Continuous Frames, Co-orbit Spaces and the Discretization Problem

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- The classical notion of *discrete frame* originated in 1952, and generalizes the notion of orthonormal basis to allow for redundant decompositions:

Definition

A sequence $\{f_n\}_{n\in\mathbb{N}}$ of elements of H a Hilbert space is a **discrete frame** for H if:

$$\exists A, B > 0 \text{ such that } \forall f \in H, \quad A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2.$$

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- The notion of *continuous frame* generalizes this, by replacing $\sum$ with $\int$ in the definition:

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Let H be a separable Hilbert space, X a locally compact Hausdorff space equipped with a positive Radon measure μ such that supp(μ) $= X$. A family $F = \{\psi_x\}_{x \in X}$ is a **continuous frame** for H if:

$$\exists\, A, B > 0 \text{ such that } \forall f \in H, \quad A\|f\|^2 \leq \int_X |\langle f, \psi_x \rangle|^2 d\mu(x) \leq B\|f\|^2.$$

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- This definition looks difficult to verify, but in fact many familiar object from harmonic analysis are continuous frames with respect to particular indexing spaces X and Hilbert spaces H.

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- Let's briefly investigate two examples of continuous frames.

- The first example of a continuous frame is the *short-time Fourier transform (STFT)*.
- Let $g \in L^2(\mathbb{R})$. Then $\{M_a T_b g\}_{a,b\in\mathbb{R}} = \{g(t-b)e^{2\pi ita}\}_{a,b\in\mathbb{R}}$ is a continuous frame for $H := L^2(\mathbb{R})$, where the space we integrate over is $X := \mathbb{R}^2$, equipped with the Lebesgue measure. In fact, it is a tight frame with $A = B = \|g\|_2^2$.
- This may be shown by defining

$$V_g f(b,a) := \int_{\mathbb{R}} f(t)\overline{g(t-b)}e^{-2\pi ita}dt = \langle f(t), g(t-b)e^{2\pi ita}\rangle_{L^2}$$

$$\Longrightarrow |V_g f(b,a)|^2 = |\langle f(t), g(t-b)e^{2\pi ita}\rangle|^2.$$

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- Suppose $\psi \in L^2(\mathbb{R})$ is such that:

$$\int_{\mathbb{R}} \frac{|\widehat{\psi(\gamma)}|^2}{|\gamma|} d\gamma < \infty.$$

We say such a ψ is *admissible*.

- In this case, define:

$$\psi^{a,b}(x) := (T_b D_a \psi)(x) = \frac{1}{\sqrt{|a|}} \psi(\frac{x-b}{a}), \; a \neq 0.$$

Then for such an admissible ψ, $\{\psi^{a,b}\}_{a,b \in \mathbb{R}, a \neq 0}$ is a continuous frame for $H := L^2(\mathbb{R})$, where the space we integrate over is $X := (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$, with measure $d\mu = \frac{1}{a^2} da\, db$, where $da\, db$ is the Lebesgue measure on $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}$.

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- The operator $W_\psi : L^2(\mathbb{R}) \times X \to \mathbb{R}$ defined as:

$$W_\psi(f)(a,b) := \langle f, \psi^{a,b} \rangle$$

is the *continuous wavelet transform* of f with respect to ψ.

- Even better than being a continuous frame, the continuous wavelet transform admits a precise reconstruction formula, the so-called *Calderón Reproducing Formula*:

$$f = \frac{1}{C_\psi} \int_X W_\psi(f)(a,b)\psi^{a,b} \frac{1}{a^2}\, da\, db,$$

where C_ψ is a constant depending only on ψ. Hence, $\{\psi^{a,b}\}_{a,b\in\mathbb{R},a\neq 0}$ is a tight frame with bounds $A = B = C_\psi$. Normalizing ψ appropriately, we may force $C_\psi = 1$. Thus, the continuous wavelet transform is a Parseval tight frame.

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- While these abstract formulas are beautiful, efficient computation demands a discretization paradigm.

- The theory of discrete frames is well-understood and frequently used in computations. The natural connection between discrete and continuous frames leads us to
  The Discretization Problem: Is there a way to sample the indexing space X of a continuous frame and acquire a discrete frame? With similar bounds?

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- This fails for a basic wavelet example indexed by $\mathbb{R}$, so a general approach must be *non-uniform*.

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- Co-orbit spaces were introduced by Feichtinger and Gröchenig in the late 1980's in order to acquire an *atomic decomposition* of function spaces.
- Their original paper studied Banach spaces invariant under the action of certain integrable group representations, and deduced decomposition results by working with these representations.
- The theory was applied to continuous frames by examining representations induced by the action of the frame.
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- Let $\{\psi_x\}_{x \in X}$ be a continuous frame for H Hilbert with respect to (X, μ).
- The associated *frame operator* is

$$S : H \to H, \qquad Sf := \int_X \langle f, \psi_x \rangle \psi_x d\mu(x).$$

- Define two operators $V, W : H \to L^2(X, \mu)$ associated to $\{\psi_x\}_{x \in X}$ as follows:

$$Vf(x) := \langle f, \psi_x \rangle,$$

$$Wf(x) := \langle f, S^{-1}\psi_x \rangle = V(S^{-1}f)(x).$$

- Here, V generalizes the notion of the analysis operator from discrete frame theory, which gives the coefficients of a discrete frame reconstruction. In the context of the short-time Fourier transform, V is V_g; for the continuous wavelet transform, V is W_ψ.

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- The definition of co-orbit spaces will involve a suitable *Banach algebra of kernels*. We make two definitions:
- The Banach algebra of kernels $\mathcal{A}_1$ is defined as the set

$$\{K : X \times X \to \mathbb{C} \mid K \text{ is measurable}, \|K\|_{\mathcal{A}_1} < \infty\},$$

with norm

$$\|K\|_{\mathcal{A}_1} = \max\{\| \int_X |K(x,y)|d\mu(y)\|_{L^\infty_x}, \| \int_X |K(x,y)|d\mu(x)\|_{L^\infty_y}\}.$$

- Multiplication of kernels is given by:

$$K_1 \circ K_2(x,y) := \int_X K_1(x,z)K_2(z,y)d\mu(z).$$

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- Furthermore, we require the particular kernel $R(x, y) := \langle \psi_y, S^{-1}\psi_x \rangle$ be contained in $\mathcal{A}_1$.
- Kernels act on functions by integration:

$$K(F)(x) := \int_X F(y)K(x, y)d\mu(y)$$

- For an appropriate weight function m, we define the Banach algebra of kernels $\mathcal{A}_m$ as:

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- Our co-orbit spaces will be defined with reference not just to a continuous frame, but to a particular space of functions Y.
- We require two properties for our space Y. First, it must be Banach with norm $\|\cdot\|_Y$ satisfying a *solidity condition*: if F is μ-measurable and $G \in Y$ is such that $|F(x)| \leq |G(x)|$ μ-almost everywhere, then $F \in Y$ and $\|F\|_Y \leq \|G\|_Y$.
- Second, there must exist an appropriate weight function m such that $\mathcal{A}_m(Y) \subset Y$ and:

$$\forall\, K \in \mathcal{A}_m, F \in Y, \; \|K(F)\|_Y \leq \|K\|_{\mathcal{A}_m}\|F\|_Y.$$

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- We now define a space of vectors whose image under W are integrable with respect to a given weight function. Our co-orbit spaces will ultimately be defined as a closed subset of the conjugate dual of these spaces.
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Here v is an appropriately chosen weight function.
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- Suppose Y is an admissible space of functions. The *co-orbit of Y with respect to the continuous frame* $\{\psi_x\}_{x\in X}$ is:

$$CoY := \{f \in (K_v^1)^\dagger \mid Vf \in Y\},$$

- A similar definition using W instead of V exists; for clarity and brevity, this presentation will focus on CoY

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Theorem

Suppose $R(Y) \subset L^\infty_{1/v}$. Then:

1) CoY is Banach with respect to the norm $\|\cdot\|_{CoY}$.

2) A function $F \in Y$ is of the form $F = Vf$ for some $f \in CoY$ if and only if $F = R(F)$.

3) The map $V : CoY \to Y$ establishes an isometric isomorphism between CoY and the closed subspace $R(Y) \subset Y$.

- We now identify certain co-orbit spaces, revealing them to be quite familiar objects.

- $CoL^{\infty}_{1/v} = (K^1_v)^{\dagger}$.

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- More interesting is how the modulation spaces appear in co-orbit space theory. Indeed, if we take as our continuous frame the short time Fourier transform, it follows from little more than definitions that $M^{p,q}_{v_s} = CoL^{p,q}_{v_s}$, where $v_s(z) := (1 + |z|)^s$, a polynomial weight.

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- Why co-orbit spaces? It gives us a way to *discretize*.

- Shorty, we will see a theorem that gives conditions under which a sampling of $\{\psi_x\}_{x \in X}$ is a *Banach frame* for CoY. In general, CoY need not be Hilbert.

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- The theorem is based on a covering of the indexing space X. If the covering is *fine enough*, we can take a representative from each covering set and acquire a discrete frame.
- A family $\mathcal{U} = \{U_i\}_{i \in I}$ of subsets of X is called a **(discrete) admissible covering** of X if:

1) Each U_i has compact closure and has non-$\emptyset$ interior.
2) $X = \bigcup_i U_i$.
3) $\exists N > 0$ such that $\sup_{j \in I} \{i \in I | U_i \cap U_j \neq \emptyset\} \leq N < \infty$.

We say such an admissible covering is **moderate** if in addition:

4) $\exists D > 0$ such that $\mu(U_i) \geq D$ for all $i \in I$.
5) $\exists \tilde{C} > 0$ such that $\mu(U_i) \leq \tilde{C} \mu(U_j)$ for all i, j such that $U_i \cap U_j \neq \emptyset$.

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- A frame $\{\psi_x\}_{x \in X}$ is said to possess property $D[\delta, m]$ if there exists a moderate admissible covering $\mathcal{U} = \mathcal{U}^\delta = \{U_i\}_{i \in I}$ such that the kernel $osc_{\mathcal{U}}$ defined by:

$$osc_{\mathcal{U}}(x, y) := \sup_{z \in Q_y} |\langle S^{-1}\psi_x, \psi_y - \psi_z \rangle| = \sup_{z \in Q_y} |R(x, y) - R(x, z)|,$$

where $Q_y := \bigcup_{\{i | y \in U_i\}} U_i$, satisfies $\|osc_{\mathcal{U}}\|_{\mathcal{A}_m} < \delta$.

- Intuitively, $D[\delta, m]$ gives a way to measure how "localized" we can make a discretely-indexed subset of our continuous frame.

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- We are now in a position to state how to achieve the discretization of certain continuous frames. This will involve discretizing a co-orbit space, which is in general only a Banach space. Consequently, we must define a notion of frame for Banach spaces, one that doesn't make use of an inner product.
- A family $\{h_i\}_{i\in I} \subset B^*$ is a **Banach frame** for $(B, \|\cdot\|_B)$ Banach if there is a BK-space $(B^\flat, \|\cdot\|_{B^\flat})$ and a bounded linear reconstruction operator $\Omega : B^\flat \to B$ such that:

  1) If $f \in B$, then $(h_i(f))_i \in B^\flat$ and there exist constants $0 < C_1, C_2 < \infty$ such that:

$$C_1\|f\|_B \leq \|(h_i(f))_{i\in I}\|_{B^\flat} \leq C_2\|f\|_B.$$

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- We see that this definition preserves the idea of a "stable reconstruction" of $f \in B$, without requiring an inner product. Indeed, in a Hilbert space, we use the inner product to determine the coefficients in a frame expansion, while in the definition of Banach frame, we resort to a general reconstruction operator Ω.

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Theorem

Assume m is an admissible weight. Suppose the frame $\{\psi_x\}_{x\in X}$ *satisfies property* $D[\delta, m]$ *for some* $\delta > 0$ *such that:*

$$\delta(\|R\|_{\mathcal{A}_m} + \max\{C_{m,\mathcal{U}}\|R\|_{\mathcal{A}_m}, \|R\|_{\mathcal{A}_m} + \delta\}) \leq 1.$$

Let $\mathcal{U}^\delta = \{U_i\}_{i\in I}$ *denote the corresponding moderate admissible covering of* X. *Here,* $C_{m,\mathcal{U}}$ *is such that* $\sup_{x,y\in U_i} m(x,y) \leq C_{m,\mathcal{U}}$. *Choose points* $(x_i)_{i\in I}$ *such that* $x_i \in U_i$. *If* $(Y, \|\cdot\|_Y)$ *is an admissible Banach space, then* $\{\psi_{x_i}\}_{i\in I} \subset \mathcal{K}^1_v$ *is a Banach frame for* CoY *with corresponding BK-space* $Y^\flat$.

- A brief sketch of the proof is as follows:
- We begin by defining a discretized version of the integral operator associated to the kernel $R(x, y) = \langle \psi_y, S^{-1}\psi_x \rangle$. More precisely, there exists a partition of unity associated to a moderate admissible covering $\mathcal{U}^\delta = \{U_i\}_{i\in I}$, call it $\{\phi_i\}_{i\in I}$.
- Given points $x_i \in U_i$, we define the operator:

$$U_\phi F(x) := \sum_{i\in I} c_i F(x_i) R(x, x_i),$$

where we define $c_i = \int_X \phi_i(x) d\mu(x)$. Note that if U_ϕ is "close enough" in norm to the operator R on $R(Y)$, then by classical functional analysis, U_ϕ is invertible on $R(Y)$, since R restricted to $R(Y)$ is the identity map.

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- Now, since $Vf \in R(Y)$ if $f \in CoY$ and $R(x, x_i) = V(S^{-1}\psi_{x_i})(x)$, we have:

$$\begin{aligned} Vf &= U_\phi^{-1} U_\phi Vf \\ &= U_\phi^{-1}\Big(\sum_{i \in I} (c_i V(f)(x_i) V(S^{-1}\psi_{x_i})\Big) \\ &= \sum_{i \in I} \langle f, \psi_{x_i} \rangle U_\phi^{-1}(c_i V(S^{-1}\psi_{x_i})). \end{aligned}$$

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- Co-orbit spaces are great. But this approach has serious drawbacks, especially for applications.
- How fine of a covering is sufficient?
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