Introduction to Compressive Sensing

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Norbert Wiener Center for Harmonic Analysis and Applications

Outline

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1. Introduction
2. Compressive Sensing in Different Basis
3. Applications to Medical Imaging
4. Applications to Background Subtraction
5. Conclusion

Initial Example

Initial Example

- Picture has 10 megapixels
  - Effectively 10 MB of information
- Can store picture as less than 1 MB using .jpg
  - Image is compressible in wavelet basis
- Camera observes in elementary basis

Questions

1) Why was it necessary to collect all 10 MB of information, but throw 9 MB away?
2) Can we measure in a different basis?

Initial Example

- Answer: For sparse objects, observe randomly in different basis.

Method

- $f \in \mathbb{C}^N$ is original signal, $\hat{f} \in \mathbb{C}^N$ is Fourier Transform
- Observe small number of random Fourier coefficients $\hat{f}(\gamma)$
- Wish to find sparsest solution $g \in \mathbb{C}^N$ such that

$$\hat{g}(\gamma) = \hat{f}(\gamma), \quad \forall\gamma \text{ randomly observed}$$

- Compressive Sensing claims that sparsest g is equal to f

Overview of the Problem

- $f \in \mathbb{C}^N$ be sparse, and choose some $\Omega \subset \mathbb{Z}_N$
- Let your measurements be y, where

$$y = \hat{f}|_\Omega$$

- Can recover "sparsest" solution by solving

$$\min_{g \in \mathbb{C}^N} \|g\|_{L^0(\mathbb{Z}_N)}, \quad \hat{g}|_\Omega = \hat{f}|_\Omega$$

Definition (L^0 norm)

$$\|f\|_{L^0(\mathbb{Z}_N)} = |\{x \in \mathbb{Z}_N : f[x] \neq 0\}|$$

Overview of the Problem

- $\|\cdot\|_{L^0}$ is not computationally efficient
  - Non-convex problem
  - NP-Hard

Main Questions

1. Is there metric other than $\|\cdot\|_{L^0}$ minimization?
2. How do we define "sparse"?
3. What is minimum size of Ω needed?

Outline

Initial Results in Fourier Space

- Candés, Romberg, and Tao proposed:

L^1 Minimization

$$\min_{g \in \mathbb{C}^N} \|g\|_{L^1}, \quad \hat{g}|_{\Omega} = \hat{f}|_{\Omega}, \tag{1}$$

- Problem can be solved using Interior Points Method
  - Modified Newton's method
- Remember: Fourier coefficients are *sampled randomly*
  - If desire m samples, choose Ω uniformly at random over all $|\Omega| = m$

Main Theorem

Theorem (Candés, Romberg, Tao)

Let $f \in \mathbb{C}^N$ be some discrete signal with support set T, where T is unknown. Choose Ω of size $|\Omega| = m$ uniformly at random. For a given accuracy parameter M, if

$$|T| \leq C_M (\log N)^{-1} |\Omega|, \tag{2}$$

then with probability exceeding $1 - O\left(N^{-M}\right)$, the minimizer to problem (1) is unique and equal to f.

- $C_M \sim O\left(\frac{1}{M}\right)$

Arbitrary Orthogonal Matrix

- Generalize to NxN orthogonal matrix U such that $U^*U = N \cdot I_N$
- Observe $y = U_\Omega f$
- Recover sparse $f \in \mathbb{C}^N$ by solving

$$\min_{g \in \mathbb{C}^N} \|g\|_{L^1}, \quad U_\Omega g = U_\Omega f. \qquad (3)$$

- Successful with high probability given

$$|\Omega| \geq C \cdot [\mu(U)]^2 \cdot |T| \cdot \log(N)$$

where

$$\mu(U) = \max_{k,j} |U_{k,j}|$$

Main Theorem

Theorem (Candés, Romberg)

Fix $T \subset \mathbb{Z}_N$. *Let* U *be an NxN orthogonal matrix with* $\mu = \max_{i,j} |U_{i,j}|$.
Choose a sign sequence $z(t)$ *for* $t \in T$, *uniformly at random.*
Choose Ω *at random such that*

$$|\Omega| \geq C_0 |T| \mu^2(U) \log(N/\delta) \quad \textit{and} \quad |\Omega| \geq C_0' \log^2(N/\delta)$$

Let $f \in \mathbb{C}^N$ *have* $\text{supp}(f) = T$ *and*

$$\text{sgn}(f)(t) = z(t), \quad \forall t \in T.$$

Then with probability exceeding $1 - \delta$, f *is the unique minimizer to (3).*

Alternate Interpretation of U

- Consider $U = \Phi\Psi$
  - Ψ is sparsity basis, $\Psi^*\Psi = I$
  - Call Φ measurement basis, $\Phi^*\Phi = N \cdot I$

Corollary (Sparsity and Measurement Basis)

Let $x \in \mathbb{C}^N$ (not necessarily sparse). Wish to recover x from

$$y = \Phi_\Omega x.$$

Assume $\exists$ sparse f such that $x = \Psi f$, so

$$y = \Phi_\Omega \Psi \cdot f = U_\Omega f.$$

If $f^\#$ minimizes (3), best estimate for x is

$$x^\# = \Psi f^\#.$$

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Theory Behind Matrix Completion

- Consider only observing rank r matrix $M \in \mathbb{C}^{N\times N}$ on some subset Ω of its indices
- Let SVD of M be $M = U\Sigma V'$
- Possible to recover M as solution to

$$\begin{aligned} \min_{X} \quad & \operatorname{rank}(X) \\ \text{such that} \quad & (UXV')_{i,j} = (U\Sigma V')_{i,j},\ (i,j) \in \Omega \end{aligned}$$

- This problem is NP-hard

Matrix Completion Approach

Definition (Nuclear Norm)

Let $\sigma_i(M)$ be the i^{th} largest singular value of M. If $\text{rank}(M) = r$, then

$$\|M\|_* := \sum_{i=1}^{r} \sigma_i(M)$$

Recovery Algorithm

Wish to recover M by solving problem (P*), which is

$$\begin{array}{ll} \min\limits_{X} & \|X\|_* \\ \text{such that} & (UXV')_{i,j} = M_{i,j}, \ (i,j) \in \Omega \end{array}$$

Applications to Nuclear Magnetic Resonance

- Nuclear Magnetic Resonance (NMR) imaging studies molecular structure.
- Multidimensional correlations found can identify and study fluid-saturated porous medium
  - Specifically can use T1-T2 relaxation times (longitudinal and traverse)
  - Collisions of spin-bearing molecules with pore walls induce more rapid relaxation
  - Simple correlation between relaxation rate and pore surface-to-volume ratio
- Best way to measure T1-T2 times is using pulse train of RF energy particles
- Problem is NMR is incredibly slow

Math Behind NMR

- Echo measurements are related to T1-T2 correlations via Laplace Transform

$$M(\tau_1,\tau_2) = \int\int (1-2e^{\tau 1/T_1})e^{\tau_2/T_2}\mathcal{F}(T_1,T_2)dT_1dT_2 + E(\tau_1,\tau_2)$$

- We'll consider more general 2D Fredholm Integral

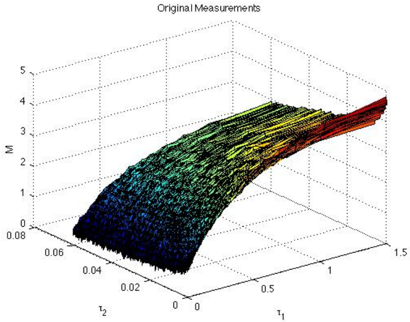
$$M(\tau_1,\tau_2) = \int\int k_1(\tau 1,T_1)k_2(\tau_2,T_2)\mathcal{F}(T_1,T_2)dT_1dT_2 + E(\tau_1,\tau_2)$$

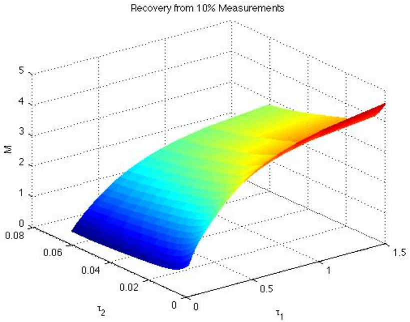
where $E(\tau_1,\tau_2) \sim \mathcal{N}(0,\epsilon)$

- Discretize to

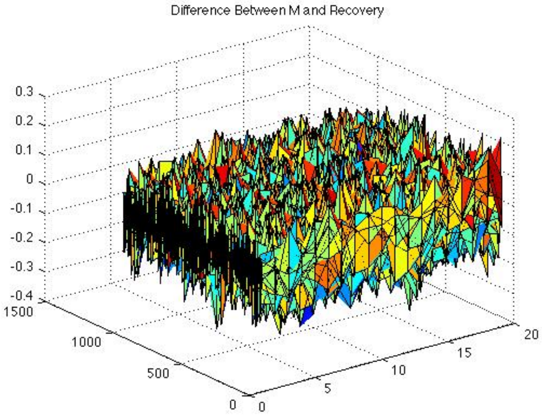
$$\mathbf{M} = \mathbf{K_1FK_2'} + \mathbf{E}$$

Recovery from Small Number of Entries





Error Analysis



Outline

Video

Video Presentation

Robust PCA

Principal Component Pursuit

Let L_0 be low rank background and S_0 be sparse foreground. Wish to recover L_0 and S_0 by solving

$$\begin{aligned} \min_{L,S} \quad & \|L\|_* + \lambda \|S\|_1 \\ \text{such that} \quad & L + S = M \end{aligned} \tag{4}$$

Video

Video Split

Outline

Final Thoughts

- Whole field based around observations being redundant
- In reality, most objects can be represented more sparsely in different way
- Still large number of applications that can benefit
- (Wojtek made me put this in) NWC has many more problems of interest