Approximating scalable frames

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Outline

Definition

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Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A set of vectors $\{\varphi_i\}_{i=1}^M \subset \mathbb{K}^N$ is called a *finite frame for* $\mathbb{K}^N$ if there are two constants $0 < A \leq B$ such that

$$A\|x\|^2 \leq \sum_{i=1}^{M} |\langle x, \varphi_i \rangle|^2 \leq B\|x\|^2, \quad \text{for all } x \in \mathbb{K}^N. \tag{1}$$

If the frame bounds A and B are equal, the frame $\{\varphi_i\}_{i=1}^M \subset \mathbb{K}^N$ is called *a finite tight frame for* $\mathbb{K}^N$.

Frames in applications

Example

- Quantum computing: construction of POVMs
- Spherical t-designs
- Classification of hyper-spectral data
- Quantization
- Phase-less reconstruction
- Compressed sensing.

Main question

Question

Given a (non-tight) frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ can one transform Φ into a tight frame? If yes can this be done algorithmically and can the class of all frames that allow such transformations be described?

Solution

1. *If Φ denotes again the $N \times M$ synthesis matrix, a solution to the above problem is the associated canonical tight frame*

$$\{(\Phi\Phi)^{-1/2}\varphi_k\}_{k=1}^M.$$

   Involves the inverse frame operator.

2. *What "transformations" are allowed?*

Choosing a transformation

Question

Given a (non-tight) frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ *can one find nonnegative numbers* $\{c_k\}_{k=1}^M \subset [0, \infty)$ *such that* $\widetilde{\Phi} = \{c_k \varphi_k\}_{k=1}^M$ *becomes a tight frame?*

Definition

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Let M, N be given such that $N \leq M$. A frame $\Phi = \{\varphi_k\}_{k=1}^M$ in $\mathbb{R}^N$ is *scalable* if there exists nonnegative scalars $\{x_k\}_{k=1}^M$ such that the system $\widetilde{\Phi_I} = \{x_k\varphi_k\}_{k=1}^M$ is a tight frame for $\mathbb{R}^N$.
If all the coefficients can be chosen to be positive, then we say that the frame is *strictly scalable*.

An extension of scalable frame

Definition

Let M, m, N be given such that $N \leq m \leq M$. A frame $\Phi = \{\varphi_k\}_{k=1}^{M}$ in $\mathbb{R}^N$ is $m-$*scalable* if there exist a subset $\Phi_I = \{\varphi_k\}_{k\in I}$ with $\#I = m$, and nonnegative scalars $\{x_k\}_{k\in I}$ such that the system $\widetilde{\Phi_I} = \{x_k\varphi_k\}_{k\in I}$ is a tight frame for $\mathbb{R}^N$.

A geometric characterization of scalable frames

Theorem (G. Kutyniok, F. Philipp, K. Tuley, K.O. (2012))

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N \setminus \{0\}$ *be a frame for* $\mathbb{R}^N$. *Then the following statements are equivalent.*

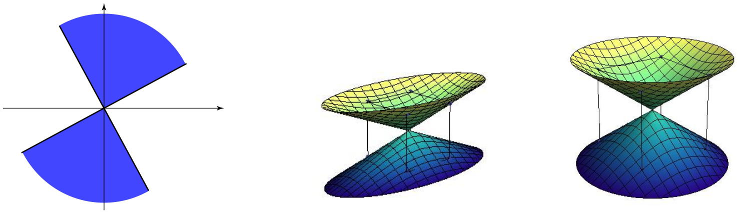
(i) Φ *is not scalable.*

(ii) *There exists a symmetric* $M \times M$ *matrix* Y *with* $trace(Y) < 0$ *such that* $\langle \varphi_j, Y\varphi_j \rangle \geq 0$ *for all* $j = 1, \ldots, M$.

(iii) *There exists a symmetric* $M \times M$ *matrix* Y *with* $trace(Y) = 0$ *such that* $\langle \varphi_j, Y\varphi_j \rangle > 0$ *for all* $j = 1, \ldots, M$.

Scalable frames in $\mathbb{R}^2$ and $\mathbb{R}^3$

Figures show sample regions of vectors of a non-scalable frame in $\mathbb{R}^2$ and $\mathbb{R}^3$.



(a) (b) (c)

Figure : (a) shows a sample region of vectors of a non-scalable frame in $\mathbb{R}^2$. (b) and (c) show examples of sets in $\mathcal{C}_3$ which determine sample regions in $\mathbb{R}^3$.

Scalable frames and Farkas's lemma

Setting

Let $F : \mathbb{R}^N \to \mathbb{R}^d$, $d := (N-1)(N+2)/2$, *defined by*

$$F(x) = \begin{pmatrix} F_0(x) \\ F_1(x) \\ \vdots \\ F_{N-1}(x) \end{pmatrix}$$

$$F_0(x) = \begin{pmatrix} x_1^2 - x_2^2 \\ x_1^2 - x_3^2 \\ \vdots \\ x_1^2 - x_N^2 \end{pmatrix}, \ldots, F_k(x) = \begin{pmatrix} x_k x_{k+1} \\ x_k x_{k+2} \\ \vdots \\ x_k x_N \end{pmatrix}$$

and $F_0(x) \in \mathbb{R}^{N-1}$, $F_k(x) \in \mathbb{R}^{N-k}$, $k = 1, 2, \ldots, N-1$.

Scalable frames and Farkas's lemma

Theorem (G. Kutyniok, F. Philipp, K.O. (2013))

$\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ is scalable if and only if $F(\Phi)u = 0$ has a nonnegative non trivial solution, where $F(\Phi)$ is the $d \times M$ matrix whose k^{th} row is $F(\varphi_k)$. This is equivalent to 0 being in the relative interior of the convex polytope whose extreme points are $\{F(\varphi_k)\}_{k=1}^M$.

Some questions

Question

Let $\Phi = \{\varphi_k\}_{k=1}^{M} \subset \mathbb{R}^N$ *be a frame.*

1. *If* Φ *is scalable, how to find the corresponding weights?*
2. *Can one find "intrinsic measures of scalability"?*
3. *In particular, if a frame is not scalable, how "far" it is to be so?*

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Fritz John's Theorem

Theorem (F. John (1948))

Let $K \subset B = B(0,1)$ be a convex body with nonempty interior. There exits a unique ellipsoid $\mathcal{E}_{min}$ of minimal volume containing K. Moreover, $\mathcal{E}_{min} = B$ if and only if there exist $\{\lambda_k\}_{k=1}^m \subset (0,\infty)$ and $\{u_k\}_{k=1}^m \subset \partial K \cap S^{N-1}$, $m \geq N+1$ such that

(i) $\sum_{k=1}^m \lambda_k u_k = 0$

(ii) $x = \sum_{k=1}^m \lambda_k \langle x, u_k\rangle u_k, \forall x \in \mathbb{R}^N$

where ∂K is the boundary of K and S^{N-1} is the unit sphere in $\mathbb{R}^N$. In particular, the points u_k are contact points of K and S^{N-1}.

F. John's characterization of scalable frames

Setting

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ *be a frame for* $\mathbb{R}^N$*. We apply F. John's theorem to the convex body* $K = P_\Phi = conv(\{\pm\varphi_k\}_{k=1}^M)$*. Let* $\mathcal{E}_\Phi$ *denote the ellipsoid of minimal volume containing* P_Φ*, and* $V_\Phi = Vol(\mathcal{E}_\Phi)/\omega_N$ *where* ω_N *is the volume of the euclidean unit ball.*

Theorem (X. Chen, R. Wang, K.O. (2014))

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ *be a frame. Then* Φ *is scalable if and only if* $V_\Phi = 1$*. In this case, the ellipsoid* $\mathcal{E}_\Phi$ *of minimal volume containing* $P_\Phi = conv(\{\pm\varphi_k\}_{k=1}^M)$ *is the euclidean unit ball* B*.*

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Theorem (X. Chen, R. Wang, K.O. (2014))

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ be a frame. Then Φ is scalable if and only if $V_\Phi = 1$. In this case, the ellipsoid $\mathcal{E}_\Phi$ of minimal volume containing $P_\Phi = conv(\{\pm\varphi_k\}_{k=1}^M)$ is the euclidean unit ball B.

Reformulating the scalability problem

Setting

$$\Phi = \{\varphi_i\}_{i=1}^M \textit{ is scalable} \iff \exists \{c_i\}_{i=1}^M \subset [0,\infty) : \Phi C \Phi^T = I,$$

where $C = \textit{diag}(c_i)$.

$$C_\Phi = \{\Phi C \Phi^T = \sum_{i=1}^M c_i \varphi_i \varphi_i^T : c_i \geq 0\}$$

is the (closed) cone generated by $\{\varphi_i \varphi_i^T\}_{i=1}^M$.

$$\Phi = \{\varphi_i\}_{i=1}^M \textit{ is scalable} \iff I \in C_\Phi.$$

$$D_\Phi := \min_{C \geq 0 \textit{ diagonal}} \left\| \Phi C \Phi^T - I \right\|_F$$

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Comparing D_Φ to the frame potential

Proposition (X. Chen, R. Wang, K.O. (2014))

(a) *Φ is scalable if and only if $D_\Phi = 0$.*

(b) *If $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ is a unit norm frame we have*

$$D_\Phi^2 \leq N - \frac{M^2}{\mathrm{FP}(\Phi)},$$

where $\mathrm{FP}(\Phi)$ is the frame potential of Φ.

Comparing the measures of scalability

Theorem (X. Chen, R. Wang, K.O. (2014))

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ is a unit norm frame, then

$$\frac{N(1 - D_\Phi^2)}{N - D_\Phi^2} \leq V_\Phi^{4/N} \leq \frac{N(N-1-D_\Phi^2)}{(N-1)(N-D_\Phi^2)} \leq 1,$$

where the leftmost inequality requires $D_\Phi < 1$. Consequently, $V_\Phi \to 1$ is equivalent to $D_\Phi \to 0$.

Values of V_Φ and D_Φ for randomly generated frames of M vectors in $\mathbb{R}^4$.

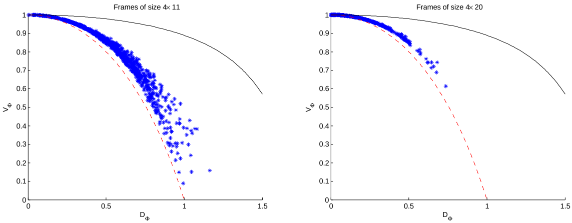

Figure : Relation between V_Φ and D_Φ with $M = 11, 20$. The black line indicates the upper bound in the last theorem, while the red dash line indicates the lower bound.

Projecting a frame onto the scalable frames

Setting

We denote the set of scalable frames of M vectors in $\mathbb{R}^N$ by $\mathcal{S}c(M,N)$. Given a unit norm frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$, let

$$d_\Phi := \inf_{\Psi \in \mathcal{S}c(M,N)} \|\Phi - \Psi\|_F.$$

Proposition (X. Chen, R. Wang, K.O. (2014))

If $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ is a unit norm frame such that $d_\Phi < 1$ then there exists $\hat{\Phi} \in \mathcal{S}c(M,N)$ such that $\|\Phi - \hat{\Phi}\|_F = d_\Phi$.

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Comparing the measures of scalability

Theorem (X. Chen, R. Wang, K.O. (2014))

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ *is a unit norm frame and assume that* $d_\Phi < 1$. *Then with* $K := \min\{M, \frac{N(N+1)}{2}\}$ *and* $\omega := D_\Phi + \sqrt{K}$ *we have*

$$\frac{D_\Phi}{\omega + \sqrt{\omega^2 - D_\Phi^2}} \leq d_\Phi \leq \sqrt{KN\left(1 - V_\Phi^{2/N}\right)}.$$

Consequently, we can bound d_Φ *below and above by expressions of* D_Φ *or expressions of* V_Φ.

Approximating with scalable frames

Theorem (X. Chen, R. Wang, K.O. (2014))

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ *is a unit norm frame and assume that* $d_\Phi \leq \frac{1}{2}(1+\sqrt{K})^{-1}$. *Let* $\hat{\Phi}$ *be given by ,* $\hat{\Phi} = arg\inf_{\Psi \in \mathcal{S}c(M,N)} \|\Phi - \Psi\|_F$, *and let* $E_\Phi = E(X)$ *be the minimal ellipsoid of* Φ, *where* $X^{-1} = \sum_{i=1}^M \rho_i \varphi_i \varphi_i^T$. *Then there exists a (constructible) scalable frame* $\widetilde{\Phi} = \{\tilde{\varphi}_i\}_{i=1}^M$ *which is a good approximation to* Φ *in the following sense:*

$$\|\widetilde{\Phi} - \Phi\|_F = K\sqrt{N}O(d_\Phi),$$

where $K = \min\{M, \frac{N(N+1)}{2}\}$.

Probability of a frame to be scalable

Theorem (X. Chen, R. Wang, K.O. (2013))

Given $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{R}^N$, *be a frame such that each frame vector* φ_i *is drawn independently and uniformly from* S^{N-1}, *if* $P_{M,N}$ *indicates the probability of* Φ *being scalable, then*
(a) $P_{M,N} = 0$, *when* $M < \frac{N(N+1)}{2}$,
(b) $P_{M,N} > 0$, *when* $M \geq \frac{N(N+1)}{2}$,
(c)

$$1 - C(N)\left(1 - A_\alpha^{N-1}\right)^M \leq P_{M,N} \leq 1 - \left(1 - A_{\arccos(1/\sqrt{N})}^{N-1}\right)^{M-N},$$

where $C(N)$ *is the number of caps with angular radius* $\frac{1}{2}\arccos\sqrt{\frac{N-1}{N}}$ *needed to cover* S^{N-1}. *Consequently,* $\lim_{M\to\infty} P_{M,N} = 1$.

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Thank You!
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