Analysis meets Graphs

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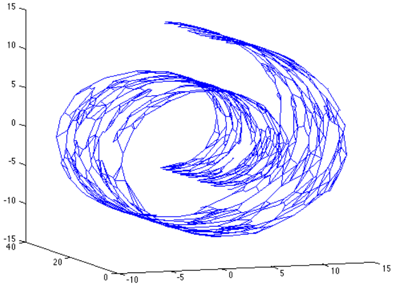
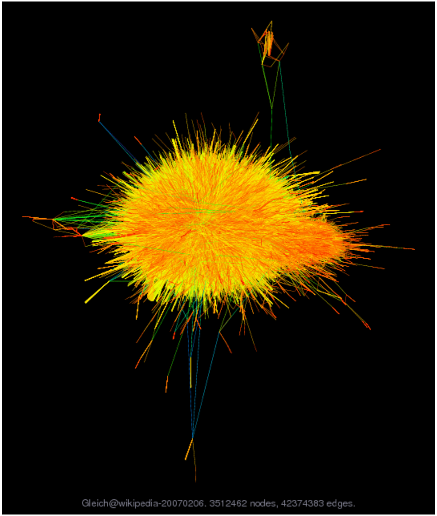
Why study graphs?

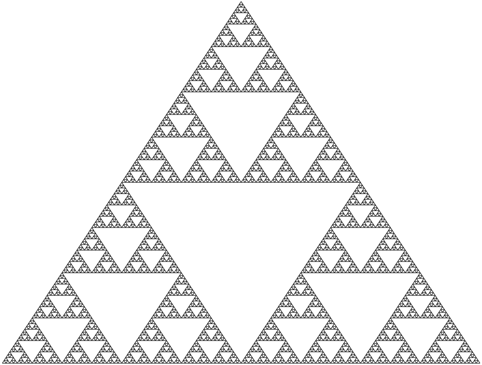
- Graph theory has developed into a useful tool in applied mathematics.
- Vertices correspond to different sensors, observations, or data points. Edges represent connections, similarities, or correlations among those points.



Wikipedia graph



Graphs in pure mathematics too!



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Why study Analysis?

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- Real Analysis
- Complex Analysis
- Partial Differential Equations
- Ordinary Differential Equations
- HARMONIC ANALYSIS
- Probability
- Differential Geometry
- Functional Analysis
- Compressive Sensing
- Stochastic Processes
- Measure Theory
- Brownian Motion
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Analysis toolbox

- Translation
- Convolution
- Modulation
- Fourier Transform

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Outline

1. Fourier Transform
2. Modulation
3. Convolution
4. Translation

Motivation

- In the classical setting, the Fourier transform on $\mathbb{R}$ is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i \xi t} \, dt = \langle f, e^{2\pi i \xi t} \rangle.$$

- This is precisely the inner product of f with the eigenfunctions of the Laplace operator.

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Graph Laplacian

Definition

The pointwise formulation for the Laplacian acting on a function $f : V \to \mathbb{R}$ is

$$\Delta f(x) = \sum_{y \sim x} f(x) - f(y).$$

- For a finite graph, the Laplacian can be represented as a matrix. Let D denote the $N \times N$ *degree matrix*, $D = \text{diag}(d_x)$. Let A denote the $N \times N$ *adjacency matrix*,

$$A(i,j) = \begin{cases} 1, & \text{if } x_i \sim x_j \\ 0, & \text{otherwise.} \end{cases}$$

Then the unweighted graph Laplacian can be written as

$$L = D - A.$$

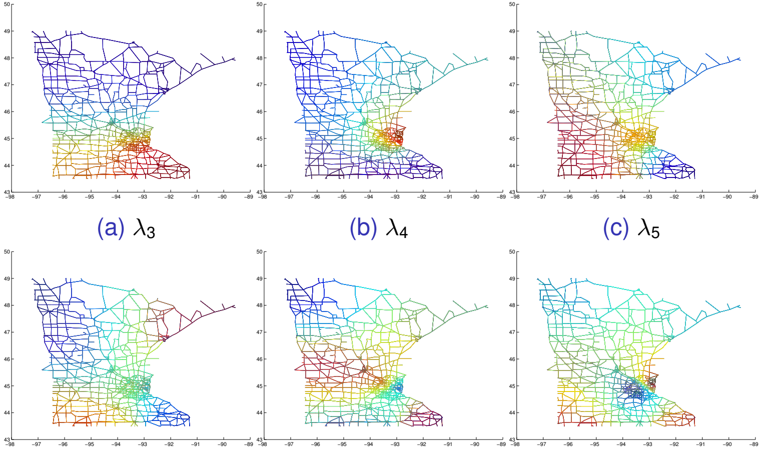
Spectrum of the Laplacian

- L is a real symmetric matrix and therefore has nonnegative eigenvalues $\{\lambda_k\}_{k=0}^{N-1}$ with associated orthonormal eigenvectors $\{\varphi_k\}_{k=0}^{N-1}$.
- If G is finite and connected, then we have

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}.$$

- Easy to show that $\varphi_0 \equiv 1/\sqrt{N}$.

Data Sets - Minnesota Road Network

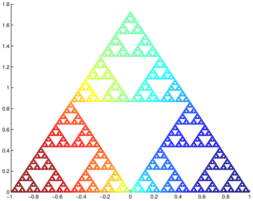


(a) λ_3 (b) λ_4 (c) λ_5

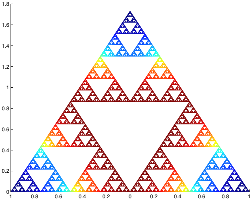
(d) λ_6 (e) λ_7 (f) λ_8

Figure : Eigenfunctions corresponding to the first six nonzero eigenvalues. Minnesota road graph (2642 vertices)

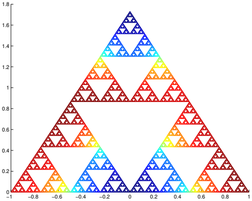
Data Sets - Sierpinski gasket graph approximation



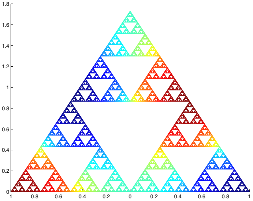
(a) λ_3



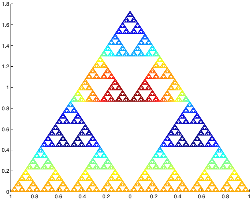
(b) λ_4



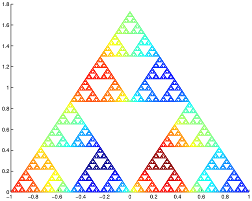
(c) λ_5



(d) λ_6



(e) λ_7



(f) λ_8

Figure : Eigenfunctions corresponding to the first six nonzero eigenvalues. Level-8 graph approximation to Sierpinski gasket (9843 vertices)

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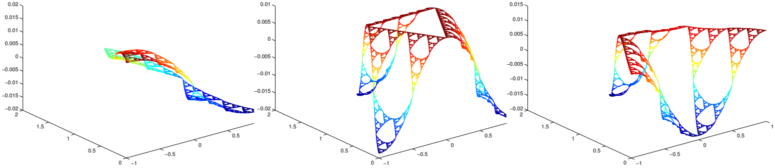
Data Sets - Sierpinski gasket graph approximation



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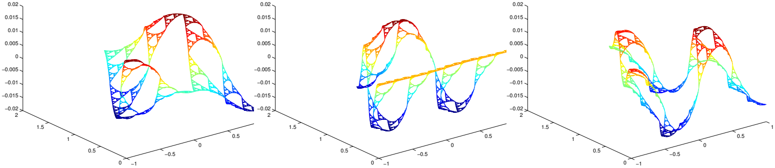


(d) λ_6 (e) λ_7 (f) λ_8

Figure : Eigenfunctions corresponding to the first six nonzero eigenvalues. Level-8 graph approximation to Sierpinski gasket (9843 vertices)

Graph Fourier Transform

Definition

The *graph Fourier transform* is defined as

$$\hat{f}(\lambda_l) = \langle f, \varphi_l \rangle = \sum_{n=1}^{N} f(n)\varphi_l^*(n).$$

Notice that the graph Fourier transform is only defined on values of $\sigma(L)$.

The *inverse Fourier transform* is then given by

$$f(n) = \sum_{l=0}^{N-1} \hat{f}(\lambda_l)\varphi_l(n).$$

Outline

Graph Modulation

- In Euclidean setting, modulation is multiplication of a Laplacian eigenfunction.

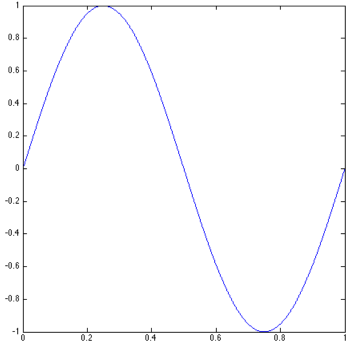
Definition

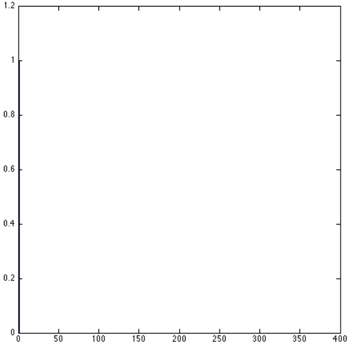
For any $k = 0, 1, ..., N-1$ the *graph modulation operator* M_k, is defined as
$$(M_k f)(n) = \sqrt{N} f(n) \varphi_k(n).$$

- On $\mathbb{R}$,

modulation in the time domain = translation in the frequency domain
$$\widehat{M_\xi f}(\omega) = \hat{f}(\omega - \xi).$$

Example - Classical Setting





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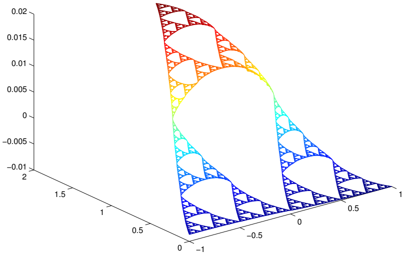
Example - Movie

$G = \mathrm{SG}_6$

$\hat{f}(\lambda_l) = \delta_2(l) \implies f = \varphi_2.$



Example - Movie

G =Minnesota

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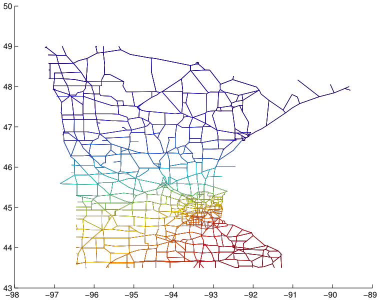


Outline

Graph Convolution - Motivation and Definition

- Classically, for signals $f, g \in L^2(\mathbb{R})$ we define the convolution as

$$f * g(t) = \int_{\mathbb{R}} f(u)g(t-u)\,du.$$

- However, there is no clear analogue of translation in the graph setting. So we exploit the property

$$(\widehat{f * g})(\xi) = \hat{f}(\xi)\hat{g}(\xi),$$

and then take inverse Fourier transform.

Definition

For $f, g : V \to \mathbb{R}$, we define the *graph convolution* of f and g as

$$f * g(n) = \sum_{l=0}^{N-1} \hat{f}(\lambda_l)\hat{g}(\lambda_l)\varphi_l(n).$$

Outline

Graph Translation

- For signal $f \in L^2(\mathbb{R})$, the translation operator, T_u, can be thought of as a convolution with δ_u.
- On $\mathbb{R}$ we can calculate $\hat{\delta}_u(k) = \int_{\mathbb{R}} \delta_u(x) e^{-2\pi ikx}\, dx = e^{-2\pi iku} (= \varphi_k^*(u))$.
- Then by taking the convolution on $\mathbb{R}$ we have

$$(T_u f)(t) = (f * \delta_u)(t) = \int_{\mathbb{R}} \hat{f}(k) \hat{\delta}_u(k) \varphi_k(t)\, dk = \int_{\mathbb{R}} \hat{f}(k) \varphi_k^*(u) \varphi_k(t)\, dk$$

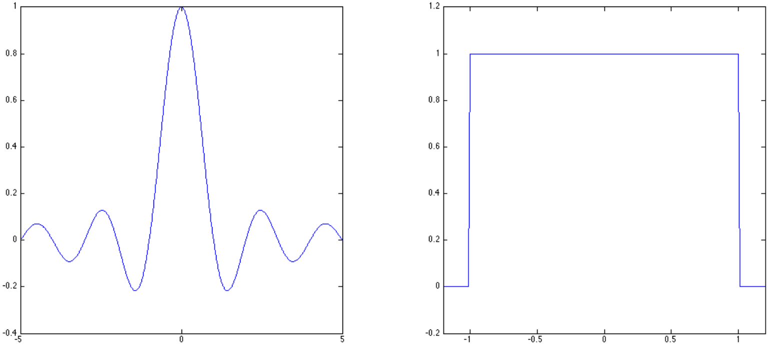
Definition

For any $f : V \to \mathbb{R}$ the *graph translation operator*, T_i, is defined to be

$$(T_i f)(n) = \sqrt{N}(f * \delta_i)(n) = \sqrt{N} \sum_{l=0}^{N-1} \hat{f}(\lambda_l) \varphi_l^*(i) \varphi_l(n).$$

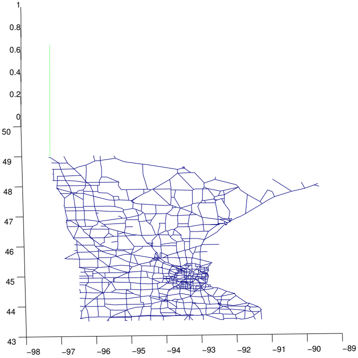
Example - Classical Setting

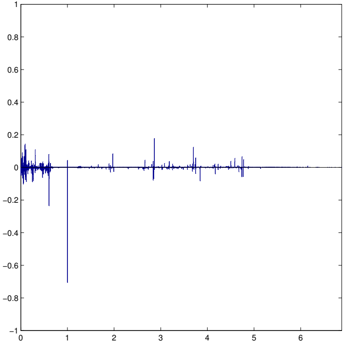
Translation in the time domain = Modulation in the frequency domain



Example - Movie

$G =$Minnesota
$f = \mathbb{1}_1$





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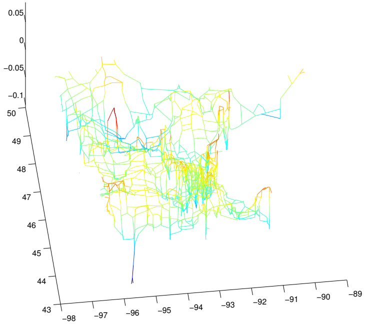
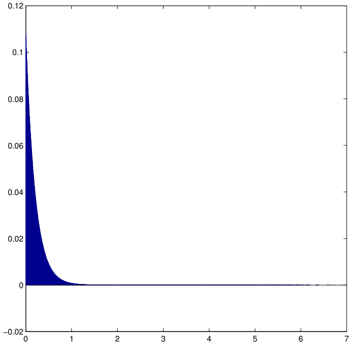
Example - Movie

G =Minnesota
$\hat{f}(\lambda_l) = e^{-5\lambda_l}$





Example - Movie

$G = \mathrm{SG}_6$
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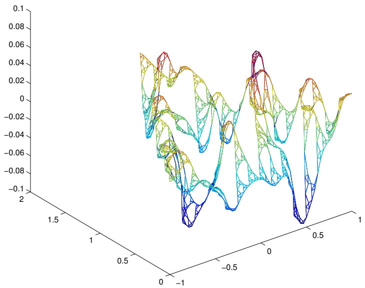

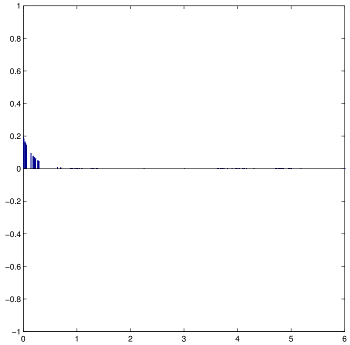

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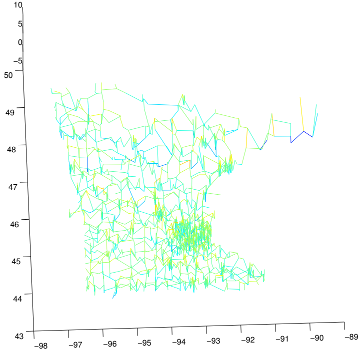
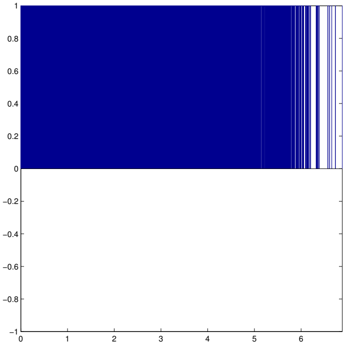
Example - Movie

$G =$ Minnesota
$\hat{f} \equiv 1$





Nice Properties of Graph Translation

Theorem

For any $f, g : V \to \mathbb{R}$ *and* $i, j \in \{1, 2, ..., N\}$ *then*

1. $T_i(f * g) = (T_i f) * g = f * (T_i g)$.
2. $T_i T_j f = T_j T_i f$.

Not-so-nice Properties of Translation operator

- The translation operator is not isometric

$$\|T_i f\|_{\ell^2} \neq \|f\|_{\ell^2}$$

- In general, the set of translation operators $\{T_i\}_{i=1}^N$ do not form a group like in the classical Euclidean setting.

$$T_i T_j \neq T_{i+j}$$

- In general, Can we even hope for $T_i T_j = T_{i \bullet j}$ for some semigroup operation, $\bullet : \{1, ..., N\} \times \{1, ..., N\} \rightarrow \{1, ..., N\}$?

Theorem (B. & O.)

Given a graph, G, with eigenvector matrix $\Phi = [\varphi_0 | \cdots | \varphi_{N-1}]$. Graph translation on G is a semigroup, i.e. $T_i T_j = T_{i \bullet j}$ for some semigroup operator $\bullet : \{1, ..., N\} \times \{1, ..., N\} \rightarrow \{1, ..., N\}$, only if $\Phi = (1/\sqrt{N})H$, where H is a Hadamard matrix.

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Center
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Why should we care?

- With these *time-frequency* operators generalized to *vertex-frequency* operators, we are able to convert many nice results and theories from harmonic analysis to the graph setting.
    - Wavelets and Wavelet Transform
    - Windowed/Short-Time Fourier Transform (STFT)
    - Windowed Fourier Frames

Why should we care?

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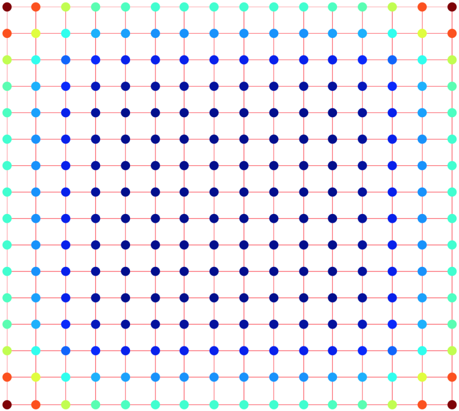

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