Invariant Subspace Perturbations or: How I Learned to Stop Worrying and Love Eigenvectors

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Outline

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Error in A Matrix

- Eigenvector decomposition is ubiquitous in mathematics
  - Principle Component Analysis
  - Quantum States
  - Fourier Analysis
  - Spectral Graph Theory

Most Common Problem in Mathematics

Find (λ, v) such that

$$Av = \lambda v$$

- One imagines computers made this problem trivial
  - $[U, S] = \text{eig}(A)$
- Question: What happens to eigenpairs if matrix has tiny errors

$$\widetilde{A} = A + E$$

Error in A Matrix

- Answer for eigenvalues: You're fine and they behave rather continuously
    - Weyl's Inequality
- Answer for eigenvectors:

Problem!

Eigenvector Perturbations

- Eigenvectors under perturbation require careful treatment
- Dependent on separation of spectrum

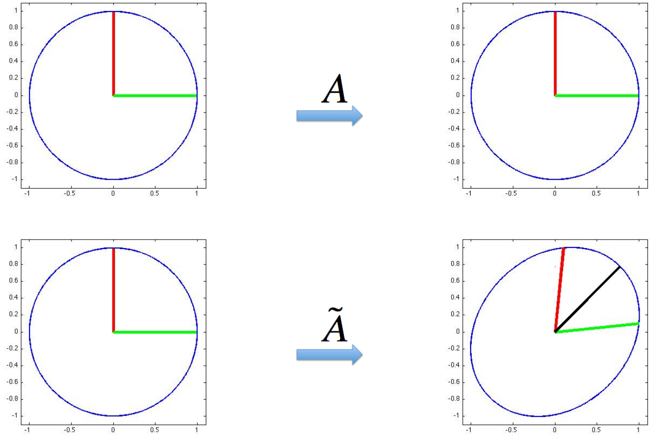
Example

Let $A = \begin{pmatrix} 1-\epsilon & 0 \\ 0 & 1+\epsilon \end{pmatrix} \implies \sigma(A) = \{1-\epsilon, 1+\epsilon\}$, $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $\widetilde{A} = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} \implies \sigma(\widetilde{A}) = \{1-\epsilon, 1+\epsilon\}$, $\widetilde{V} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

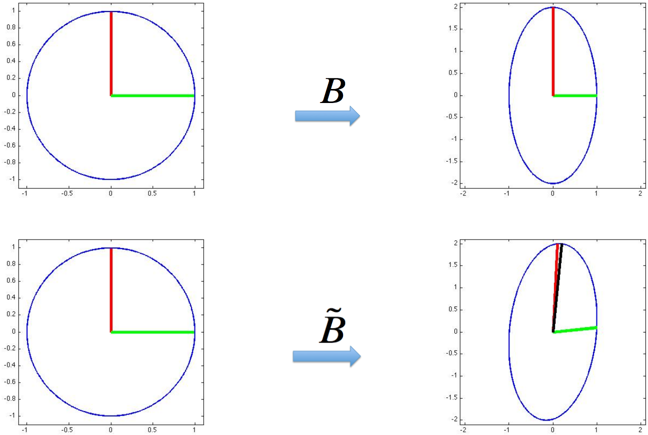
- V and $\widetilde{V}$ are as far apart as possible

Eigenvector Perturbations Geometrically



Problem is that image of A is rotationally symmetric

Stable Eigenvector Geometrically



Lack of symmetry allows for robust perturbations

Eigenvector Perturbations

- Separation of spectrum creates stable perturbations

Example

Let $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \implies \sigma(B) = \{1, 2\}$, $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $\widetilde{B} = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 2 \end{pmatrix} \implies \sigma(\widetilde{B}) = \{\frac{3-\sqrt{1+16\epsilon^2}}{2}, \frac{3+\sqrt{1+16\epsilon^2}}{2}\}$,

$$\widetilde{V} = \begin{pmatrix} 0.995 & 0.099 \\ -0.099 & 0.995 \end{pmatrix} \text{ for } \epsilon = .1$$

- Rotation between V and $\widetilde{V}$ is $\approx 5^\circ$

Outline

1. Introduction to Eigenvector Perturbation
2. Eigenvalue Separation and Rigorous Arguments
3. Eigenvalue Concentration

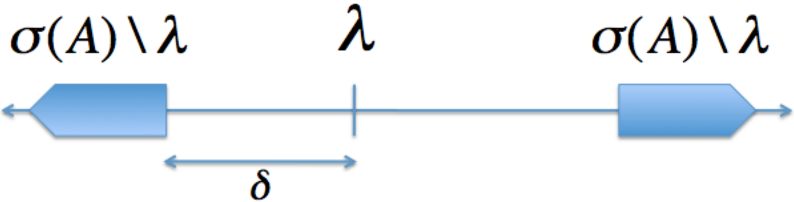
Eigenvalue Separation

- Consider symmetric matrices $A, E \in \mathbb{R}^{n\times n}$
- Eigenvectors of $A + E$ depend on separation of spectrum $\sigma(A)$

Definition (Separation of Spectrum)

The separation of the spectrum $\sigma(A)$ at λ is

$$\text{sep}(\lambda, \sigma(A) \setminus \lambda) = \min\{|\lambda - \gamma| : \gamma \in \sigma(A) \setminus \lambda\}$$



Invariant Subspace Perturbations I

Theorem (Davis, 1963)

Let $A, E \in \mathbb{C}^{n \times n}$ be Hermetian. Let (λ, x) be an eigenpair of A such that

$$\text{sep}(\lambda, \sigma(A) \setminus \lambda) = \delta.$$

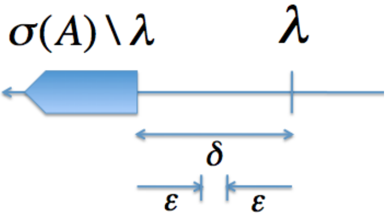
Let

- *P be a spectral projector of A such that $Px = x$*
- *P' be the corresponding spectral projector of $A + E$, and*
- *$\overline{P'}$ be the orthogonal complement $\overline{P'}z = z - P'z$.*

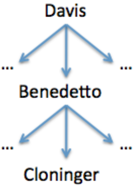
Then if $\|E\| \leq \epsilon \leq \delta/2$,

$$\|\overline{P'}P\| \leq \frac{\epsilon}{\delta - \epsilon}.$$

Invariant Subspace Perturbations II

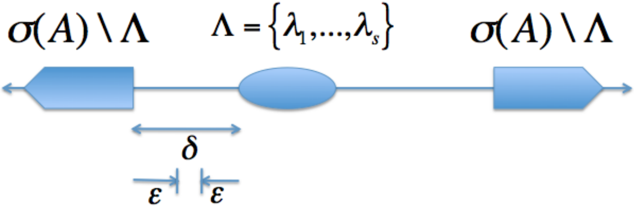


Side Note on Advisers:

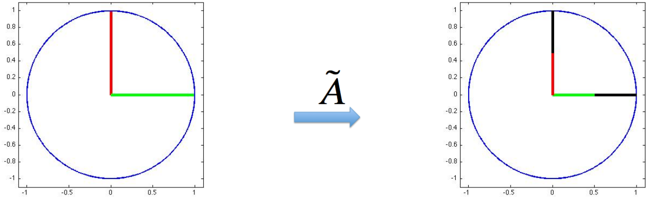


Similar Perturbation Theorems

Davis, Kahan (1970): Clustered Subspaces are Preserved



Stewart (1973): Take Direction of Error Into Account



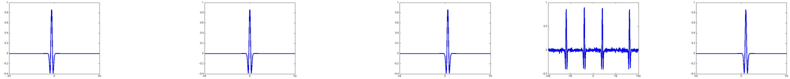
Outline

Concentration As Opposed to Angle Similarity

- Previous theory defined similarity by angle $\Theta[V, \widetilde{V}]$
  - Not only way to consider "similarity"
- Can also consider eigenvector "localization"
- Important when:
  - $\sigma(A)$ has high density in interval
  - A is adjacency matrix for network graph

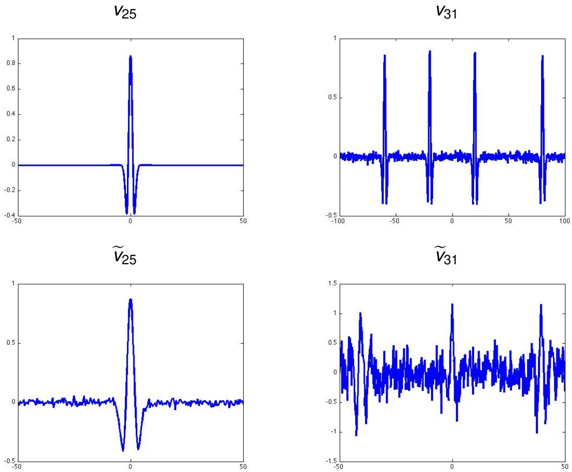
Plot of Eigenvectors for $A \in \mathbb{R}^{1000 \times 1000}$

$\lambda_{20} = 0.0287$ ... $\lambda_{25} = 0.0371$ ... $\lambda_{30} = 0.0479$ $\lambda_{31} = 0.0481$ $\lambda_{32} = 0.0494$



Concentration Pertrubation

Plot of Eigenvectors for $A, \widetilde{A} \in \mathbb{R}^{1000\times1000}$, $\widetilde{A} = A + E$



Eigenvector Localization

Theorem (C., 2014)

Let $A \in \mathbb{R}^{n \times n}$ be symmetric with eigendecomposition $A = V \Sigma V^*$. Let (λ_i, v_i) be an eigenpair of A. Assume

- *Partition $V = [V_1, V_2, v_i, V_3, V_4]$ where $V_2, V_3 \in \mathbb{R}^{n \times s}$, ordered by $\lambda_1 \leq ... \leq \lambda_n$*
- *$\exists C \subsetneq \{1, ..., n\}$ such that supp$(v_i) \subset C$ and supp$(v_j) \subset C$ where v_j is a column of V_2, V_3.*
- *Let $(\widetilde{\lambda}, x)$ an eigenvector of the perturbed matrix $\widetilde{A} = A + E$, where $x = [x_1, ..., x_n]$.*

Then

$$\sum_{j \in C^c} |x_j|^2 \leq \frac{\|(\widetilde{\lambda} - \lambda_i)x - Ex\|_2^2}{\min(\lambda_i - \lambda_{i-s}, \lambda_{i+s} - \lambda_i)^2}.$$

Conclusions

- Eigenvector perturbation depends on inverse of spectrum separation
- Led to Nobel Prize in Physics for particle localization (Anderson 1977)
- Concentration / localization is lesser restriction on eigenvectors
  - Depends on cluster of vectors concentrated in similar area
- Applications in eigenstate localization on data-dependent graphs