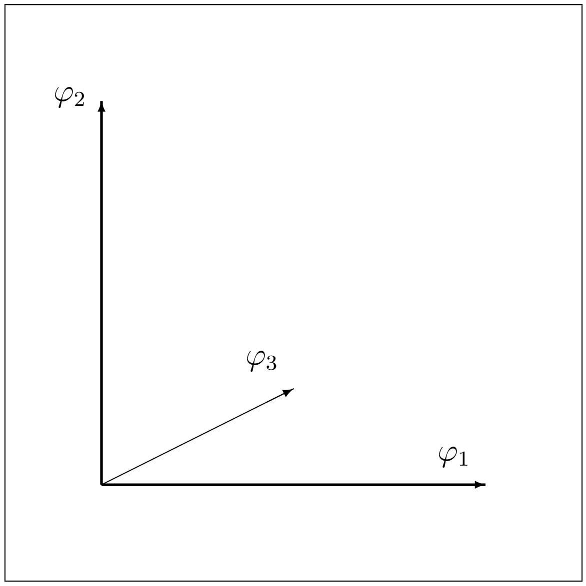


Figure 3.1: The frame $\Phi_{\alpha,\beta}$ from Example 3.2.7 for $\alpha = 1/2$ and $\beta = 1/4$. This frame has no frame multiplications.

Of particular interest, Lemma 3.2.5 tells us the canonical dual frame $\{S^{-1}\varphi_j\}_{j\in J}$ and the canonical tight frame $\{S^{-1/2}\varphi_j\}_{j\in J}$ share the same multiplications as the original frame Φ. Because of this, we will focus our attention on tight frames. An invertible map $U \in \mathcal{B}(\mathcal{H})$ mapping a λ-tight frame $\Phi = \{\varphi_j\}$ to a λ'-tight frame $\Psi = \{\psi_j\}$, as in Lemma 3.2.5, is a positive multiple of a unitary operator. Indeed,

$$\lambda \|U^*x\|^2 = \sum_j |\langle U^*x, \varphi_j\rangle|^2 = \sum_j |\langle x, U\varphi_j\rangle|^2 = \sum_j |\langle x, \psi_j\rangle|^2 = \lambda' \|x\|^2 .$$

This leads us to a notion of equivalence for tight frames that sounds stronger than similarity but is actually just the restriction of similarity to the class of tight frames.

Definition 3.2.8 (Equivalence of tight frames)**.** Tight frames $\Phi = \{\varphi_j\}_{j\in J}$ and $\Psi = \{\psi_j\}_{j\in J}$ for $\mathcal{H}$ are *unitarily equivalent* if there is a unitary map U and a positive constant c such that

$$\forall j \in J, \quad \varphi_j = cU\psi_j.$$

Whenever we speak of equivalence classes for tight frames we will mean under unitary equivalence. For finite frames unitary equivalence can be stated in terms of Gramians.

Proposition 3.2.9 (Lemma 2.7 of [40])**.** *Suppose* $\Phi = (\varphi_1, \ldots, \varphi_n)$ *and* $\Psi = (\psi_1, \ldots, \psi_n)$ *are sequences of vectors, and suppose* $\operatorname{span}(\Phi) = \mathcal{H}$*. There exists a unitary* U *such that for every* $i = 1, \ldots, n$, $U\varphi_i = \psi_i$, *if and only if*

$$\forall i, j, \quad \langle \varphi_i, \varphi_j \rangle = \langle \psi_i, \psi_j \rangle\,,$$

i.e., the Gram matrices of Φ *and* Ψ *are equal.*

From the above proposition we have that tight frames Φ and Ψ are unitarily equivalent if and only if one of their Gramians is a scaled version of the other. In the case where both Φ and Ψ are equivalent Parseval tight frames their Gramians are equal.

We have chosen to follow in the footsteps of D. Han and D. Larson [25] with our definitions of similarity and unitary equivalence, which are somewhat restrictive. In particular, the ordering of the frame and not just the unordered set of frame elements is important. This choice was made in concert with the way in which we have defined frame multiplication, i.e., with a fixed index for our frame. Also, we have made no attempt to define equivalence for frames indexed by different sets. This can be done and results then proven about the correspondence of frame multiplications between similar or equivalent frames (under the new definition), but allowing frames with two different index sets (of the same cardinality!) to be considered similar only obfuscates our theorems.

Theorem 3.2.10 (Multiplications of equivalent frames)**.** *Let* $\Phi = \{\varphi_j\}_{j\in J}$ *and* $\Psi = \{\psi_j\}_{j\in J}$ *be finite tight frames for* $\mathcal{H}$*. If* Φ *is unitarily equivalent to* Ψ*, then* $mult\,(\Phi) = mult\,(\Psi)$*.*

Proof. Since Φ and Ψ are unitarily equivalent they are similar. Therefore, by Lemma 3.2.5, $\bullet : J \times J \to J$ defines a frame multiplication on Φ if and only if it defines a frame multiplication on Ψ. That is, $mult\,(\Phi) = mult\,(\Psi)$. ■

The converse of the above theorem does not hold. The multiplications of a tight frame provide a coarser equivalence relation than unitary equivalence. In fact, we may have uncountably many equivalence classes of tight frames which share the same multiplications.

Example 3.2.11. Let $\{\alpha_i\}_{i=1,2}$ and $\{\beta_i\}_{i=1,2}$ be real numbers such that $\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > 0$, $\alpha_1 + \beta_1 < 1$, and $\alpha_2 + \beta_2 < 1$. Define Φ_{α_1,β_1} and Φ_{α_2,β_2} as in Example 3.2.7. Then $mult\,(\Phi_{\alpha_1,\beta_1}) = mult\,(\Phi_{\alpha_2,\beta_2}) = \emptyset$. It can be easily shown, by checking the six cases of where to map $(1,0)^t$ and $(0,1)^t$, that there is no invertible operator A such that $A\Phi_{\alpha_1,\beta_1} = \Phi_{\alpha_2,\beta_2}$ as sets. Therefore, there is no $c > 0$ and $U \in \mathcal{U}(\mathbb{R}^2)$ such that cU maps between the canonical tight frames $S_1^{-1/2}\Phi_{\alpha_1,\beta_1}$ and $S_2^{-1/2}\Phi_{\alpha_2,\beta_2}$ (for any reordering of the elements) and $S_1^{-1/2}\Phi_{\alpha_1,\beta_1}$ and $S_2^{-1/2}\Phi_{\alpha_2,\beta_2}$ are not unitarily equivalent. Hence, there are uncountably many equivalence classes of tight frames which share the same empty set of frame multiplications.

In contrast to the above example, we will see in Chapter 4 that if a tight frame has a particular frame multiplication, a group operation, then it belongs to one of only finitely many equivalence classes of tight frames which share that same group operation as a frame multiplication. At the moment, we have a characterization of bases in terms of their multiplications (once we exclude the degenerate one dimensional case where you can have a frame of a single repeated vector).

Proposition 3.2.12. *Let $\Phi = \{\varphi_j\}_{j \in J}$ be a finite frame for a Hilbert space $\mathcal{H}$ and suppose dim$(\mathcal{H}) > 1$. If mult $(\Phi) = \{$all functions $\bullet : J \times J \to J\}$, then Φ is a basis. If in addition Φ is a tight (Parseval) frame, then Φ is an orthogonal (orthonormal) basis.*

Proof. Suppose that $\sum_i a_i \varphi_i = 0$, $j_0 \in J$, and $\varphi_{j_1}, \varphi_{j_2} \in \Phi$ are linearly independent. Let $\bullet : J \times J \to J$ be the function sending all products to j_2 except

$$\forall j \in J, \quad j_0 \bullet j := j_1.$$

By assumption, $\bullet \in$ *mult* (Φ); therefore, by Proposition 3.2.3,

$$\forall j \in J, \quad 0 = \sum_i a_i \varphi_{i \bullet j} = a_{j_0} \varphi_{j_1} + \sum_{i \neq j_0} a_i \varphi_{j_2}.$$

Since φ_{j_1} and φ_{j_2} are linearly independent, $a_{j_0} = 0$, and since j_0 was arbitrary, Φ is a linearly independent set. The last statement of the proposition follows from the elementary fact that a basis which satisfies Parseval's identity (or a scaled version) is an orthogonal set. ∎

3.3 Permutation Operations and Quasigroups

Proposition 3.2.9 characterizes unitary equivalence in terms of Gramians; therefore, we should be able to determine all properties of an equivalence class of tight frames from their shared Gramian. Theorem 3.2.10 says that unitarily equivalent frames share the same multiplications, but we have also shown that two frames with the same set of multiplications are not necessarily unitarily equivalent. What we can take away from this is that we should be able to determine the multiplications of a frame from its Gramian, but knowing a frame has a particular frame multiplication does not in general tell us anything about its Gramian. In this section we make progress on describing the multiplications of a frame in terms of its Gramian.

Given a tight frame $\Phi = \{\varphi_j\}_{j\in J}$, if we restrict our attention to binary operations such that multiplication on the left and right by any fixed element is onto, then we are able to characterize these operations which define a frame multiplication by the Gramian of Φ. This is the contents of Theorem 3.3.2. Because the frames we are interested in are finite, a function $J \to J$ which is onto must be a permutation. A set J equipped with a binary operation $\bullet$ such that the functions

$$k \mapsto j \bullet k, \quad k \mapsto k \bullet j \tag{3.7}$$

are permutations for every $j \in J$ is called a quasigroup. The property (3.7) above is referred to as the *Latin square property* because it ensures the multiplication table of $(J, \bullet)$ is a Latin square.

Lemma 3.3.1. *Let $\Phi = \{\varphi_j\}_{j\in J}$ be a tight frame for a finite dimensional Hilbert space $\mathcal{H}$, and let $\bullet : J \times J \to J$ be a frame multiplication for Φ with extension $*$ to* *all of $\mathcal{H}$. If the functions*

$$k \mapsto j \bullet k, \quad k \mapsto k \bullet j$$

are permutations for every $j \in J$, i.e., $(J, \bullet)$ is a quasigroup, then the functions $L_j : \mathcal{H} \to \mathcal{H}$ defined by,

$$L_j(x) = \varphi_j * x$$

and $R_j : \mathcal{H} \to \mathcal{H}$ defined by

$$R_j(x) = x * \varphi_j$$

are unitary linear operators for every $j \in J$.

Proof. Linearity of L_j follows from bilinearity of $*$. To show L_j is unitary, let A be the frame bound for Φ. Then

$$\begin{aligned}
A \left\| L_j^*(x) \right\|^2 &= \sum_{k \in J} \left| \left\langle L_j^*(x), \varphi_k \right\rangle \right|^2 \\
&= \sum_{k \in J} \left| \langle x, L_j(\varphi_k) \rangle \right|^2 \\
&= \sum_{k \in J} \left| \langle x, \varphi_j * \varphi_k \rangle \right|^2 \quad \text{(def. of } L_j\text{)} \\
&= \sum_{k \in J} \left| \langle x, \varphi_{j \bullet k} \rangle \right|^2 \quad \text{(def. of } *\text{)} \\
&= \sum_{k \in J} \left| \langle x, \varphi_k \rangle \right|^2 \quad \text{(reordering terms)} \\
&= A \left\| x \right\|^2 .
\end{aligned}$$

Therefore, L_j^* is an isometry, and since $\mathcal{H}$ is finite dimensional, it follows that L_j is unitary. Similarly, we can show R_j is unitary. ■

Theorem 3.3.2 (Quasigroup frame multiplications)**.** *Let $\Phi = \{\varphi_j\}_{j\in J}$ be a tight frame for a finite dimensional Hilbert space $\mathcal{H}$, and let $\bullet : J \times J \to J$ be a binary operation such that $(J, \bullet)$ is a quasigroup. Then $\bullet$ is a frame multiplication for Φ if and only if*

$$\forall i, j, k \in J, \quad \langle \varphi_i, \varphi_j \rangle = \langle \varphi_{i\bullet k}, \varphi_{j\bullet k} \rangle = \langle \varphi_{k\bullet i}, \varphi_{k\bullet j} \rangle . \tag{3.8}$$

Proof. Suppose first that (3.8) holds. If $\sum_i a_i\varphi_i = 0$, then for any $j, k \in J$ we have

$$\begin{aligned} 0 &= \left\langle \sum_i a_i\varphi_i, \varphi_j \right\rangle = \sum_i a_i \langle \varphi_i, \varphi_j \rangle \\ &= \sum_i a_i \langle \varphi_{i\bullet k}, \varphi_{j\bullet k} \rangle = \left\langle \sum_i a_i\varphi_{i\bullet k}, \varphi_{j\bullet k} \right\rangle . \end{aligned}$$

Allowing j to vary over all of J shows that $\sum_i a_i\varphi_{i\bullet k} = 0$. Similarly we can use the fact that $\langle \varphi_i, \varphi_j \rangle = \langle \varphi_{k\bullet i}, \varphi_{k\bullet j} \rangle$ to show $\sum_i a_i\varphi_{k\bullet i} = 0$. Hence, by Proposition 3.2.3, $\bullet$ is a frame multiplication for Φ. For the converse, assume $\bullet$ is a frame multiplication for Φ. By Lemma 3.3.1, the operators $\{L_j\}_{j\in J}$ and $\{R_j\}_{j\in J}$ defined as left and right multiplication by φ_j are unitary. Hence,

$$\forall i, j, k \in J, \quad \langle \varphi_i, \varphi_j \rangle = \langle R_k(\varphi_i), R_k(\varphi_j) \rangle = \langle \varphi_{i\bullet k}, \varphi_{j\bullet k} \rangle ,$$

and

$$\forall i, j, k \in J, \quad \langle \varphi_i, \varphi_j \rangle = \langle L_k(\varphi_i), L_k(\varphi_j) \rangle = \langle \varphi_{k\bullet i}, \varphi_{k\bullet j} \rangle . \qquad \blacksquare$$

Chapter 4

Group Frame Multiplication

In this chapter we investigate the special case of frame multiplications defined by binary operations $\bullet : J \times J \to J$ which are group operations, i.e., when $J = \mathcal{G}$, a group, and $\bullet$ is the group operation. As is customary in harmonic analysis on groups we omit the $\bullet$ and write the group operation by juxtaposition or addition. If $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ is a frame for $\mathcal{H}$ and the group operation of $\mathcal{G}$ is a frame multiplication for Φ, then we say that $\mathcal{G}$ defines a frame multiplication for Φ (remember we are omitting $\bullet$!). Throughout the chapter $\mathcal{H}$ is a finite dimensional Hilbert space over $\mathbb{R}$ or $\mathbb{C}$ unless stated otherwise.

4.1 G-frames

Definition 4.1.1 (G-frame 1)**.** Let $\mathcal{G}$ be a finite group. A finite frame Φ for $\mathcal{H}$ is a *G-frame* if there exists $\pi : \mathcal{G} \to \mathcal{U}(\mathcal{H})$, a unitary representation of $\mathcal{G}$, and $\varphi \in \mathcal{H}$ such that

$$\Phi = \{\pi(g)\varphi\}_{g\in\mathcal{G}},$$

where the equality is in terms of multisets.

Not to be confused with the abbreviated form of generalized frames, G-frames are one of several related classes of frames and codes that have been the object of recent study. Bölcskei and Eldar [18] define *geometrically uniform* frames as the orbit of a generating vector under an abelian group of unitary matrices. A signal space code is called *geometrically uniform* by Forney [21] or a *group code* by Slepian [37] if its symmetry group (a group of isometries) acts transitively. *Harmonic frames* are projections of the rows or columns of the character table (Fourier matrix) of an abelian group (cf. [8], [10], [30]). In [40] it was shown that harmonic frames and geometrically uniform tight frames are equivalent and can be characterized by their Gramian. We take our first definition for a G-frame from Han [24] where the associated representation π is called a *frame representation.* Frame representations were introduced by Han and Larson in [25].

If Φ is a G-frame, then Φ is generated by the orbit of any one of its elements under the action of $\mathcal{G}$, and if Φ contains N unique vectors, then each element of Φ is repeated $|\mathcal{G}|/N$ times. We fix an identity element of the frame φ_e and write $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ where $\varphi_g := \pi(g)\varphi_e$. From this we see that G-frames are frames for which there exists an indexing by the group $\mathcal{G}$ such that

$$\pi(g)\varphi_h = \pi(g)\pi(h)\varphi_e = \pi(gh)\varphi_e = \varphi_{gh}.$$

This leads to a second (essentially equivalent) definition of a G-frame for a frame already indexed by $\mathcal{G}$. This is the definition used by Vale and Waldron in [41].

Definition 4.1.2 (G-frame 2)**.** Let $\mathcal{G}$ be a finite group. A finite frame $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ for a Hilbert space $\mathcal{H}$ is a *G-frame* if there exists $\pi : \mathcal{G} \to \mathcal{U}(\mathcal{H})$, a unitary representation of $\mathcal{G}$, such that

$$\forall g, h \in \mathcal{G}, \quad \pi(g)\varphi_h = \varphi_{gh}.$$

The difference in the second definition is that we begin with a frame as a sequence indexed by $\mathcal{G}$ and ask whether a particular type of representation exists. In the former definition we began with only a multiset of vectors and asked whether an indexing exists such that the latter definition holds. For example, let $\mathcal{G} = \mathbb{Z}/4\mathbb{Z} = (\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}, +)$ and consider the frame $\Phi = \{\varphi_{\bar{0}} = 1, \varphi_{\bar{1}} = -1, \varphi_{\bar{2}} = i, \varphi_{\bar{3}} = -i\}$ for $\mathbb{C}$. Φ would pass as a G-frame under the first definition because there are two one-dimensional representations of $\mathcal{G}$ that generate Φ (this is clear from the Fourier matrix of $\mathbb{Z}/4\mathbb{Z}$). However, it would not qualify as a G-frame under definition two because the representation π would have to satisfy $\pi(\bar{1})\varphi_{\bar{0}} = \varphi_{\bar{1}}$, i.e., $\pi(\bar{1})1 = -1$. There is one one-dimensional representation of $\mathbb{Z}/4\mathbb{Z}$ which satisfies this, but it does not generate Φ. Indeed, it is defined by $\pi(\bar{0}) = 1, \pi(\bar{1}) = -1, \pi(\bar{2}) = 1, \pi(\bar{3}) = -1$. In keeping with our view that a frame is a sequence with a fixed index, we will use the second definition from here on.

Vale and Waldron noted in [41] that if $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ is a G-frame, then its Gramian has a special form

$$(G_{g,h}) = \langle \varphi_h, \varphi_g \rangle = \langle \pi(h)\varphi_e, \pi(g)\varphi_e \rangle = \left\langle \varphi_e, \pi(h^{-1}g)\varphi_e \right\rangle, \tag{4.1}$$

i.e., the g-h-entry is a function of $h^{-1}g$. This is what is called a G-matrix.

Definition 4.1.3 (G-matrix)**.** Let $\mathcal{G}$ be a finite group. A matrix $A = (a_{g,h})_{g,h\in\mathcal{G}}$ is called a *G-matrix* if there exists a function $\nu : \mathcal{G} \to \mathbb{C}$ such that

$$\forall g, h \in \mathcal{G}, \quad a_{g,h} = \nu(h^{-1}g).$$

Vale and Waldron were then able to prove a version of the following theorem using an argument that hints at a connection to frame multiplication. We include a version of their proof and highlight the connections to our theory.

Theorem 4.1.4 (Vale and Waldron, [41]). *Let $\mathcal{G}$ be a finite group. A frame $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ for a Hilbert space $\mathcal{H}$ is a G-frame if and only if its Gramian is a G-matrix.*

Proof. If Φ is a G-frame, then (4.1) implies its Gramian is the G-matrix defined by $\nu(g) = \langle \varphi_e, \pi(g)\varphi_e\rangle$.

For the converse, suppose the Gramian of Φ is a G-matrix. Let S be the frame operator, and let $\widetilde{\varphi}_g := S^{-1}\varphi_g$ be the canonical dual frame elements. Each $x \in \mathcal{H}$ has the frame decomposition

$$x = \sum_{h\in\mathcal{G}} \langle x, \widetilde{\varphi}_h\rangle \varphi_h. \tag{4.2}$$

For each $g \in \mathcal{G}$, define a linear operator $U_g : \mathcal{H} \to \mathcal{H}$ by

$$\forall x \in \mathcal{H}, \quad U_g(x) := \sum_{h\in\mathcal{G}} \langle x, \widetilde{\varphi}_h\rangle \varphi_{gh}.$$

Since the Gramian of Φ is a G-matrix, we have

$$\forall g, h, k \in \mathcal{G}, \quad \langle \varphi_{gh}, \varphi_{gk}\rangle = \nu((gh)^{-1}gk) = \nu(h^{-1}k) = \langle \varphi_h, \varphi_k\rangle . \tag{4.3}$$

A calculation shows that U_g is unitary; it follows from (4.2) and (4.3) that

$$\begin{aligned}
\langle U_g(x), U_g(y)\rangle &= \left\langle \sum_{h\in\mathcal{G}} \langle x, \widetilde{\varphi}_h\rangle \varphi_{gh}, \sum_{k\in\mathcal{G}} \langle y, \widetilde{\varphi}_k\rangle \varphi_{gk} \right\rangle \\
&= \sum_{h\in\mathcal{G}} \sum_{k\in\mathcal{G}} \langle x, \widetilde{\varphi}_h\rangle \overline{\langle y, \widetilde{\varphi}_k\rangle} \langle \varphi_{gh}, \varphi_{gk}\rangle \\
&= \sum_{h\in\mathcal{G}} \sum_{k\in\mathcal{G}} \langle x, \widetilde{\varphi}_h\rangle \overline{\langle y, \widetilde{\varphi}_k\rangle} \langle \varphi_h, \varphi_k\rangle \\
&= \left\langle \sum_{h\in\mathcal{G}} \langle x, \widetilde{\varphi}_h\rangle \varphi_h, \sum_{k\in\mathcal{G}} \langle y, \widetilde{\varphi}_k\rangle \varphi_k \right\rangle \\
&= \langle x, y\rangle .
\end{aligned}$$

Also, for every $h, k \in \mathcal{G}$

$$\begin{aligned}
\langle U_g(\varphi_h) - \varphi_{gh}, \varphi_{gk} \rangle &= \langle U_g(\varphi_h), \varphi_{gk} \rangle - \langle \varphi_{gh}, \varphi_{gk} \rangle \\
&= \left\langle \sum_{m \in \mathcal{G}} \langle \varphi_h, \widetilde{\varphi}_m \rangle \varphi_{gm}, \varphi_{gk} \right\rangle - \langle \varphi_{gh}, \varphi_{gk} \rangle \\
&= \sum_{m \in \mathcal{G}} \langle \varphi_h, \widetilde{\varphi}_m \rangle \langle \varphi_{gm}, \varphi_{gk} \rangle - \langle \varphi_{gh}, \varphi_{gk} \rangle \\
&= \sum_{m \in \mathcal{G}} \langle \varphi_h, \widetilde{\varphi}_m \rangle \langle \varphi_m, \varphi_k \rangle - \langle \varphi_h, \varphi_k \rangle \\
&= \left\langle \sum_{m \in \mathcal{G}} \langle \varphi_h, \widetilde{\varphi}_m \rangle \varphi_m, \varphi_k \right\rangle - \langle \varphi_h, \varphi_k \rangle \\
&= \langle \varphi_h, \varphi_k \rangle - \langle \varphi_h, \varphi_k \rangle \\
&= 0.
\end{aligned}$$

By allowing k to vary over all of $\mathcal{G}$ it follows that $U_g(\varphi_h) = \varphi_{gh}$. This implies $\pi : g \mapsto U_g$ is a unitary representation, since

$$\forall g_1, g_2, h \in \mathcal{G}, \quad U_{g_1 g_2}\varphi_h = \varphi_{g_1 g_2 h} = U_{g_1}\varphi_{g_2 h} = U_{g_1}U_{g_2}\varphi_h$$

and $\{\varphi_h\}_{h \in \mathcal{G}}$ spans $\mathcal{H}$. Hence, π is a unitary representation of $\mathcal{G}$ such that $\pi(g)\varphi_h = \varphi_{gh}$, i.e., Φ is a G-frame for $\mathcal{H}$. ■

The operators $\{U_g\}_{g \in \mathcal{G}}$ are essentially frame multiplication on the left by φ_g, but there may not exist a bilinear product on all of $\mathcal{H}$ which agrees with or properly joins the operators $\{U_g\}_{g \in \mathcal{G}}$. We show in Lemma 4.2.1 that when these operators do arise from a frame multiplication defined by $\mathcal{G}$, then they are unitary as above when the Gramian is a G-matrix. Equation 4.3 is reminiscent of Theorem 3.3.2, and together they are enough to show, for an *abelian* group $\mathcal{G}$, if the Gramian of

$\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ is a G-matrix (or by the above theorem if Φ is a G-frame), then $\mathcal{G}$ defines a frame multiplication for Φ. We can say more than this. We can tell the equivalence class of frames Φ belongs to (we show it is equivalent to a harmonic frame) and what type of multiplication is defined by $\mathcal{G}$. We do this in Section 4.2.

Example 4.1.5. For $\mathcal{G}$ a cyclic group, a G-matrix is a circulant matrix. If we consider $\mathcal{G} = \mathbb{Z}/4\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ with the natural ordering, then all G-matrices (corresponding to this choice of $\mathcal{G}$) are of the form

$$\begin{pmatrix} \nu(\bar{0}) & \nu(\bar{3}) & \nu(\bar{2}) & \nu(\bar{1}) \\ \nu(\bar{1}) & \nu(\bar{0}) & \nu(\bar{3}) & \nu(\bar{2}) \\ \nu(\bar{2}) & \nu(\bar{1}) & \nu(\bar{0}) & \nu(\bar{3}) \\ \nu(\bar{3}) & \nu(\bar{2}) & \nu(\bar{1}) & \nu(\bar{0}) \end{pmatrix}$$

for some $\nu : \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\} \to \mathbb{C}$, which is a regular 4×4 circulant matrix.

Example 4.1.6. For a non-circulant example of a G-matrix, let $\mathcal{G} = D_3$, the dihedral group of order 6. If we use the presentation

$$D_3 =< r, s : r^3 = e, s^2 = e, rs = sr^2 >$$

and order the elements e, r, r^2, s, sr, sr^2, then every G-matrix has the form

$$\begin{array}{c@{}c} & \begin{array}{cccccc} e & r & r^2 & s & sr & sr^2 \end{array} \\ \begin{array}{c} e \\ r \\ r^2 \\ s \\ sr \\ sr^2 \end{array} & \begin{pmatrix} \nu(e) & \nu(r^2) & \nu(r) & \nu(s) & \nu(sr) & \nu(sr^2) \\ \nu(r) & \nu(e) & \nu(r^2) & \nu(sr) & \nu(sr^2) & \nu(s) \\ \nu(r^2) & \nu(r) & \nu(e) & \nu(sr^2) & \nu(s) & \nu(sr) \\ \nu(s) & \nu(sr) & \nu(sr^2) & \nu(e) & \nu(r^2) & \nu(r) \\ \nu(sr) & \nu(sr^2) & \nu(s) & \nu(r) & \nu(e) & \nu(r^2) \\ \nu(sr^2) & \nu(s) & \nu(sr) & \nu(r^2) & \nu(r) & \nu(e) \end{pmatrix} \end{array}$$

for some $\nu : D_3 \to \mathbb{C}$.

It is a well known result that the rows and columns of the character table of an abelian group are orthogonal. This fact combined with one direction of Naimark's theorem (Theorem 1.4.6), that the orthogonal projection of an orthogonal basis is a tight frame, gives a class of frames called *harmonic frames*.

Definition 4.1.7 (Harmonic frames)**.** A equal-norm frame Φ of N vectors for $\mathcal{H}$ is a *harmonic frame* if it comes from the character table of an abelian group, i.e., is given by the columns of a submatrix obtained by taking d rows of the character table of an abelian group of order N.

By their definition it is clear that there are only finitely many inequivalent harmonic frames for a particular abelian group. It is known, see the following theorem, that if a tight frame is generated by an abelian group of symmetries, then it is equivalent to a harmonic frame. Hence, there are only finitely many equivalence classes of tight frames generated by a fixed abelian group.

Theorem 4.1.8 (Theorem 5.4 of [40])**.** *A tight frame Φ of n vectors for $\mathcal{H}$ is unitarily equivalent to a harmonic frame if and only if it is generated by an abelian group $\mathcal{G} \subset \mathcal{U}(\mathcal{H})$, i.e., $\Phi = \mathcal{G}\varphi$, $\forall \varphi \in \Phi$.*

4.2 Abelian Frame Multiplications

We begin with a naive exploration of the cyclic group case and show how it quickly leads to something very familiar to harmonic analysts. Let $\Phi = \{\varphi_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ be a linearly dependent ($N > d$) frame. Suppose $* : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$ is a bilinear product such that $\varphi_m * \varphi_n = \varphi_{m+n}$, i.e., $\mathbb{Z}/N\mathbb{Z}$ defines a frame multiplication for Φ. By linear dependence, there exists coefficients $\{a_k\}_{k=0}^{N-1} \subset \mathbb{C}$ not all zero such that

$$\sum_{k=0}^{N-1} a_k \varphi_k = 0.$$

Multiplying on the left by φ_m and utilizing the aforementioned properties of $*$ yields

$$0 = \varphi_m * \left(\sum_{k=0}^{N-1} a_k \varphi_k \right) = \sum_{k=0}^{N-1} a_k \left(\varphi_m * \varphi_k \right) = \sum_{k=0}^{N-1} a_k \varphi_{m+k} \quad \forall m \in \mathbb{Z}/N\mathbb{Z}. \tag{4.4}$$

It is convenient to rewrite (4.4) with the index on the coefficients varying with m.

$$\sum_{k=0}^{N-1} a_{k-m} \varphi_k = 0 \quad \forall m \in \mathbb{Z}/N\mathbb{Z} \tag{4.5}$$

This makes it easy to write down the matrix formulation of (4.5). Let $\mathbf{a} = (a_k)_{k=0}^{N-1}$, let A be the $N \times N$ circulant matrix generated by the vector $\mathbf{a}$, and let Φ be the

$N \times d$ matrix with vectors φ_k as its rows. In symbols,

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & \dots & a_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \vdots \\ \varphi_{N-1} \end{pmatrix}.$$

Equation (4.5) in matrix form is

$$A\Phi = 0.$$

We can easily see from this formulation that the columns of Φ are in the nullspace of a circulant matrix A. A consequence of this and the fact that the DFT diagonalizes circulant matrices is that the columns of Φ are linear combinations of some subset of at least d (the rank of Φ is d) columns of the DFT matrix, and if $\omega_j = e^{2\pi ij/N}$, then $a_0 + a_{N-1}\omega_j + a_{N-2}\omega_j^2 + \ldots + a_1\omega_j^{N-1} = \lambda_j$ (the eigenvalues of A) are zero for at least d choices of $j \in \{0, 1, \ldots, N-1\}$. Assuming the existence of a frame multiplication for a spanning set (a frame) and a cyclic group has lead to a condition involving the discrete Fourier transform; this is a promising development. In what follows we improve upon the discussion here by proving a result for general finite abelian group operations and tight frames.

As a matter of good bookkeeping, we present next an extension of Lemma 3.3.1 to infinite dimensional Hilbert spaces when $\mathcal{G}$ is a group and not just a quasigroup. The key additional quality of a group is the existence of left and right inverses for each element $g \in \mathcal{G}$. This occurs more generally for a loop, a quasigroup with identity, and in that case the following lemma holds as well.

Lemma 4.2.1. *Let $\mathcal{G}$ be a countable group, and let $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ be a tight frame for a separable Hilbert space $\mathcal{H}$ over $\mathbb{F}$ which may be infinite dimensional. If $\mathcal{G}$ defines a frame multiplication for Φ with continuous extension $*$ to all of $\mathcal{H}$, then the functions $L_g : \mathcal{H} \to \mathcal{H}$ defined by,*

$$L_g(x) = \varphi_g * x,$$

and $R_g : \mathcal{H} \to \mathcal{H}$ defined by

$$R_g(x) = x * \varphi_g$$

are unitary linear operators for every $g \in \mathcal{G}$.

Proof. Let $x \in \mathcal{H}$, $g \in \mathcal{G}$, and A be the frame constant for Φ. Linearity and continuity of L_g follows easily from bilinearity and continuity of $*$. To show L_g is unitary, we compute

$$\begin{aligned} A\left\|L_g^*(x)\right\|^2 &= \sum_{h\in\mathcal{G}} \left|\left\langle L_g^*(x), \varphi_h\right\rangle\right|^2 \\ &= \sum_{h\in\mathcal{G}} \left|\langle x, L_g(\varphi_h)\rangle\right|^2 \\ &= \sum_{h\in\mathcal{G}} \left|\langle x, \varphi_g * \varphi_h\rangle\right|^2 \quad \text{(def. of } L_g) \\ &= \sum_{h\in\mathcal{G}} \left|\langle x, \varphi_{gh}\rangle\right|^2 \\ &= \sum_{h\in\mathcal{G}} \left|\langle x, \varphi_h\rangle\right|^2 \quad \text{(reordering terms)} \\ &= A\left\|x\right\|^2. \end{aligned}$$

Therefore L_g^* is an isometry. If $\mathcal{H}$ is finite dimensional, this is equivalent to L_g^* and L_g being unitary. For the infinite dimensional case we also need that L_g is an isometry (this is one of many equivalent characterizations of unitary operators). To show L_g is an isometry we first show it is invertible and $L_g^{-1} = L_{g^{-1}}$.

Define $D = \{\sum_h a_h\varphi_h : |\{a_h : a_h \neq 0\}| < \infty\}$, i.e., D is all finite linear combinations of frame elements from Φ. It follows from the frame reconstruction formula that D is dense in $\mathcal{H}$. Now, for any $g \in \mathcal{G}$, L_g maps D onto D, and for every $x = \sum_h a_h\varphi_h$ in D,

$$\begin{aligned} L_{g^{-1}}L_g(x) &= L_{g^{-1}}L_g\left(\sum_h a_h\varphi_h\right) \\ &= L_{g^{-1}}\left(\sum_h a_h\varphi_{gh}\right) \\ &= \sum_h a_h\varphi_h = x. \end{aligned}$$

In short, $L_{g^{-1}}L_g$ is linear, bounded, and the identity on a dense subspace of $\mathcal{H}$, therefore it is the identity on all of $\mathcal{H}$.

Now we show L_g is an isometry. From general operator theory we have equalities

$$\left\|L_g^{-1}\right\|_{op} = \|L_{g^{-1}}\|_{op} = \left\|L_{g^{-1}}^*\right\|_{op} = 1,$$

and

$$\|L_g\|_{op} = \left\|L_g^*\right\|_{op} = 1.$$

Employing these and the definition of operator norm yields

$$\|L_g(x)\| \leq \|x\| \quad \text{and} \quad \|x\| = \left\|L_g^{-1}L_g(x)\right\| \leq \left\|L_g^{-1}\right\|_{op}\|L_g(x)\| = \|L_g(x)\|.$$

Therefore, $\|L_g(x)\| = \|x\|$. The same proof shows R_g is unitary. ■

Theorem 4.2.2 (Abelian frame multiplications 1)**.** *Let $\mathcal{G}$ be a finite abelian group, and let $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ be a tight frame for $\mathcal{H}$. Then $\mathcal{G}$ defines a frame multiplication for Φ if and only if Φ is a G-frame.*

Proof. Suppose $\mathcal{G}$ defines a frame multiplication for Φ and the bilinear product given on $\mathcal{H}$ is $*$. For each $g \in \mathcal{G}$ define an operator $U_g : \mathcal{H} \to \mathcal{H}$ by the formula

$$U_g(x) = \varphi_g * x.$$

By Lemma 4.2.1, $\{U_g\}_{g\in\mathcal{G}}$ is a family of unitary operators on $\mathcal{H}$. Define a map $\pi : g \mapsto U_g$. That π is a unitary representation of $\mathcal{G}$ follows from the calculation

$$U_g U_h \varphi_k = U_g(\varphi_h * \varphi_k) = U_g(\varphi_{hk}) = \varphi_g * \varphi_{hk} = \varphi_{ghk} = U_{gh}\varphi_k,$$

and that $\{\varphi_k\}_{k\in\mathcal{G}}$ spans $\mathcal{H}$. Finally, we have $\pi(g)\varphi_h = \varphi_{gh}$ showing Φ is a G-frame.

Conversely, suppose $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ is a G-frame. Then there exists a unitary representation π of $\mathcal{G}$ such that $\pi(g)\varphi_h = \varphi_{gh}$. It follows from the fact $\pi(g)$ is unitary and $\mathcal{G}$ is abelian that

$$\forall g, h_1, h_2 \in \mathcal{G}, \quad \langle \varphi_{h_1}, \varphi_{h_2} \rangle = \langle \pi(g)\varphi_{h_1}, \pi(g)\varphi_{h_2} \rangle = \langle \varphi_{gh_1}, \varphi_{gh_2} \rangle = \langle \varphi_{h_1 g}, \varphi_{h_2 g} \rangle .$$

Hence, Theorem 3.3.2 implies $\mathcal{G}$ defines a frame multiplication for Φ. ■

When $\mathcal{H} = \mathbb{C}^d$ and $\mathcal{G}$ is a finite abelian group, we can describe the form of frame multiplications defined by $\mathcal{G}$ and the equivalence class of the associated frames. Given the connection between G-frames, frames generated by groups of operators, and frame multiplications defined by abelian groups, the following theorem is an extension of Theorem 4.1.8.

Theorem 4.2.3 (Abelian frame multiplications 2)**.** *Let $\mathcal{G}$ be a finite abelian group, and let $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ be a tight frame for $\mathbb{C}^d$. If $\mathcal{G}$ defines a frame multiplication for Φ, then Φ is unitarily equivalent to a harmonic frame and there exists $U \in \mathcal{U}(\mathbb{C}^d)$ and $c > 0$ such that*

$$cU\left(\varphi_g * \varphi_h\right) = cU\left(\varphi_g\right) cU\left(\varphi_h\right), \tag{4.6}$$

where the product on the right is vector pointwise multiplication and $*$ is the frame multiplication given by $\mathcal{G}$, i.e., $\varphi_g * \varphi_h := \varphi_{gh}$.

Remark 4.2.4. Strictly speaking, we may cancel a c from both sides of Equation 4.6. We leave them in place because, as we shall see in the proof, cU maps the tight frame Φ to a harmonic frame. Therefore, it is made clearer what (4.6) means when all the c's are in place, i.e., performing the frame multiplication defined by $\mathcal{G}$ and then mapping to the harmonic frame is the same as first mapping to the harmonic frame and then multiplying pointwise.

Proof. For each $g \in \mathcal{G}$ define an operator $U_g : \mathbb{C}^d \to \mathbb{C}^d$ by the formula

$$U_g(x) = \varphi_g * x.$$

By Theorem 4.2.2, $\{U_g\}_{g\in\mathcal{G}}$ is an abelian group of unitary operators which generates

$$\Phi = \{U_g(\varphi_e) : g \in \mathcal{G}\}.$$

Furthermore, since the U_g are unitary, we have

$$\forall g \in \mathcal{G}, \quad \left\|\varphi_e\right\|_2 = \left\|U_g(\varphi_e)\right\|_2 = \left\|\varphi_g\right\|_2,$$

which shows Φ is equal-norm. For the next step we use a technique found in the proof of Theorem 5.4 of [40]. A commuting family of diagonalizable operators, such as $\{U_g\}_{g\in\mathcal{G}}$, is simultaneously diagonalizable, i.e., there is a unitary operator V for which

$$\forall g \in \mathcal{G}, \quad D_g = VU_gV^*$$

is a diagonal matrix; see [31, Th. 6.5.8]. This is a consequence of Schur's lemma and Maschke's theorem. Since $\{U_g\}_{g\in\mathcal{G}}$ is an abelian group of operators, all the invariant subspaces are one dimensional, and orthogonally decomposing $\mathbb{C}^d$ into the invariant subspaces of $\{U_g\}_{g\in\mathcal{G}}$ simultaneously diagonalizes the operators U_g. The operators D_g are unitary, and consequently, they have diagonal entries of modulus 1. Define a new frame, generated by the diagonal operators D_g, Ψ by

$$\psi = V(\varphi_e), \quad \Psi = \{D_g(\psi) : g \in \mathcal{G}\}.$$

Clearly, then

$$\Phi = \{U_g(\varphi_e) : g \in \mathcal{G}\} = V^*\Psi.$$

Let $(D_g\psi)_j$ be the j-th component of the vector $D_g\psi$. Form the $d \times |\mathcal{G}|$ matrix with columns the elements of Ψ, i.e.,

$$\begin{pmatrix} (D_g\psi)_0 & \dots & (D_h\psi)_0 \\ (D_g\psi)_1 & \dots & (D_h\psi)_1 \\ \vdots & \ddots & \vdots \\ (D_g\psi)_{d-1} & \dots & (D_h\psi)_{d-1} \end{pmatrix}. \tag{4.7}$$

Since Ψ is the image of Φ under V, it is a equal-norm tight frame, and the matrix (4.7) has orthogonal rows of equal length. We compute the norm of row j to be

$$\left(\sum_g |(D_g)_j(\psi)_j|^2\right)^{1/2} = \sqrt{|\mathcal{G}|}|(\psi)_j|.$$

Therefore, the components of ψ have equal modulus, and if we let W^* be the diagonal matrix with the entries of ψ on its diagonal, then there exists $c > 0$ such that cW^* is a unitary matrix. Now, we have

$$\Phi = \frac{1}{c} U^* \{D_g \mathbf{1} : g \in \mathcal{G}\}, \quad \text{where } \mathbf{1} = (1, 1, \ldots, 1)^t \text{ and } U^* = cV^*W^* \text{ is unitary.}$$

It is important to note that we have more than just the equality of sets of vectors stated above; the g's on both sides coincide under the transformation, i.e.,

$$\begin{aligned}\frac{1}{c} U^*(D_g \mathbf{1}) &= V^* W^* D_g(\mathbf{1}) = V^* D_g(\psi) \\ &= U_g V^*(\psi) = U_g(\varphi_e) = \varphi_g.\end{aligned}$$

We have found our unitary operator U and positive constant c such that $cU\varphi_g = D_g \mathbf{1}$. It remains to show that $\{D_g \mathbf{1} : g \in \mathcal{G}\}$ is harmonic and that the product $*$ behaves as claimed. Proving $\{D_g \mathbf{1} : g \in \mathcal{G}\}$ is harmonic amounts to showing for $j = 0, 1, \ldots, d-1$,

$$\gamma_j : \mathcal{G} \to \mathbb{C}$$

defined by

$$\gamma_j(g) = (D_g \mathbf{1})_j = (D_g)_{jj}$$

is a character of the group $\mathcal{G}$. This follows easily since

$$\forall j, \quad \gamma_j(gh) = (D_{gh})_{jj} = (D_g)_{jj}(D_h)_{jj} = \gamma_j(g)\gamma_j(h).$$

Finally, we compute

$$\begin{aligned} cU(\varphi_g * \varphi_h) &= cU(\varphi_{gh}) \\ &= D_{gh}\mathbf{1} \\ &= (D_g\mathbf{1})(D_h\mathbf{1}) \\ &= cU(\varphi_g)cU(\varphi_h). \end{aligned} \qquad \blacksquare$$

In the discussion motivating this section we supposed there was a bilinear product on $\mathbb{C}^d$ and a frame Φ such that $\varphi_m * \varphi_n = \varphi_{m+n}$, i.e., our underlying group was $\mathbb{Z}/N\mathbb{Z}$. By strengthening our assumptions on Φ to be a tight frame, we may apply Theorem 4.2.2 to show Φ is a G-frame for the abelian group $\mathbb{Z}/N\mathbb{Z}$, and furthermore, by Theorem 4.2.3, Φ is unitarily equivalent to a DFT frame (harmonic frame with $\mathcal{G} = \mathbb{Z}/N\mathbb{Z}$). In summary, we have the following corollary.

Corollary 4.2.5. *Let $\Phi = \{\varphi_n\}_{n \in \mathbb{Z}/N\mathbb{Z}} \subseteq \mathbb{C}^d$ be a tight frame. If $\mathbb{Z}/N\mathbb{Z}$ defines a frame multiplication for Φ, then Φ is unitarily equivalent to a DFT frame.*

Example 4.2.6. Consider the group $\mathbb{Z}/4\mathbb{Z}$, and let

$$\Phi = \left\{ \varphi_{\bar{0}} = \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}, \varphi_{\bar{1}} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \varphi_{\bar{2}} = \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}, \varphi_{\bar{3}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}.$$

$\Phi = \{\varphi_g\}_{g \in \mathbb{Z}/4\mathbb{Z}}$ is a tight frame for $\mathbb{C}^2$, and the Gramian of Φ is

$$G = \begin{pmatrix} 4 & 2+2i & 0 & 2-2i \\ 2-2i & 4 & 2+2i & 0 \\ 0 & 2-2i & 4 & 2+2i \\ 2+2i & 0 & 2-2i & 4 \end{pmatrix}.$$

It is easy to see that G is a G-matrix for $\mathbb{Z}/4\mathbb{Z}$, and therefore, by Theorems 4.1.4 and 4.2.2, $\mathbb{Z}/4\mathbb{Z}$ defines a frame multiplication for Φ. Hence, by Theorem 4.2.3, there exists a unitary matrix U and positive constant c such that $cU\Phi$ is a harmonic frame. Indeed, if we let

$$c = \frac{1}{\sqrt{2}}, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix},$$

then

$$\Psi := cU\Phi = \left\{ \psi_{\overline{0}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \psi_{\overline{1}} = \begin{pmatrix} 1 \\ i \end{pmatrix}, \psi_{\overline{2}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \psi_{\overline{3}} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$$

is a harmonic frame, and

$$\forall g, h \in \mathbb{Z}/4\mathbb{Z}, \quad cU(\varphi_{gh}) = cU(\varphi_g)cU(\varphi_h).$$

4.3 General Group Frame Multiplications

For a general finite group $\mathcal{G}$ (not necessarily abelian) we do not have Theorem 4.2.2 in full or the description of possible frame multiplications given by Theorem 4.2.3. What makes the situation different is that for $\mathcal{G}$ non-abelian and $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ a G-frame, while we still have

$$\forall h_1, h_2, g \in \mathcal{G}, \quad \langle \varphi_{h_1}, \varphi_{h_2} \rangle = \langle \varphi_{gh_1}, \varphi_{gh_2} \rangle, \tag{4.8}$$

it does not follow that

$$\forall h_1, h_2, g \in \mathcal{G}, \quad \langle \varphi_{h_1}, \varphi_{h_2} \rangle = \langle \varphi_{h_1 g}, \varphi_{h_2 g} \rangle. \tag{4.9}$$

By Theorem 3.3.2, (4.9) is necessary if $\mathcal{G}$ is to define a frame multiplication for Φ. Even the smallest non-abelian group can exhibit a failure of (4.9).

Example 4.3.1. Let $\mathcal{G}$ be the dihedral group of order 6,

$$\mathcal{G} = D_3 =< r, s : r^3 = 1, s^2 = 1, rs = sr^2 > .$$

Define a representation $\pi : D_3 \to \mathcal{U}(\mathbb{R}^2)$ on the generators of D_3 by

$$\pi(r) := \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad \pi(s) := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If we take $\varphi = (1,0)^t$, then $\Phi = \{\varphi_g := \pi(g)\varphi : g \in D_3\}$ is a G-frame for $\mathbb{R}^2$. The frame elements are $\varphi_e = (1,0)^t$, $\varphi_r = (-\frac{1}{2}, \frac{\sqrt{3}}{2})^t$, $\varphi_{r^2} = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})^t$, $\varphi_s =$

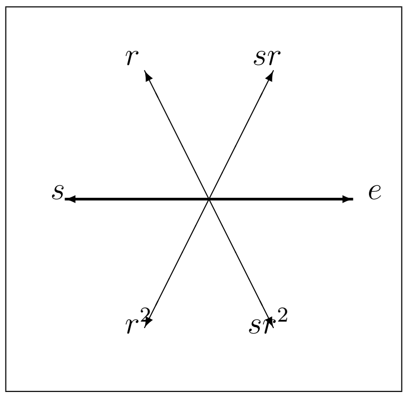


Figure 4.1: The elements of Φ, $\{\varphi_g\}_{g \in D_3}$.

$(-1,0)^t, \varphi_{sr} = (\frac{1}{2}, \frac{\sqrt{3}}{2})^t$, and $\varphi_{sr^2} = (\frac{1}{2}, -\frac{\sqrt{3}}{2})^t$.

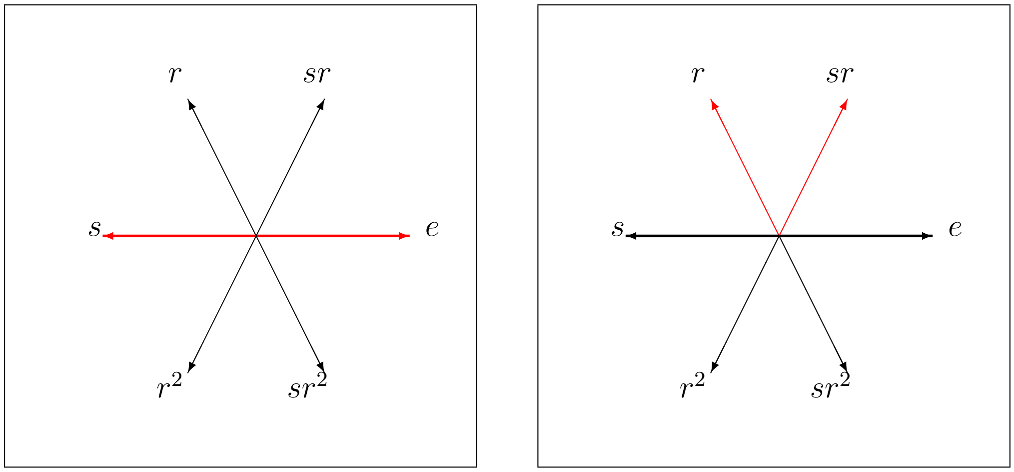
As expected, since Φ is a G-frame and (4.8) holds, we have

$$\langle \varphi_e, \varphi_s \rangle = -1 = \langle \varphi_r, \varphi_{rs} \rangle ,$$

but multiplying on the right by r yields

$$\langle \varphi_e, \varphi_s \rangle = -1 \neq \frac{1}{2} = \langle \varphi_r, \varphi_{sr} \rangle .$$

Therefore, D_3 cannot define a frame multiplication for Φ. We can show this another way by exploiting the linear relationships that exist between the frame elements à



(a) $\varphi_e + \varphi_s = 0$.

(b) $\varphi_r + \varphi_{sr} \neq 0$.

Figure 4.2: Proposition 3.2.3 in action.

la Proposition 3.2.3. Suppose that $*$ is the extension of a frame multiplication for Φ given by D_3. Then

$$\varphi_e + \varphi_s = 0;$$

therefore,

$$\begin{aligned} 0 &= (\varphi_e + \varphi_s) * \varphi_r \\ &= \varphi_e * \varphi_r + \varphi_s * \varphi_r \\ &= \varphi_r + \varphi_{sr} \\ &\neq 0. \end{aligned}$$

This is a contradiction, so no such $*$ can exist.

4.3.1 The Regular Representations

Definition 4.3.2 (Regular representation)**.** Let $\mathcal{G}$ be a finite group. The representation $(\ell^2(\mathcal{G}), \lambda)$ defined by

$$\forall g, h \in \mathcal{G}, \quad (\lambda(g)f)(h) = f(g^{-1}h)$$

is called the *left regular representation* of $\mathcal{G}$. The representation $(\ell^2(\mathcal{G}), \rho)$ defined by

$$\forall g, h \in \mathcal{G}, \quad (\rho(g)f)(h) = f(hg)$$

is called the *right regular representation* of $\mathcal{G}$.

If $\{e_g\}_{g\in\mathcal{G}}$ is the standard orthonormal basis for $\ell^2(\mathcal{G})$, i.e.,

$$e_g(h) = \begin{cases} 1 & h = g \\ 0 & h \neq g, \end{cases}$$

then

$$\lambda(g)e_h = e_{gh} \quad \text{and} \quad \rho(g)e_h = e_{hg^{-1}}.$$

The characters χ_λ and χ_ρ are

$$\chi_\lambda(g) = \begin{cases} |\mathcal{G}| & g = e \\ 0 & g \neq e \end{cases} \quad \text{and} \quad \chi_\rho(g) = \begin{cases} |\mathcal{G}| & g = e \\ 0 & g \neq e, \end{cases} \tag{4.10}$$

which follows from $\mathrm{Tr}(I) = |\mathcal{G}|$ and that for every $g \neq e$, $\lambda(g)$ and $\rho(g)$ permute $\{e_g\}_{g\in\mathcal{G}}$ and do not fix any e_g, i.e., their matrix forms with respect to the standard orthonormal basis have all zeroes on the diagonal. Since λ and ρ have the same character, we know they are unitarily equivalent and share the same isotypic decomposition; see Theorem 1.3.13.

Proposition 4.3.3. *The decomposition of the regular representations of $\mathcal{G}$ into isotypic components is*

$$\lambda = \rho = \bigoplus_{i=1}^{N} d_i \pi_i, \tag{4.11}$$

where N is the number of conjugacy classes of $\mathcal{G}$, $\pi_1, \pi_2, \ldots, \pi_N$ are the irreducible representations of $\mathcal{G}$, and $d_i = \dim \pi_i$. The equalities in (4.11) denote membership in the same equivalence class.

Proof. From Theorem 1.3.13, we have

$$\lambda = \bigoplus_{i=1}^{N} (\chi_\lambda | \chi_{\pi_i}) \pi_i,$$

and by (4.10), $(\chi_\lambda | \chi_{\pi_i}) = \overline{\chi_{\pi_i}(e)} = \dim \pi_i$. ■

4.3.2 Characterization of Group Frame Multiplications

Proposition 4.3.4. *Let $\mathcal{G}$ be a finite group with left regular representation λ and right regular representation ρ. Then $\mathcal{A}_\lambda' = \mathcal{A}_\rho$ and $\mathcal{A}_\rho' = \mathcal{A}_\lambda$.*

Proof. Let $\{e_g\}_{g \in \mathcal{G}}$ be the standard orthonormal basis for $\ell^2(\mathcal{G})$. We have

$$\forall g, h, k \in \mathcal{G}, \quad \rho(g)\lambda(h)e_k = \rho(g)e_{hk} = e_{hkg^{-1}} = \lambda(h)e_{kg^{-1}} = \lambda(h)\rho(g)e_k.$$

Showing $\lambda(h)$ and $\rho(g)$ commute; hence, $\mathcal{A}_\rho \subseteq \mathcal{A}_\lambda'$ (and $\mathcal{A}_\lambda \subseteq \mathcal{A}_\rho'$). For the other inclusion, suppose $T \in \mathcal{A}_\lambda'$. We must show that T can be written as a linear combination of operators in $\rho(\mathcal{G})$. Consider the operator $\widetilde{T} \in \mathcal{A}_\rho$ defined by

$$\widetilde{T} = \sum_{g \in \mathcal{G}} \langle T e_1, e_{g^{-1}} \rangle \rho(g).$$

We claim that $T = \widetilde{T}$. Indeed,

$$
\begin{aligned}
\forall h, k \in \mathcal{G}, \quad \left\langle \left(\widetilde{T} - T \right) e_h, e_k \right\rangle &= \left\langle \left(\sum_{g \in \mathcal{G}} \langle Te_1, e_{g^{-1}} \rangle \, \rho(g) - T \right) e_h, e_k \right\rangle \\
&= \left\langle \sum_{g \in \mathcal{G}} \langle Te_1, e_{g^{-1}} \rangle \, e_{hg^{-1}} - Te_h, e_k \right\rangle \\
&= \left\langle \sum_{g \in \mathcal{G}} \langle Te_1, e_{g^{-1}} \rangle \, e_{hg^{-1}}, e_k \right\rangle - \langle Te_h, e_k \rangle \\
&= \sum_{g \in \mathcal{G}} \langle Te_1, e_{g^{-1}} \rangle \, \langle e_{hg^{-1}}, e_k \rangle - \langle Te_h, e_k \rangle \\
&= \left\langle Te_1, e_{(k^{-1}h)^{-1}} \right\rangle - \langle Te_h, e_k \rangle \\
&= \left\langle Te_1, \lambda(h^{-1}) e_k \right\rangle - \langle Te_h, e_k \rangle \\
&= \langle \lambda(h) Te_1, e_k \rangle - \langle Te_h, e_k \rangle \\
&= \langle Te_h, e_k \rangle - \langle Te_h, e_k \rangle \quad (\text{since } T \in \mathcal{A}'_\lambda) \\
&= 0,
\end{aligned}
$$

and since $\{e_g\}_{g \in \mathcal{G}}$ is a basis this implies $\widetilde{T} - T = 0$. Therefore, $\mathcal{A}'_\lambda \subseteq \mathcal{A}_\rho$. Similarly to above, we can show directly that $\mathcal{A}'_\rho \subset \mathcal{A}_\lambda$, or we can appeal to the von Neumann double commutant theorem. That is, since $\mathcal{A}_\lambda$ is closed in the strong operator topology (in fact, it is finite dimensional), Theorem 1.2.5 implies $\mathcal{A}''_\lambda = \mathcal{A}_\lambda$; therefore, $\mathcal{A}'_\rho = \mathcal{A}''_\lambda = \mathcal{A}_\lambda$. ■

Theorem 4.3.5 (Group frame multiplications 1)**.** *Let $\mathcal{G}$ be a finite group, and let $\Phi = \{\varphi_g\}_{g \in \mathcal{G}}$ be a Parseval tight frame for $\mathcal{H}$. $\mathcal{G}$ defines a frame multiplication for Φ if and only if the Gramian of Φ is in $\mathcal{A}'_\lambda \cap \mathcal{A}'_\rho$.*

Proof. Let L be the analysis operator, and let $G = LL^*$ be the Gramian operator for the Parseval tight frame Φ. Note that since Φ is Parseval, we have $L^*L = S = I$. By

Naimark's theorem (Theorem 1.4.6) and the subsequent remark, G is the orthogonal projection onto $L(\mathcal{H}) \subseteq \ell^2(\mathcal{G})$, and if $\{e_g\}_{g\in\mathcal{G}}$ is the standard orthonormal basis for $\ell^2(\mathcal{G})$, then for every $g \in \mathcal{G}$, $L(\varphi_g) = G(e_g)$. Suppose $G \in \mathcal{A}'_\lambda \cap \mathcal{A}'_\rho$. Then

$$\begin{aligned}
\forall g, h, k \in \mathcal{G}, \quad \langle \varphi_g, \varphi_h \rangle &= \langle L\varphi_g, L\varphi_h \rangle \quad (L^*L = I) \\
&= \langle Ge_g, Ge_h \rangle \\
&= \langle \lambda(k)Ge_g, \lambda(k)Ge_h \rangle \quad \text{(since } \lambda(k) \text{ is unitary)} \qquad (4.12) \\
&= \langle G\lambda(k)e_g, G\lambda(k)e_h \rangle \quad \text{(since } G \in \mathcal{A}'_\lambda) \\
&= \langle Ge_{kg}, Ge_{kh} \rangle \\
&= \langle \varphi_{kg}, \varphi_{kh} \rangle .
\end{aligned}$$

By applying $\rho(k^{-1})$ in place of $\lambda(k)$ in (4.12) and using that $G \in \mathcal{A}'_\rho$, we can show $\langle \varphi_g, \varphi_h \rangle = \langle \varphi_{gk}, \varphi_{hk} \rangle$. Hence, Theorem 3.3.2 implies $\mathcal{G}$ defines a frame multiplication for Φ.

For the converse, assume $\mathcal{G}$ defines a frame multiplication for Φ. By Theorem 3.3.2, it follows that

$$\forall g, h, k \in \mathcal{G}, \quad \langle \varphi_g, \varphi_h \rangle = \langle \varphi_{gk}, \varphi_{hk} \rangle = \langle \varphi_{kg}, \varphi_{kh} \rangle . \qquad (4.13)$$

Consequently,

$$
\begin{aligned}
\forall g,h,k \in \mathcal{G}, \quad \langle \lambda(g)Ge_h, e_k\rangle &= \left\langle Ge_h, \lambda(g^{-1})e_k\right\rangle \\
&= \langle Ge_h, e_{g^{-1}k}\rangle \\
&= \langle Ge_h, Ge_{g^{-1}k}\rangle \quad (G \text{ is an orthogonal projection}) \\
&= \langle L\varphi_h, L\varphi_{g^{-1}k}\rangle \\
&= \langle \varphi_h, \varphi_{g^{-1}k}\rangle \quad (L^*L = I) \\
&= \langle \varphi_{gh}, \varphi_k\rangle \quad (\text{by } (4.13)) \\
&= \langle L\varphi_{gh}, L\varphi_k\rangle \\
&= \langle Ge_{gh}, Ge_k\rangle \\
&= \langle Ge_{gh}, e_k\rangle \\
&= \langle G\lambda(g)e_h, e_k\rangle .
\end{aligned}
$$

Since $\{e_g\}_{g\in\mathcal{G}}$ is a basis, we have for every $g \in \mathcal{G}$, $\lambda(g)G = G\lambda(g)$. Similarly, only utilizing the other half of (4.13), we can show for every $g \in \mathcal{G}$, $\rho(g)G = G\rho(g)$; therefore, $G \in \mathcal{A}'_\lambda \cap \mathcal{A}'_\rho$ as desired. ■

Proposition 4.3.6. *Let $(\mathcal{H}, \pi)$ be an irreducible representation of $\mathcal{G}$ and $f : \mathcal{G} \to \mathbb{C}$ a function. Consider the operator π_f on $\mathcal{H}$ defined by*

$$\pi_f = \sum_{g\in\mathcal{G}} f(g)\pi(g),$$

and recall a function $f : \mathcal{G} \to \mathbb{C}$ is a class function if it is constant on each conjugacy class of $\mathcal{G}$. If f is a class function, then

$$\pi_f = \frac{|\mathcal{G}|(f|\overline{\chi_\pi})}{\dim \pi} I.$$

Proof. Suppose f is a class function. Then

$$\forall h \in \mathcal{G}, \quad \pi(h)\pi_f = \pi(h)\sum_g f(g)\pi(g)$$
$$= \sum_g f(g)\pi(hg)$$
$$= \sum_g f(h^{-1}g)\pi(g)$$
$$= \sum_g f(gh^{-1})\pi(g)$$
$$= \sum_g f(g)\pi(gh)$$
$$= \pi_f \pi(h). \tag{4.14}$$

By (4.14) and Schur's lemma (Lemma 1.3.9), there is a $\xi \in \mathbb{C}$ such that $\pi_f = \xi I$. Also,

$$\operatorname{Tr}\pi_f = \sum_{g\in\mathcal{G}} f(g)\operatorname{Tr}\pi(g) = \sum_{g\in\mathcal{G}} f(g)\chi_\pi(g) = |\mathcal{G}|(f|\overline{\chi_\pi}),$$

and the result follows. ∎

Lemma 4.3.7. *Let $\mathcal{G}$ be a finite group, let $(\ell^2(\mathcal{G}), \lambda)$ be the left regular representation of $\mathcal{G}$, and let $\pi_1, \pi_2, \ldots, \pi_N$ be the unique irreducible representations of $\mathcal{G}$. For each i, $1 \le i \le N$, define P_i to be the orthogonal projection onto $\mathcal{H}_i$, the support of the i-th isotypic component of λ. If $P : \ell^2(\mathcal{G}) \to \ell^2(\mathcal{G})$ is an orthogonal projection, then $P \in \mathcal{A}'_\lambda \cap \mathcal{A}'_\rho$ if and only if there exist $I \subset \{1, 2, \ldots, N\}$ such that*

$$P = \sum_{i\in I} P_i. \tag{4.15}$$

Proof. If $I \subseteq \{1, 2, \ldots, N\}$ and $P = \sum_{i\in I} P_i$, then P is an orthogonal projection onto an invariant subspace of λ and ρ. This easily implies $P \in \mathcal{A}'_\lambda \cap \mathcal{A}'_\rho$. Conversely, suppose P is an orthogonal projection on $\ell^2(\mathcal{G})$, and suppose $P \in \mathcal{A}'_\lambda \cap \mathcal{A}'_\rho$. Since $P \in \mathcal{A}'_\rho$ Proposition 4.3.4 implies there exist coefficients $\{a_g\}_{g\in\mathcal{G}}$ such that

$$P = \sum_{g\in\mathcal{G}} a_g \lambda(g).$$

Furthermore,

$$\begin{aligned}
\forall h \in \mathcal{G}, \quad \langle Pe_1, e_h\rangle &= \left\langle \sum_g a_g \lambda(g) e_1, e_h \right\rangle \\
&= \sum_g a_g \langle e_g, e_h\rangle \\
&= a_h.
\end{aligned}$$

Because $P \in \mathcal{A}'_\lambda \cap \mathcal{A}'_\rho$, it follows

$$\forall g \in \mathcal{G}, \quad \langle Pe_1, e_h\rangle = \langle Pe_1, e_{ghg^{-1}}\rangle,$$

i.e., the function $h \mapsto \langle Pe_1, e_h\rangle$ is a class function on $\mathcal{G}$.

Decompose $\ell^2(\mathcal{G})$ into a direct sum of the supports of the isotypic components of λ, i.e.,

$$\ell^2(\mathcal{G}) = \bigoplus_{i=1}^{N} \mathcal{H}_i = \bigoplus_{i=1}^{N} \bigoplus_{j=1}^{d_i} \mathcal{K}_{i,j}.$$

Let $i \in \{1, 2, \ldots, N\}$, $x_i \in \mathcal{H}_i$, and let $x_i = \sum_{j=1}^{d_i} x_{i,j}$, where $x_{i,j} \in \mathcal{K}_{i,j}$. Then

$$\begin{aligned}
Px_i &= \sum_{g\in\mathcal{G}} \langle Pe_1, e_g\rangle \lambda(g) \left(\sum_{j=1}^{d_i} x_{i,j} \right) \\
&= \sum_{j=1}^{d_i} \left(\sum_{g\in\mathcal{G}} \langle Pe_1, e_g\rangle \lambda(g) x_{i,j} \right) \\
&= \sum_{j=1}^{d_i} \left(\sum_{g\in\mathcal{G}} \langle Pe_1, e_g\rangle \pi_i(g) x_{i,j} \right).
\end{aligned} \tag{4.16}$$

Since π_i is irreducible and $g \mapsto \langle Pe_1, e_g \rangle$ is a class function, applying Proposition 4.3.6 yields

$$\sum_{g \in \mathcal{G}} \langle Pe_1, e_g \rangle \pi_i(g) = \frac{|\mathcal{G}|(\langle Pe_1, e_g \rangle | \overline{\chi_{\pi_i}})}{d_i} I_{\mathcal{K}_{i,j}}.$$

Continuing where we left off in (4.16)

$$Px_i = \sum_{j=1}^{d_i} \left(\sum_{g \in \mathcal{G}} \langle Pe_1, e_g \rangle \pi_i(g) x_{i,j} \right) = \frac{|\mathcal{G}|(\langle Pe_1, e_g \rangle | \overline{\chi_{\pi_i}})}{d_i} x_i.$$

Therefore, x_i is an eigenvector of P. Since P is a projection, it can only have eigenvalues 0 and 1, and consequently, we may define $I \subseteq \{1, 2, \dots, N\}$ by $i \in I$ if and only if $Px_i = x_i$. Clearly, then

$$P = \sum_{i \in I} P_i. \qquad \blacksquare$$

We have an explicit formula for the projections $P \in \mathcal{A}'_\lambda \cap \mathcal{A}'_\rho$, i.e., those of the form (4.15) from Lemma 4.3.7. Since $\{\overline{\chi_{\pi_i}}\}_{i=1}^N$ is an orthonormal basis for the space of class functions on $\mathcal{G}$ (Theorem 1.3.12) and $g \mapsto \langle Pe_1, e_g \rangle$ is a class function,

$$\langle Pe_1, e_g \rangle = \sum_{i=1}^{N} (\langle Pe_1, e_g \rangle | \overline{\chi_{\pi_i}}) \overline{\chi_{\pi_i}(g)}.$$

Therefore, we can write P as

$$\begin{aligned} P = \sum_{g \in \mathcal{G}} \langle Pe_1, e_g \rangle \lambda(g) &= \sum_{g \in \mathcal{G}} \sum_{i=1}^{N} (\langle Pe_1, e_g \rangle | \overline{\chi_{\pi_i}}) \overline{\chi_{\pi_i}(g)} \lambda(g) \\ &= \sum_{i=1}^{N} \left(\sum_{g \in \mathcal{G}} (\langle Pe_1, e_g \rangle | \overline{\chi_{\pi_i}}) \overline{\chi_{\pi_i}(g)} \lambda(g) \right) \\ &= \sum_{i \in I} \frac{d_i}{|\mathcal{G}|} \left(\sum_{g \in \mathcal{G}} \overline{\chi_{\pi_i}(g)} \lambda(g) \right). \end{aligned}$$

The last equality follows by defining I as in Lemma 4.3.7. The operators

$$\frac{d_i}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \overline{\chi_{\pi_i}(g)} \lambda(g)$$

are precisely the projections P_i onto the support of the istopyic components of λ, cf. Theorem 4.1 of [32]. Combining this observation with Theorem 4.3.5 and Lemma 4.3.7, we have a general formula for the Gramian operator of a Parseval frame for which $\mathcal{G}$ defines a frame multiplication. In particular, this formula proves for a finite group $\mathcal{G}$ and Hilbert space $\mathcal{H}$, $\mathcal{G}$ defines a frame multiplication for only finitely many equivalence classes of tight frames for $\mathcal{H}$.

Theorem 4.3.8 (Group frame multiplications 2)**.** *Let $\mathcal{G}$ be a finite group, let $(\ell^2(\mathcal{G}), \lambda)$ be the left regular representation of $\mathcal{G}$, and let $\pi_1, \pi_2, \ldots, \pi_N$ be the unique irreducible representations of $\mathcal{G}$. If $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ is a Parseval tight frame for $\mathcal{H}$, then $\mathcal{G}$ defines a frame multiplication for Φ if and only if there exist $I \subset \{1, 2, \ldots, N\}$ such that*

$$G = \sum_{i\in I} \frac{d_i}{|\mathcal{G}|} \left(\sum_{g\in\mathcal{G}} \overline{\chi_{\pi_i}(g)} \lambda(g) \right) \tag{4.17}$$

is the Gramian operator of Φ.

Proof. Let G be the Gramian of the Parseval frame Φ. By Theorem 4.3.5, $\mathcal{G}$ defines a frame multiplication for Φ if and only if $G \in \mathcal{A}'_\lambda \cap \mathcal{A}'_\rho$, and by Lemma 4.3.7 and the subsequent remarks, $G \in \mathcal{A}'_\lambda \cap \mathcal{A}'_\rho$ if and only if there exist $I \subset \{1, 2, \ldots, N\}$ such that (4.17) holds. ∎

Corollary 4.3.9. *Let $\mathcal{G}$ be a finite group, and let $\pi_1, \pi_2, \ldots, \pi_N$ be the unique irreducible representations of $\mathcal{G}$. Denote the dimension of π_i by d_i. There exists a tight frame $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ for $\mathcal{H}$ such that $\mathcal{G}$ defines a frame multiplication for Φ if and* *only if there exists* $I \subseteq \{1, 2, \ldots, N\}$ *such that*

$$\dim \mathcal{H} = \sum_{i \in I} d_i^2. \tag{4.18}$$

Proof. By Proposition 4.3.3, the dimension of the support of the i-th isotypic component of λ is d_i^2. Suppose $\mathcal{G}$ defines a frame multiplication for Φ. By the above Theorem, the Gramian of Φ is a sum of projections onto the supports of the isotypic components of λ. Hence, the rank of the Gramian of Φ is of the form (4.18). Conversely, by Naimark's Theorem (Theorem 1.4.6), the projection of the standard orthonormal basis for $\ell^2(\mathcal{G})$ onto any sum of the supports of the isotypic components of λ is a Parseval frame $\Phi = \{\varphi_g\}_{g \in \mathcal{G}}$ for its span, and by the above Theorem, $\mathcal{G}$ defines a frame multiplication on Φ. ■

Example 4.3.10. Let $\mathcal{G} = D_3$, the dihedral group of order 6. If we order the standard basis elements for $\ell^2(D_3)$ e_1, e_r, e_{r^2}, e_s, e_{sr}, e_{sr^2}, then the generators of the left regular representation have matrix forms

$$\lambda(r) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \lambda(s) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

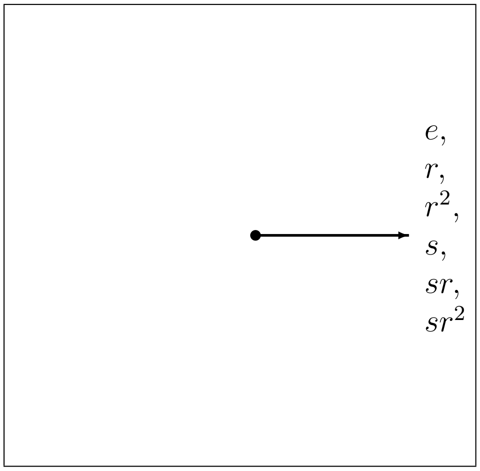
Since D_3 has three conjugacy classes, $\{1\}$, $\{r, r^2\}$, and $\{s, sr, sr^2\}$, and $6 = 1^2 + 1^2 + 2^2$ is the only decomposition of 6 into three squares, we can deduce it has two irreducible representations of dimension 1 and one irreducible representation of dimension 2. Corollary 4.3.9 tells us D_3 defines frame multiplications in dimensions $1^2 = 1$, $1^2 + 1^2 = 2$, $2^2 = 4$, $1^2 + 2^2 = 5$ and $1^2 + 1^2 + 2^2 = 6$. In order to find the frames associated with these frame multiplications we must decompose $\ell^2(D_3)$ into the support of the isotypic components of λ. With a little work we find orthonormal bases for the invariant subspaces of λ: $\mathcal{K}_{1,1}$, $\mathcal{K}_{2,1}$, $\mathcal{K}_{3,1}$, and $\mathcal{K}_{3,2}$.

$$
\begin{array}{c}
\begin{array}{cccc} \overbrace{\qquad}^{\mathcal{K}_{1,1}} & \overbrace{\qquad}^{\mathcal{K}_{2,1}} & \overbrace{\qquad\qquad}^{\mathcal{K}_{3,1}} & \overbrace{\qquad\qquad}^{\mathcal{K}_{3,2}} \end{array} \\
\begin{pmatrix}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{3}}{6} & \frac{1}{2} & -\frac{\sqrt{3}}{6} & -\frac{1}{2} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{3}}{6} & -\frac{1}{2} & -\frac{\sqrt{3}}{6} & \frac{1}{2} \\
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{3}}{3} & 0 \\
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{\sqrt{3}}{6} & -\frac{1}{2} & \frac{\sqrt{3}}{6} & -\frac{1}{2} \\
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{\sqrt{3}}{6} & \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{1}{2}
\end{pmatrix}
\end{array}.
$$

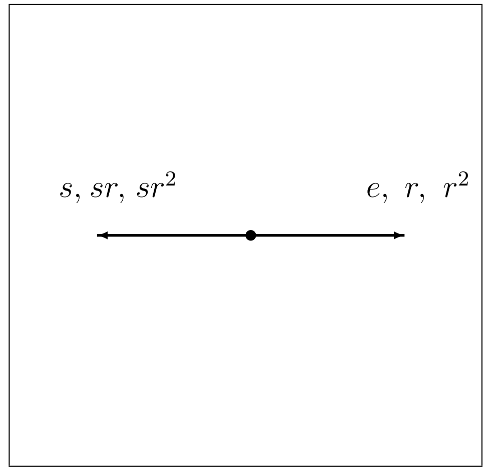
The support of the isotypic components of λ are

$$\mathcal{H}_1 = \mathcal{K}_{1,1}, \quad \mathcal{H}_2 = \mathcal{K}_{2,1}, \quad \text{and} \quad \mathcal{H}_3 = \mathcal{K}_{3,1} \oplus \mathcal{K}_{3,2}.$$

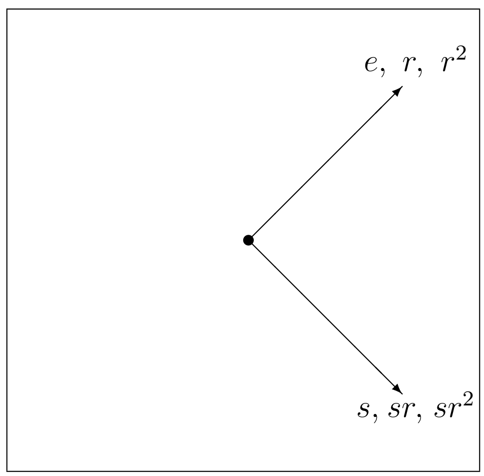
Projecting the standard orthonormal basis onto direct sums of the $\mathcal{H}_i$, we obtain the Parseval frames for which D_3 defines a frame multiplication. Listing the number of such projections, we see there are two frames for $\mathbb{C}^1$, one for $\mathbb{C}^2$, one for $\mathbb{C}^4$, two for $\mathbb{C}^5$, and one for $\mathbb{C}^6$. We can visualize the 1 and 2 dimensional frames in $\mathbb{R}$ and $\mathbb{R}^2$; see Figure 4.3.



(a) Projection onto $\mathcal{H}_1$: six copies of the vector $\left(\frac{1}{\sqrt{6}}\right)$



(b) Projection onto $\mathcal{H}_2$: three copies of $\left(\frac{1}{\sqrt{6}}\right)$ and three copies of $\left(-\frac{1}{\sqrt{6}}\right)$



(c) Projection onto $\mathcal{H}_1 \oplus \mathcal{H}_2$: three copies of $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and three copies of $\left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$

Figure 4.3: The 1 and 2 dimensional frames for which D_3 defines a frame multiplication.

4.3.3 Equivalence with Central G-frames

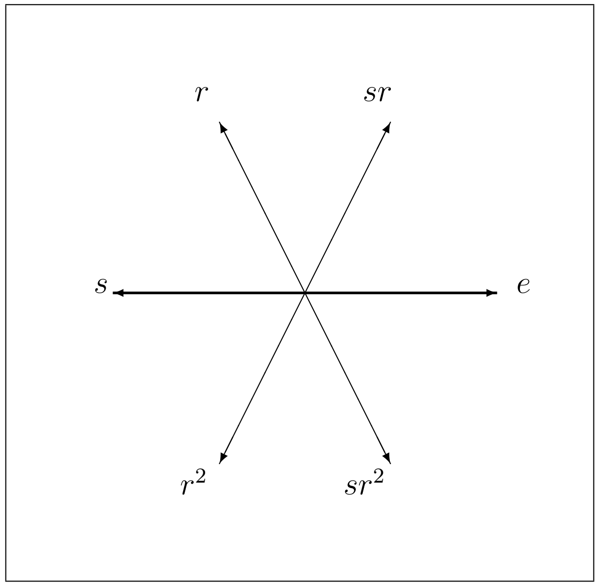
Unlike the case for abelian groups, where Theorem 4.1.8 implies there can only be finitely many inequivalent G-frames, there may be uncountably many inequivalent G-frames for $\mathcal{G}$ non-abelian.

Example 4.3.11. Let $\mathcal{G}$ be the dihedral group of order 6 from Example 4.3.1. For $0 \leq \theta \leq \pi/2$ define the representation $\pi_\theta : D_3 \to \mathcal{U}(\mathbb{R}^2)$ on the generators of D_3 by

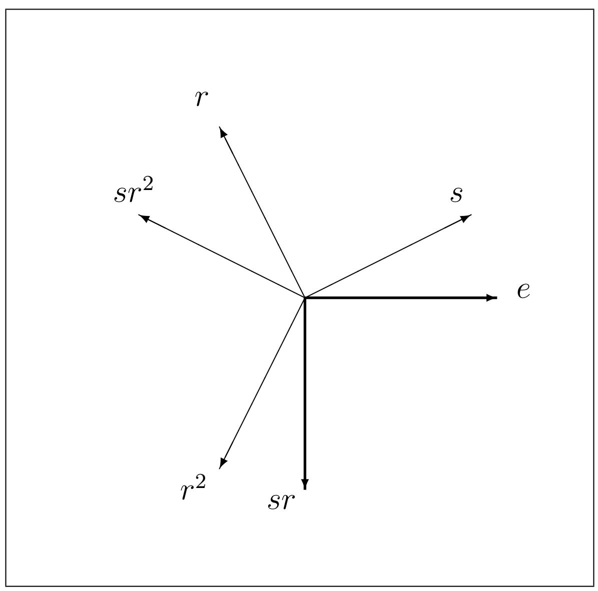
$$\pi_\theta(r) := \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad \pi_\theta(s) := \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}.$$

$\pi_\theta(r)$ is rotation in the plane by $2\pi/3$ radians and $\pi_\theta(s)$ is reflection across the line that makes an angle θ with the x-axis in the first quadrant. The representation π from Example 4.3.1 is the same as $\pi_{\pi/2}$. If we take $\varphi = (1,0)^t$, then $\Phi^\theta = \{\varphi_g^\theta := \pi_\theta(g)\varphi : g \in D_3\}$ is a G-frame for $\mathbb{R}^2$. That Φ^θ is generated by π_θ is obvious from its definition; that it is a tight frame can be most easily seen from the fact that it is a union of two tight frames each unitarily equivalent (via a rotation) to the Mercedes Benz frame. If $0 \leq \theta_1 < \theta_2 \leq \pi/6$, then the tight frames Φ^{θ_1} and Φ^{θ_2} are inequivalent. This follows from the fact that unitary operators preserve angles.

Example 4.3.11 describes an uncountable family of inequivalent G-frames for D_3. A desire to find classes of G-frames for $\mathcal{G}$ non-abelian with only finitely many equivalence classes, as in the case for harmonic frames, led Vale and Waldron to define central G-frames in [41]. From Theorem 4.1.4 we know that the Gramian of a G-frame is a G-matrix; central G-frames are the subset of G-frames such that their Gramians are G-matrices defined by a class function.



(a) $\theta = \pi/2$



(b) $\theta = \pi/12$

Figure 4.4: The elements of Φ^θ from Example 4.3.11 for two choices of θ.

Definition 4.3.12 (Central G-frames)**.** A G-frame $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ is *central* if the function $\nu : \mathcal{G} \to \mathbb{C}$, $g \mapsto \langle \varphi_e, \varphi_g \rangle$ is a class function, i.e., if ν is constant on each conjugacy class of $\mathcal{G}$.

In an abelian group every conjugacy class has just a single element and every G-frame is a central G-frame. When $\mathcal{G}$ is non-abelian the conjugacy classes can be large, and the central G-frames will occupy a proper subset of all G-frames. Vale and Waldron showed in [41] that for a finite group $\mathcal{G}$ there are only finitely many equivalence classes of central G-frames, i.e., the behavior exhibited in Example 4.3.11 does not occur. They also gave an explicit formula for the function ν defining the G-matrices that arise as the Gramians of central G-frames. In this section we prove that for any finite group $\mathcal{G}$ the central G-frames are precisely the frames for which $\mathcal{G}$ defines a frame multiplication. Hence, our work in Section 4.3.2, where we showed there are only finitely many inequivalent frames sharing the same group frame multiplication, corroborates the work done by Vale and Waldron.

Example 4.3.13. Let $\mathcal{G} = D_3$, the dihedral group of order 6. We described all G-matrices for D_3 in Example 4.1.6. Using the same presentation,

$$D_3 =< r, s : r^3 = e, s^2 = e, rs = sr^2 >,$$

we have the conjugacy classes are $\{e\}$, $\{r, r^2\}$, and $\{s, sr, sr^2\}$. If $\Phi = \{\varphi_g\}_{g\in D_3}$ is a central G-frame, then its Gramian has the form

$$\begin{array}{c c} & \begin{array}{cccccc} e & r & r^2 & s & sr & sr^2 \end{array} \\ \begin{array}{c} e \\ r \\ r^2 \\ s \\ sr \\ sr^2 \end{array} & \begin{pmatrix} \nu(e) & \nu(r) & \nu(r) & \nu(s) & \nu(s) & \nu(s) \\ \nu(r) & \nu(e) & \nu(r) & \nu(s) & \nu(s) & \nu(s) \\ \nu(r) & \nu(r) & \nu(e) & \nu(s) & \nu(s) & \nu(s) \\ \nu(s) & \nu(s) & \nu(s) & \nu(e) & \nu(r) & \nu(r) \\ \nu(s) & \nu(s) & \nu(s) & \nu(r) & \nu(e) & \nu(r) \\ \nu(s) & \nu(s) & \nu(s) & \nu(r) & \nu(r) & \nu(e) \end{pmatrix} \end{array} \tag{4.19}$$

for some ν defined on the conjugacy classes of D_3, or, as we have written it here, representatives of the conjugacy classes: e, r, and s.

In the above example we found the form of the Gramian for a central G-frame for the group D_3. Our first follow up question might be whether a frame with such a Gramian even exists. Trivially, an orthonormal basis for $\mathbb{C}^6$ has such a Gramian, but what about in lower dimensions? It turns out it is not hard to tell which G-matrices arise from central G-frames. Any self-adjoint positive-semidefinite matrix P of the form (4.19) whose only eigenvalues are 0 and $c > 0$ is the Gramian of a central G-frame for D_3. This follows from the fact that any self-adjoint positive-semidefinite matrix is the Gramian of some set of vectors, call them Φ, and the condition on the eigenvalues of P guarantees it is a scaled orthogonal projection. These are both shown by diagonalizing the self-adjoint matrix P. Then Naimark's theorem implies Φ is a tight frame, and Theorem 4.1.4 tells us it is a G-frame. Finally, since we began with a matrix of the form (4.19), Φ is a central G-frame.

Summarizing the above, if we would like to know for what dimensions central G-frames exist for D_3 and how many there are (up to equivalence), we need only count positive-semidefinite G-matrices of the form (4.19) with a single positive eigenvalue (and zero). This is relevant to our frame multiplication problem because of the following theorem, a result of which is that the matrices described in this paragraph are precisely the projections from Section 4.3.2.

Theorem 4.3.14 (Equivalence with central G-frames)**.** *If $\mathcal{G}$ is a finite group and $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ is a tight frame for $\mathcal{H}$, then $\mathcal{G}$ defines a frame multiplication for Φ if and only if Φ is a central G-frame.*

Proof. Suppose $\mathcal{G}$ is a finite group and Φ is a tight frame for $\mathcal{H}$. By Theorem 3.3.2, $\mathcal{G}$ defines a frame multiplication for Φ if and only if

$$\forall g,h,k \in \mathcal{G}, \quad \langle \varphi_g, \varphi_h \rangle = \langle \varphi_{kg}, \varphi_{kh} \rangle = \langle \varphi_{gk}, \varphi_{hk} \rangle . \tag{4.20}$$

Equation (4.20) holds if and only if

$$\forall g,h \in \mathcal{G}, \quad \langle \varphi_h, \varphi_g \rangle = \langle \varphi_e, \varphi_{h^{-1}g} \rangle \tag{4.21}$$

and

$$\forall g,h \in \mathcal{G}, \quad \langle \varphi_e, \varphi_{h^{-1}gh} \rangle = \langle \varphi_e, \varphi_g \rangle . \tag{4.22}$$

Equation (4.21) says the Gramian of Φ is a G-matrix defined by the function $g \mapsto \langle \varphi_e, \varphi_g \rangle$, and by Theorem 4.1.4 this implies Φ is a G-frame. Finally, (4.22) says Φ is central. ■

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