Sampling: ambiguity, balayage, and classification

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Narrow band ambiguity functions and CAZAC codes

Discrete ambiguity functions

Let $u : \{0, 1, \ldots, N-1\} \to \mathbb{C}$.

- $u_p : \mathbb{Z}_N \to \mathbb{C}$ is the N-periodic extension of u.
- $u_a : \mathbb{Z} \to \mathbb{C}$ is an aperiodic extension of u:

$$u_a[m] = \begin{cases} u[m], & m = 0, 1, \ldots, N-1 \\ 0, & \text{otherwise.} \end{cases}$$

- The *discrete periodic ambiguity function* $A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{C}$ of u is

$$A_p(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} u_p[m+k]\overline{u_p[k]} e^{2\pi i k n/N}.$$

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CAZAC sequences

- $u : \mathbb{Z}_N \to \mathbb{C}$ is *Constant Amplitude Zero Autocorrelation (CAZAC)*:

$$\forall m \in \mathbb{Z}_N, \quad |u[m]| = 1, \text{ (CA)}$$

and

$$\forall m \in \mathbb{Z}_N \setminus \{0\}, \quad A_p(u)(m, 0) = 0. \text{ (ZAC)}$$

- Empirically, the (ZAC) property of CAZAC sequences u leads to phase coded waveforms w with low *aperiodic autocorrelation* $\mathcal{A}(w)(t, 0)$.
- Are there only finitely many non-equivalent CAZAC sequences?
  - "Yes" for N prime and "No" for $N = MK^2$,
  - Generally unknown for N square free and not prime.

Björck CAZAC codes and ambiguity function comparisons

Legendre symbol

Let N be a prime and $(k, N) = 1$.

- k is a quadratic residue mod N if $x^2 = k \pmod N$ has a solution.
- k is a quadratic non–residue mod N if $x^2 = k \pmod N$ has no solution.
- The Legendre symbol:

$$\left(\frac{k}{N}\right) = \begin{cases} 1, & \text{if } k \text{ is a quadratic residue mod } N\ , \\ -1, & \text{if } k \text{ is a quadratic non–residue mod } N. \end{cases}$$

The diagonal of the product table of $\mathbb{Z}_N$ gives values $k \in \mathbb{Z}$ which are squares. As such we can program Legendre symbol computation.

Example: $N = 7$. $(\frac{k}{N}) = 1$ if $k = 1, 2, 4$.

Definition

Let N be a prime number. A *Björck CAZAC sequence* of length N is

$$u[k] = e^{i\theta_N(k)}, \quad k = 0, 1, \ldots, N-1,$$

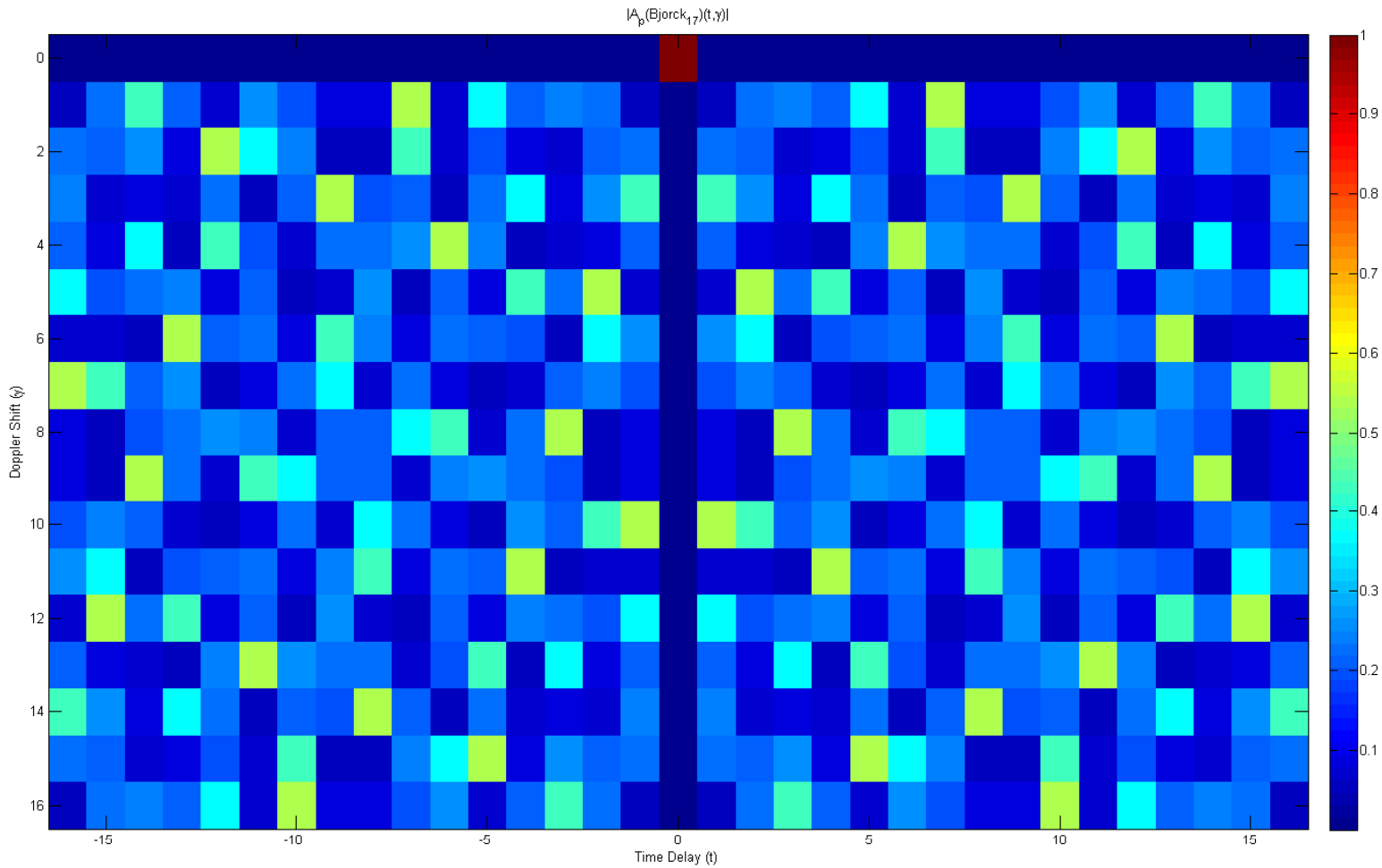
where, for $N = 1 \pmod 4$,

$$\theta_N(k) = \arccos\left(\frac{1}{1+\sqrt{N}}\right)\left(\frac{k}{N}\right),$$

and, for $N = 3 \pmod 4$,

$$\theta_N(k) = \frac{1}{2}\arccos\left(\frac{1-N}{1+N}\right)[(1-\delta_k)\left(\frac{k}{N}\right) + \delta_k].$$

δ_k is Kronecker delta and $\left(\frac{k}{N}\right)$ is Legendre symbol.



Absolute value of Bjorck code of length 17

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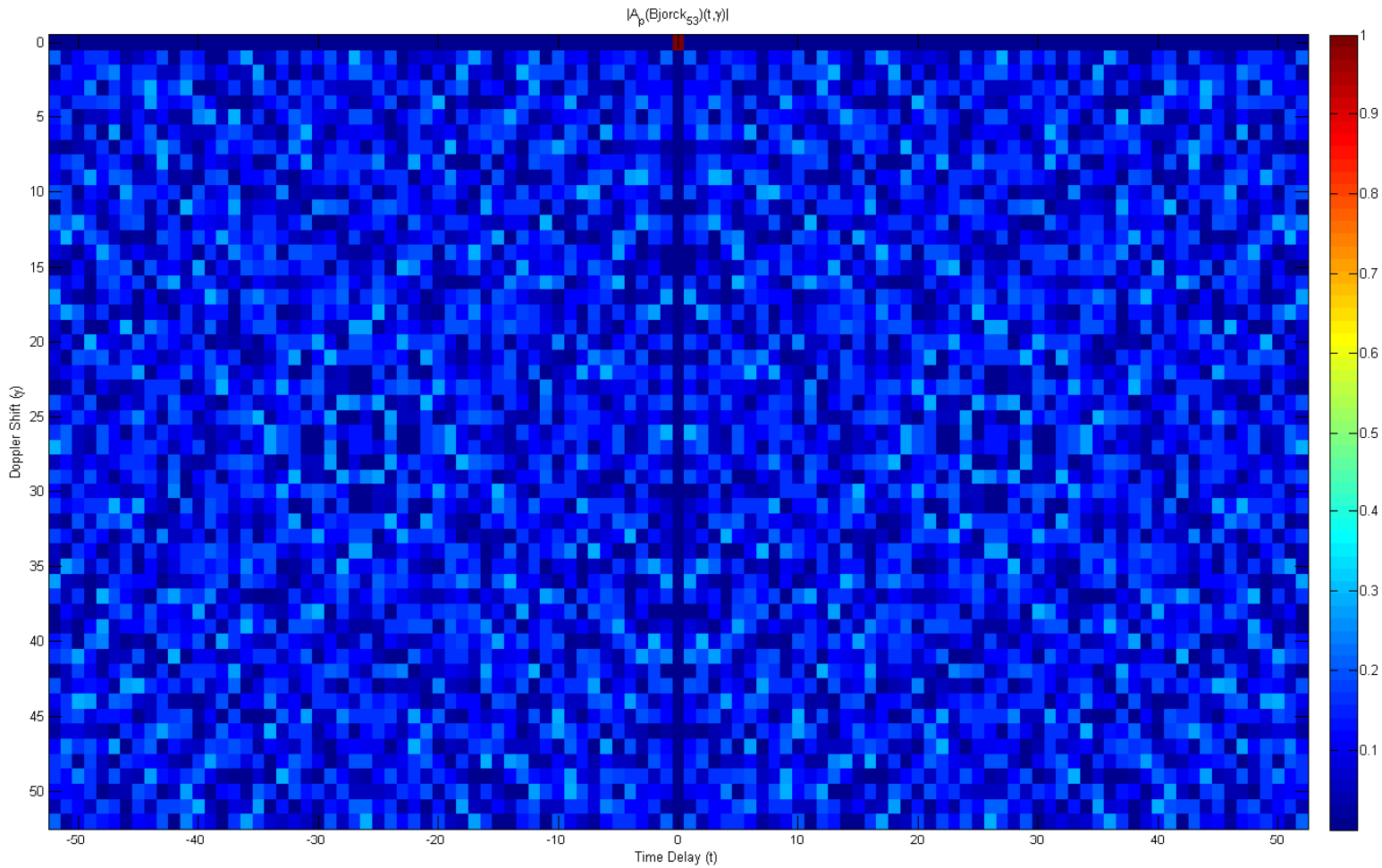

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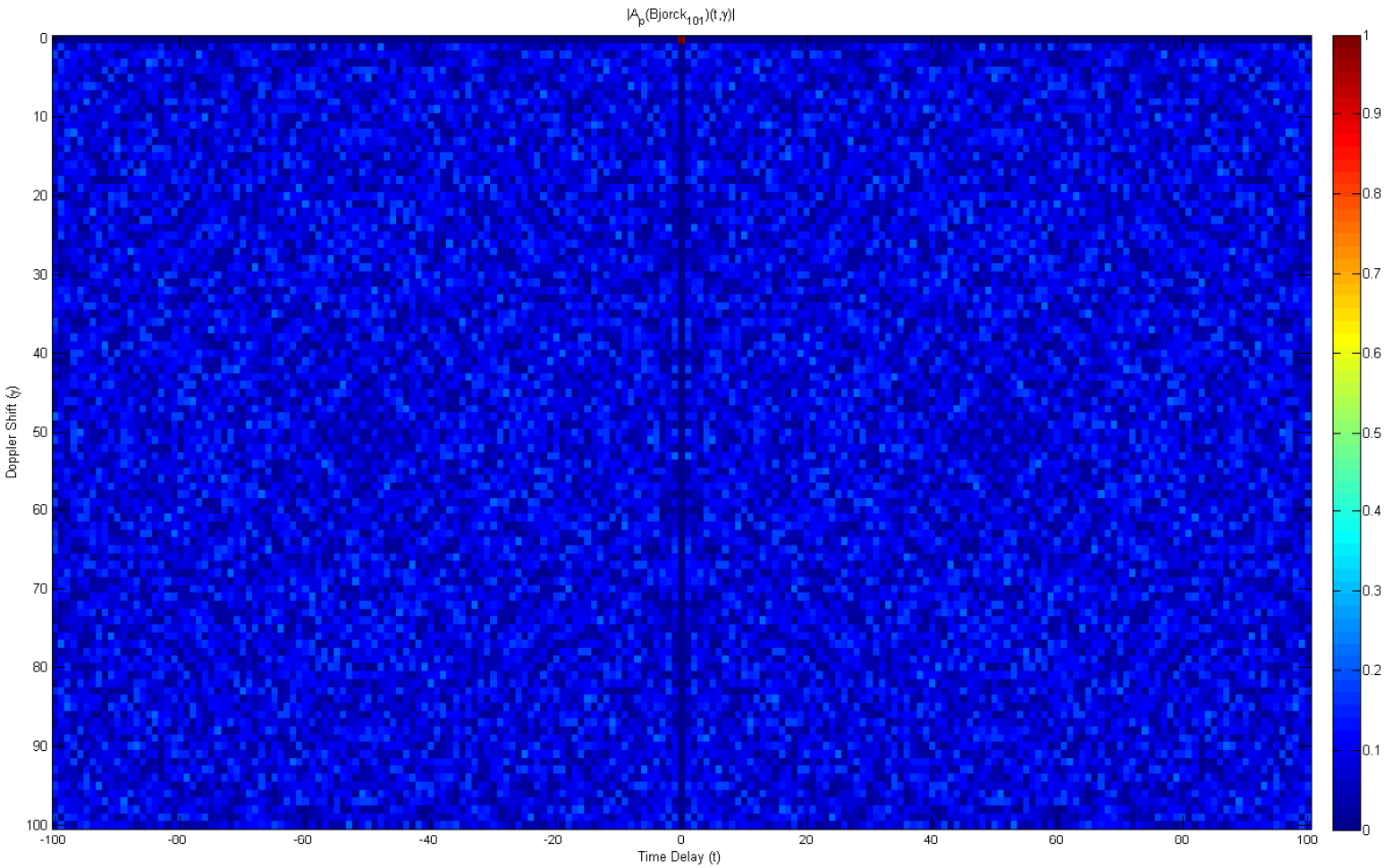

Absolute value of Bjorck code of length 101

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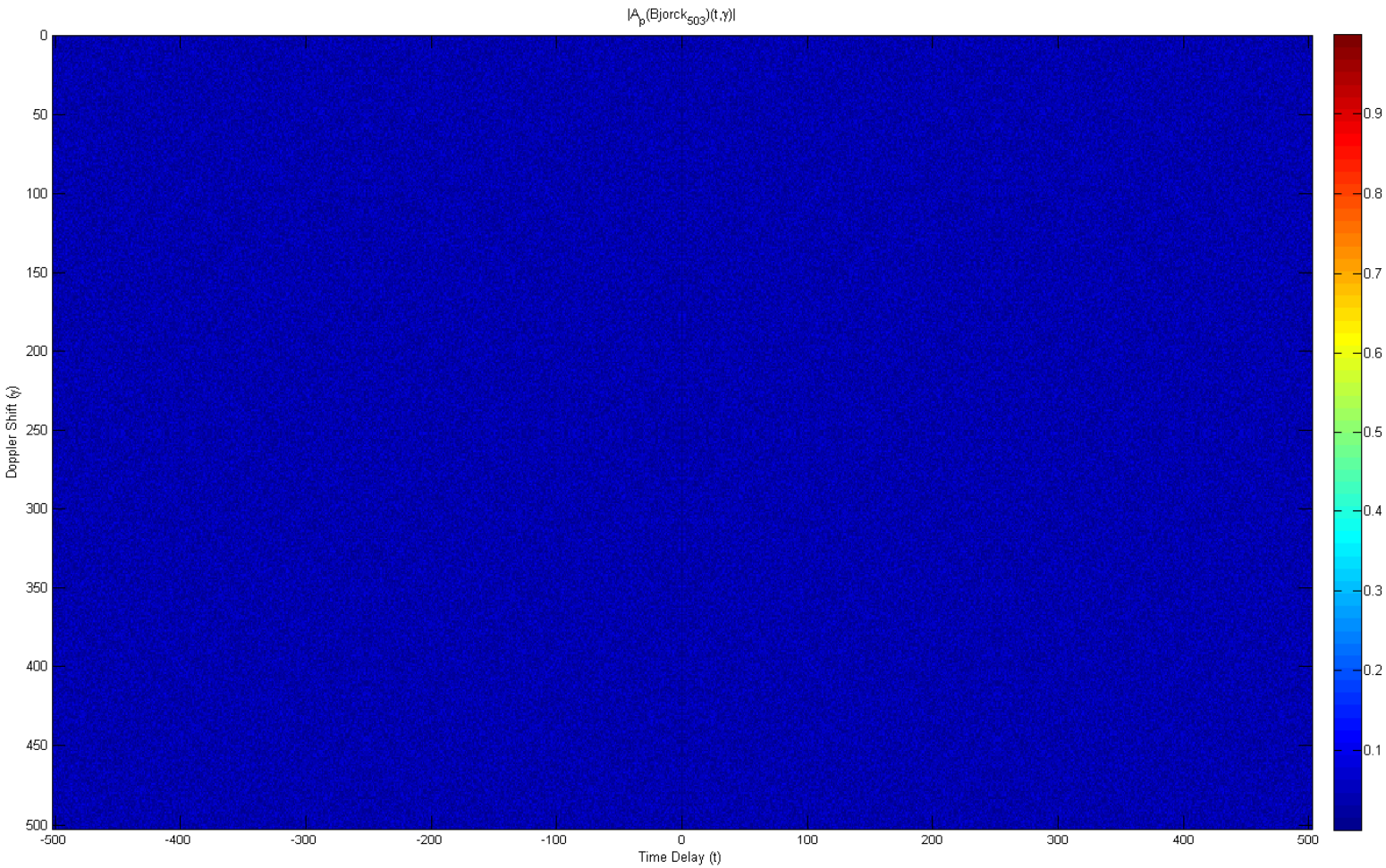

Absolute value of Bjorck code of length 503

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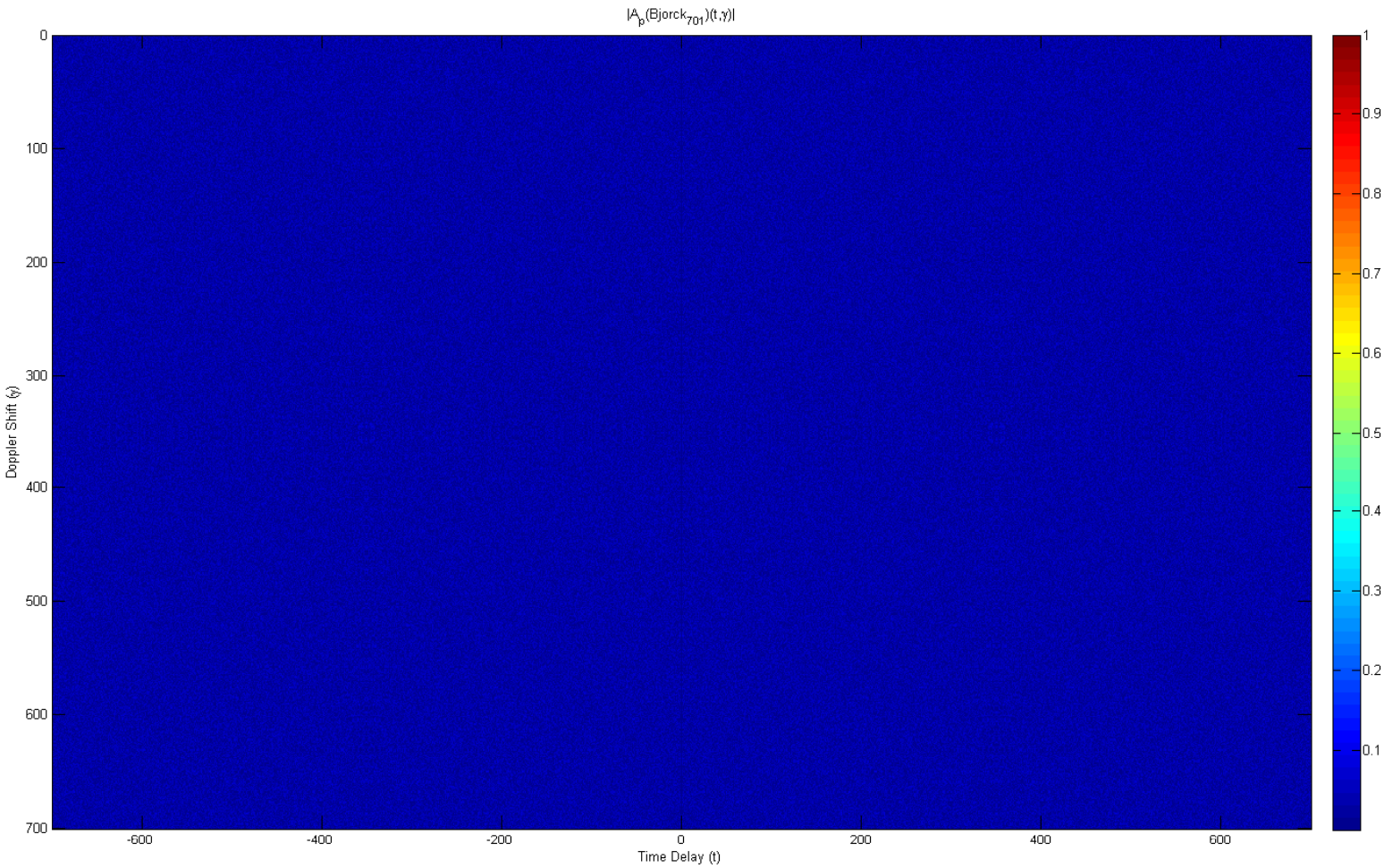

Absolute value of Bjorck code of length 701

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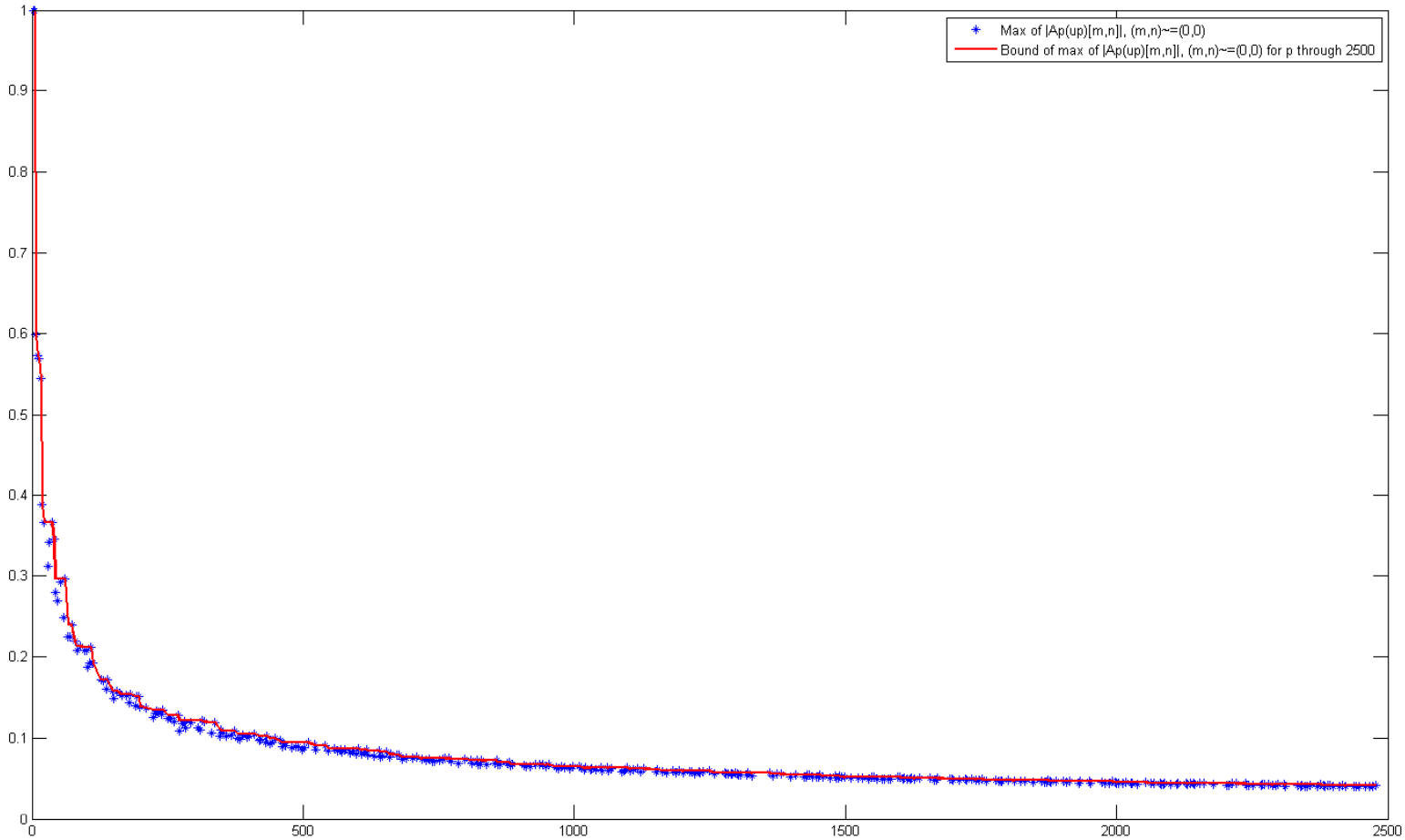

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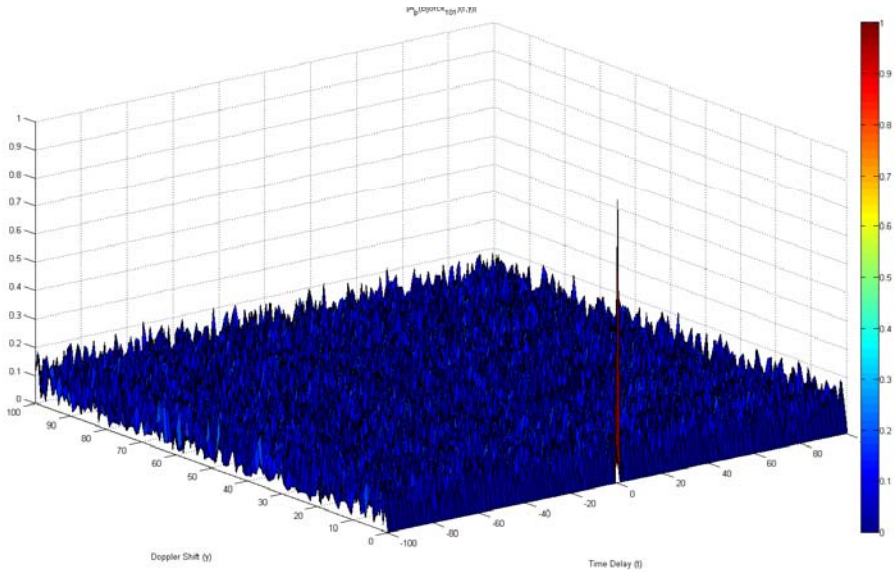
Björck CAZAC Discrete Narrow-band Ambiguity Function

Let u_p denote the Björck CAZAC sequence for prime p, and let $A(u_p)$ be its discrete narrow-band ambiguity function defined on $\mathbb{Z}_N \times \mathbb{Z}_N$.

Theorem (Benedetto-Woodworth) $\forall \epsilon > 0, \exists p_\epsilon$, a prime number which can be calculated, such that $\forall p \geq p_\epsilon$, p prime, and $\forall (m,n) \in \mathbb{Z}_N \times \mathbb{Z}_N \backslash (0,0)$, $|A(u_p)(m,n)| < \epsilon$.



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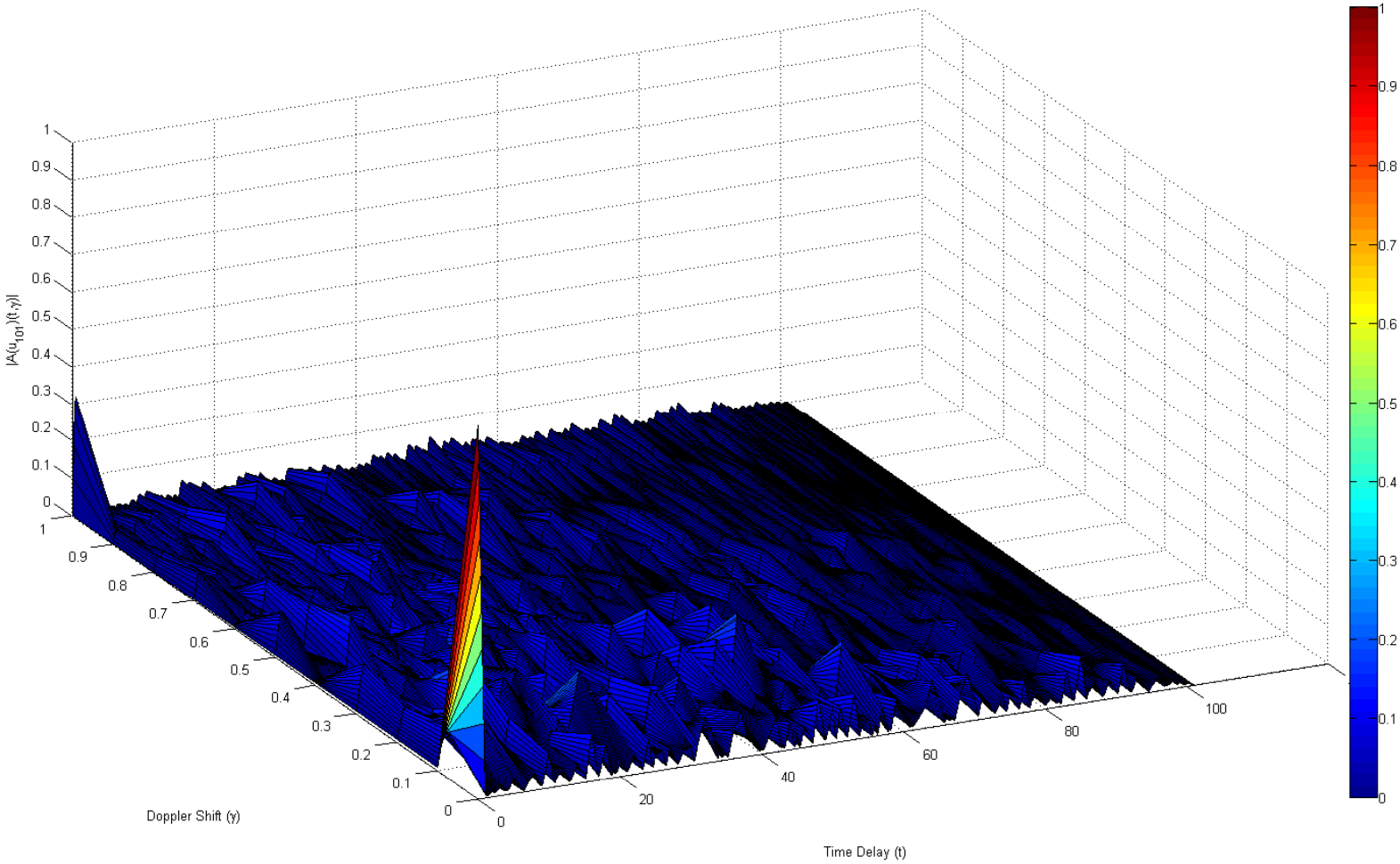

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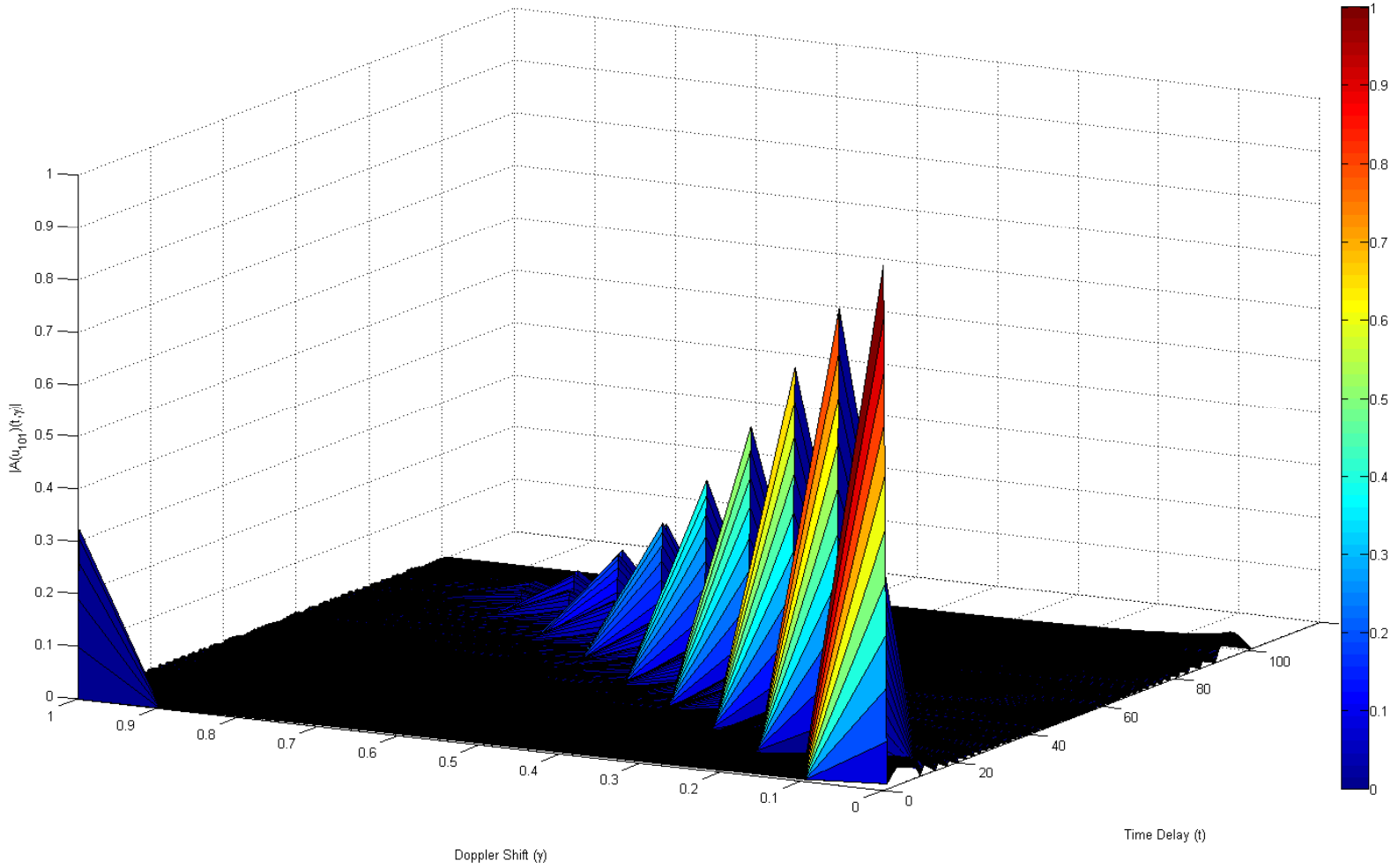

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Perspective

Sequences for coding theory, cryptography, phase-coded waveforms, and communications (synchronization, fast start-up equalization, frequency hopping) include the following in the periodic case:

- Gauss, Wiener (1927), Zadoff (1963), Schroeder (1969), Chu (1972), Zhang and Golomb (1993)
- Frank (1953), Zadoff and Abourezk (1961), Heimiller (1961)
- Milewski (1983)
- Bjørck (1985) and Golomb (1992),

and their generalizations, both periodic and aperiodic.

The general problem of using codes to generate signals leads to frames.

Balayage, Fourier frames, and sampling theory

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Fourier frames, goal, and a litany of names

Definition

$\mathsf{E} = \{x_n\} \subseteq \mathbb{R}^d, \Lambda \subseteq \widehat{\mathbb{R}}^d$. E is a *Fourier frame* for $L^2(\Lambda)$ if

$\exists A, B > 0, \forall F \in L^2(\Lambda),$

$$A \, ||F||^2_{L^2(\Lambda)} \leq \sum_n | < F(\gamma), e^{-2\pi i x_n \cdot \gamma} > |^2 \leq B \, ||F||^2_{L^2(\Lambda)}.$$

- *Goal* Formulate a general theory of Fourier frames and non-uniform sampling formulas parametrized by the space $M(\mathbb{R}^d)$ of bounded Radon measures.
- *Motivation* Beurling theory (1959-1960).
- *Names* Riemann-Weber, Dini, G.D. Birkhoff, Paley-Wiener, Levinson, Duffin-Schaeffer, Beurling-Malliavin, Beurling, H.J. Landau, Jaffard, Seip, Ortega-Certà–Seip.

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Balayage

- Let $M(G)$ be the algebra of bounded Radon measures on the LCAG G.
- Balayage in potential theory was introduced by Christoffel (early 1870s) and Poincaré (1890).

Definition

(Beurling) Balayage is possible for $(E, \Lambda) \subseteq G \times \widehat{G}$, a LCAG pair, if

$$\forall \mu \in M(G), \exists \nu \in M(E) \text{ such that } \hat{\mu} = \hat{\nu} \text{ on } \Lambda.$$

We write balayage (E, Λ).

- The set, Λ, of group characters is the analogue of the original role of Λ in balayage as a collection of potential theoretic kernels.
- Kahane formulated balayage for the harmonic analysis of restriction algebras.

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Spectral synthesis

Definition

(Wiener, Beurling) Closed $\Lambda \subseteq \widehat{G}$ is a set of *spectral synthesis* (S-set) if
$\forall \mu \in M(G), \forall f \in C_b(G)$,
supp$(\widehat{f}) \subseteq \Lambda$ and $\hat{\mu} = 0$ on $\Lambda \Longrightarrow \int_G f \; d\mu = 0$.

$(\forall T \in A'(\widehat{G}), \forall \phi \in A(\widehat{G}), \quad$ supp$(T) \subseteq \Lambda$ and $\phi = 0$ on $\Lambda \Rightarrow T(\phi) = 0.)$

- Ideal structure of $L^1(G)$ - the Nullstellensatz of harmonic analysis
- $T \in D'(\widehat{\mathbb{R}}^d), \phi \in C_c^\infty(\widehat{\mathbb{R}}^d)$, and $\phi = 0$ on supp$(T) \Rightarrow T(\phi) = 0$, with same result for $M(\widehat{\mathbb{R}}^d)$ and $C_0(\widehat{\mathbb{R}}^d)$.
- $S^2 \subseteq \widehat{\mathbb{R}}^3$ is not an S-set (L. Schwartz), and every non-discrete $\widehat{G}$ has non-S-sets (Malliavin).
- Polyhedra are S-sets. The $\frac{1}{3}$-Cantor set is an S-set with non-S-subsets.

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Strict multiplicity

Definition

$\Gamma \subseteq \widehat{G}$ is a set of *strict multiplicity* if

$$\exists\ \mu \in M(\Gamma)\backslash\{0\} \text{ such that } \check{\mu} \text{ vanishes at infinity in } G.$$

- Riemann and sets of uniqueness in the wide sense.
- Menchov (1916): $\exists$ closed $\Gamma \subseteq \widehat{\mathbb{R}}/\mathbb{Z}$ and $\mu \in M(\Gamma)\backslash\{0\}$,
$$|\Gamma| = 0 \text{ and } \check{\mu}(n) = O((\log|n|)^{-1/2}), |n| \to \infty.$$
- 20th century history to study rate of decrease: Bary (1927), Littlewood (1936), Salem (1942, 1950), Ivašev-Mucatov (1957), Beurling.

Assumption

$\forall\ \gamma \in \Lambda$ *and* $\forall\ N(\gamma)$, *compact neighborhood,* $\Lambda \cap N(\gamma)$ *is a set of strict multiplicity.*

A theorem of Beurling

Definition

$\mathsf{E} = \{x_n\} \subseteq \mathbb{R}^d$ is *separated* if

$$\exists\, r > 0,\ \forall m, n,\ m \neq n \Rightarrow ||x_m - x_n|| \geq r.$$

Theorem

Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact S-set, symmetric about $0 \in \widehat{\mathbb{R}}^d$, and let E $\subseteq \mathbb{R}^d$ be separated. If balayage (E,Λ), then

E is a Fourier frame for $L^2(\Lambda)$.

- Equivalent formulation in terms of

$$PW_\Lambda = \{f \in L^2(\mathbb{R}^d) : \mathrm{supp}(\hat{f}) \subseteq \Lambda\}.$$

- $\forall F \in L^2(\Lambda), \qquad F = \sum_{x \in \mathsf{E}} < F, S^{-1}(e_x) >_\Lambda e_x$ in $L^2(\Lambda)$.
- For $\mathbb{R}^d$ and other generality beyond Beurling's theorem in $\mathbb{R}$, the result above was formulated by Hui-Chuan Wu and JB (1998), see Landau (1967).

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Semi-discrete Gabor frames

Let $G \in L^2(\widehat{\mathbb{R}}^d)$ satisfy $||G||_{L^2(\widehat{\mathbb{R}}^d)} = 1$; let $\Lambda \subset \widehat{\mathbb{R}}^d$ be an S-set, symmetric about 0; and let $E \subset \mathbb{R}^d$ be separated. Define

(STFT) $\quad \forall F \in L^2(\Lambda), \quad V_G F(x,\gamma) = \int_\Lambda F(\lambda) G(\lambda - \gamma) e^{2\pi i x \cdot \lambda}\, d\lambda.$

Theorem

If balayage (E,Λ), then

$$\exists A, B > 0, \quad \forall F \in L^2(\Lambda),$$

$$A\, ||F||^2_{L^2(\Lambda)} \leq \int_{\widehat{\mathbb{R}}^d} \sum_{x \in E} |V_G F(x,\gamma)|^2\, d\gamma \leq B\, ||F||^2_{L^2(\Lambda)}.$$

Remark There are basic problems to be resolved and there have been fundamental recent advances.

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Examples of balayage

1. Let E $\subseteq \mathbb{R}^d$ be separated. Define

$$r = r(E) = \sup_{x \in \mathbb{R}^d} \text{dist}(x, E).$$

   If $r\rho < \frac{1}{4}$, then balayage (E, $\bar{B}(0, \rho)$). $\frac{1}{4}$ is the best possible.

2. If balayage (E, Λ) and $\Lambda_0 \subseteq \Lambda$, then balayage (E, Λ_0).

3. Let E $= \{x_n\}$ be a Fourier frame for PW$_\Lambda$. Then for all $\Lambda_0 \subseteq \Lambda$ with dist$(\Lambda_0, \Lambda^c\ ) > 0$, we have balayage (E, Λ_0).

4. In $\mathbb{R}^1$, for a separated set E, Beurling lower density $> \rho$ is necessary and sufficient for balayage (E, $[\frac{-\rho}{2}, \frac{\rho}{2}]$).

Remark In $\mathbb{R}^1$, if E is uniformly dense in the sense of Duffin-Schaeffer, then $D^-(E), D^+(E)$, and $D_u(E)$ coincide.
So Beurling's result $\Rightarrow$ Duffin-Schaeffer's result on Fourier frames.

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ΦDOs and the Kohn-Nirenberg correspondence

Definition/notation for $\Lambda \subseteq \widehat{\mathbb{R}}^d$

- $\forall \gamma \in \Lambda$, $g_\gamma \in C_b(\mathbb{R}^d)$ and $supp(\widehat{g_\gamma}) \subseteq \Lambda$
- $s(x,\gamma) = e^{2\pi i x\cdot\gamma} g_\gamma(x)$

The *Kohn-Nirenberg correspondence*

$$s \longmapsto H_s$$

with *symbol* H_s is defined by the *Hörmander operator*

$$H_s : L^2(\widehat{\mathbb{R}}^d) \to L^2(\Lambda) \subseteq L^2(\widehat{\mathbb{R}}^d)$$

$$H_s(\hat{f})(\gamma) = \int_{\mathbb{R}^d} s(x,\gamma) f(x) e^{-2\pi i x\cdot\gamma}\, dx$$

Remark

Classically, the symbol is σ and integration is over $\widehat{\mathbb{R}}^d$.

ΦDOs and generalized Fourier frames for non-uniform sampling

Theorem

Assume balayage (E, Λ) where $\Lambda \subseteq \widehat{\mathbb{R}}^d$ is a compact, symmetric S-set. Assume $E = \{x_n\}$ is separated. Let $s(x, \gamma) = e^{2\pi i x \cdot \gamma} g_\gamma(x)$, where

$$\{g_\gamma : \gamma \in \Lambda\} \subseteq C_b(\mathbb{R}^d)$$

and

$$\forall \gamma \in \Lambda, \quad supp(\widehat{g_\gamma}) \subseteq \Lambda$$

Let $f \in X_s \subseteq L^2(\mathbb{R}^d)$ if $H_s(\widehat{f}) = F \in L^2(\Lambda)$ and $suppF \subseteq \Lambda$, then

$$\exists A > 0 \quad \text{such that} \quad \forall f \in X_s$$

$$A \frac{\int_\Lambda |F(\gamma)|^2 \, d\gamma}{\|f\|_{L^2(\mathbb{R}^d)}} \leq \left(\sum_{n \in \mathbb{Z}} \left| \int_\Lambda \overline{F(\gamma)} s(x_n, \gamma) e^{2\pi i x_n \cdot \gamma} \, d\gamma \right|^2 \right)^{1/2}.$$

Classification

- Dimension reduction
- Finite frames and frame potential energy
- Frame potential energy classification algorithm
- Hyperspectral image processing

Dimension reduction

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Kernel dimension reduction

Given data space X of N vectors in $\mathbb{R}^D$. (N is the number of pixels in the hypercube, D is the number of spectral bands.)

Two Steps:

1. Construction of an $N \times N$ symmetric, positive semi–definite kernel, K, from these N data points in $\mathbb{R}^D$.
2. Diagonalization of K, and then choosing $d \leq D$ *significant orthogonal* eigenmaps of K.

Motivation

- Different classes of interest may not be orthogonal to each other; however, they may be captured by different frame elements. It is plausible that classes may correspond to elements in a frame but not elements in a basis.
- A *frame* generalizes the concept of an orthonormal basis. Frame elements are non–orthogonal.

Dimension reduction paradigm

- Given data space X of N vectors $x_m \in \mathbb{R}^D$, and let

$$K : X \times X \to \mathbb{R}$$

be a symmetric ($K(x, y) = K(y, x)$), positive semi–definite kernel.

- We map X to a low dimensional space via the following mapping:

$$X \longrightarrow K \longrightarrow \mathbb{R}^d(K), \quad d < D$$

$$x_m \mapsto y_m = (y[m, n_1], y[m, n_2], \ldots, y[m, n_d]) \in \mathbb{R}^d(K),$$

where $y[\cdot, n] \in \mathbb{R}^N$ is an eigenvector of K.

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Laplacian Eigenmaps

- Consider the data points X as the nodes of a graph.
- Define a metric $\rho : X \times X \longrightarrow \mathbb{R}^+$, e.g., $\rho(x_m, x_n) = \|x_m - x_n\|$ is the Euclidean distance.
- Choose $q \in \mathbb{N}$.
- For each x_i choose the q nodes x_n closest to x_i in the metric ρ, and place an edge between x_i and each of these nodes.
- This defines $N'(x_i)$, viz., $N'(x_i) = \{x \in X \ : \ \exists \text{ an edge between } x \text{ and } x_i.\}$.
- To define the weights on the edges, we compute:

$$W_{ij} = \begin{cases} \exp(-\|x_i - x_j\|^2/\sigma) & \text{if } x_j \in N'(x_i) \text{ or } x_i \in N'(x_j) \\ 0 & \text{otherwise} \end{cases}$$

- Set $K = D - W$, where $D_{ii} = \sum_j W_{ij}$ and $D_{ij} = 0$ for $i \neq j$;
- Diagonalize K.
- K is symmetric and positive semi–definite.

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Finite frames and frame potential energy

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FUNTF

- A set $F = \{e_j\}_{j \in J} \subseteq \mathbb{F}^d$ is a *frame* for $\mathbb{F}^d$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, if

$$\exists\, A, B > 0 \quad \text{such that} \quad \forall\, x \in \mathbb{F}^d, \quad A\|x\|^2 \leq \sum_{j \in J} |\langle x, e_j \rangle|^2 \leq B\|x\|^2.$$

- F *tight* if $A = B$. A finite unit-norm tight frame F is a FUNTF.
- N row vectors from any fixed $N \times d$ submatrix of the $N \times N$ DFT matrix, $\frac{1}{\sqrt{d}}(e^{2\pi imn/N})$, is a FUNTF for $\mathbb{C}^d$.
- If F is a FUNTF for $\mathbb{F}^d$, then

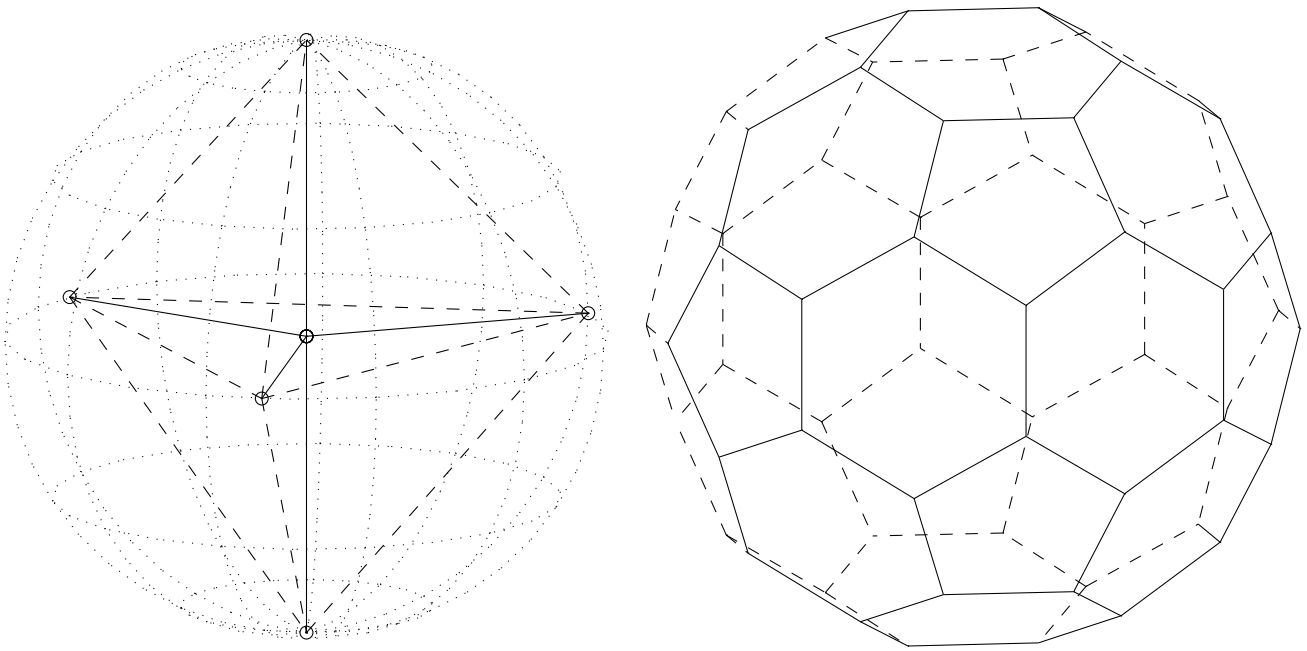
$$\forall x \in \mathbb{F}^d, \quad x = \frac{d}{N} \sum_{j=1}^{N} \langle x, e_j \rangle e_j.$$

- Frames: redundant representation, compensate for hardware errors, inexpensive, numerical stability, minimize effects of noise.

Recent applications of FUNTFs

- Robust transmission of data over erasure channels such as the internet [Casazza, Goyal, Kelner, Kovačević]
- Multiple antenna code design for wireless communications [Hochwald, Marzetta,T. Richardson, Sweldens, Urbanke]
- Multiple description coding [Goyal, Heath, Kovačević, Strohmer,Vetterli]
- Quantum detection [Bölcskei, Eldar, Forney, Oppenheim, Kebo, B]
- Grassmannian ”min-max” waveforms [Calderbank, Conway, Sloane, et al., Kolesar, B]

Examples of frames



(a) Non–FUNTF (b) FUNTF

Frame force and potential energy

$$F : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R}^d$$

$$P : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R},$$

where $P(a,b) = p(\|a-b\|), \qquad p'(x) = -xf(x)$

- Coulomb force

$$CF(a,b) = (a-b)/\|a-b\|^3, \quad f(x) = 1/x^3$$

- Frame force

$$FF(a,b) = \langle a,b\rangle(a-b), \quad f(x) = 1 - x^2/2$$

- Total potential energy for the frame force

$$TFP(\{x_n\}) = \sum_{m=1}^{N}\sum_{n=1}^{N} |\langle x_m, x_n\rangle|^2$$

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Characterization of FUNTFs

Theorem

Let $N \leq d$. The minimum value of *TFP*, for the frame force and N variables, is N; and the *minimizers* are precisely the orthonormal sets of N elements for $\mathbb{R}^d$.

Let $N \geq d$. The minimum value of *TFP*, for the frame force and N variables, is N^2/d; and the *minimizers* are precisely the FUNTFs of N elements for $\mathbb{R}^d$.

Problem

Find FUNTFs analytically, effectively, computationally.

Frame potential energy classification algorithm

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Optimization problem: maximal separation

Goal: Construct a FUNTF $\{\Psi_k\}_{k=1}^s$ such that each Ψ_k is associated to only one classifiable material.

For $\{\theta_k\}_{k=1}^s \in S^{d-1} \times \cdots \times S^{d-1}$ and $n = 1, \ldots, s$, set

$$p(\theta_n) = \sum_{m=1}^{N} |\langle y_m, \theta_n \rangle|$$

and consider the maximal separation

$$\sup_{\{\theta_j\}_{j=1}^s} \min\{|p(\theta_k) - p(\theta_n)| : k \neq n\}.$$

Optimization problem: FUNTF construction

Combine maximal separation with frame potential to construct a pseudo-FUNTF $\Psi = \{\psi_k\}_{k=1}^s$ by solving the minimization problem:

$$\sup\left\{\min\{|p(\theta_k) - p(\theta_n)| : k \neq n\} : \{\theta_j\} \in \{\arg\min_{\Phi} TFP(\Phi)\right\}, \quad (1)$$

where $\Phi = \{\phi_k\}_{k=1}^s$.

- (1) is solved using a new, fast gradient descent method for products of spheres.
- Nate Strawn created the method and developed new geometric ideas for such computation.

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Frame coefficient images

Given $\Psi = \{\Psi_n\}_{n=1}^{s}$ and $m \in \{1, \ldots, N\}$. Consider the set of frame decompositions

$$y_m = \sum_{n=1}^{s} c_{m,n}^{\alpha} \Psi_n, \quad \text{indexed by } \alpha \in \mathbb{R}.$$

- If Ψ is a FUNTF then $\alpha = 0$ designates the canonical dual, i.e.,

$$c_{m,n}^{0} = \frac{d}{s} \langle y_m, \Psi_n \rangle.$$

Frame coefficient images (continued)

- For each $m \in \{1, \ldots, N\}$ choose an ℓ^1 sparse decomposition

$$y_m = \sum_{n=1}^{s} c_{m,n}^{\alpha(m)} \Psi_n$$

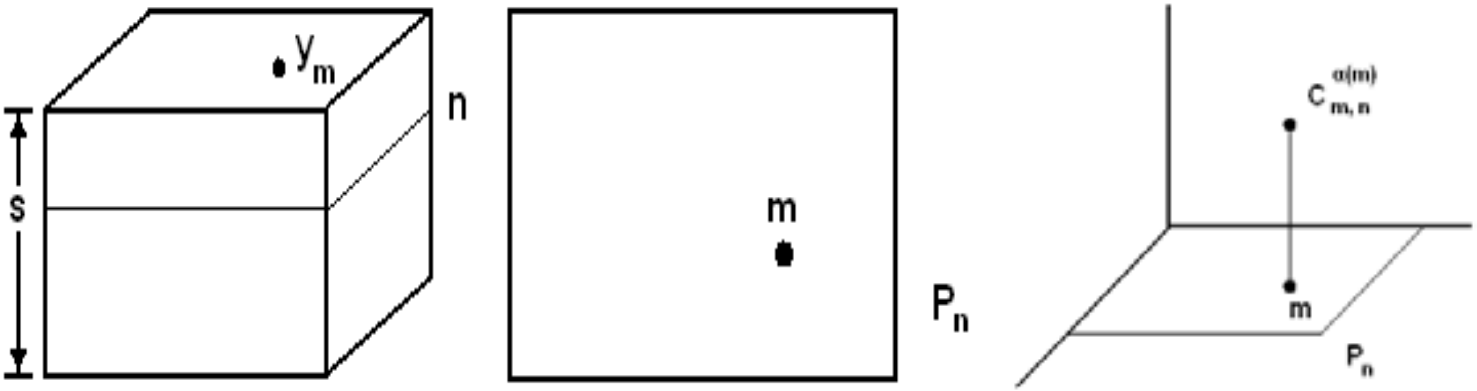
defined by the inequality,

$$\forall\, \alpha, \quad \sum_{n=1}^{s} |c_{m,n}^{\alpha(m)}| \leq \sum_{n=1}^{s} |c_{m,n}^{\alpha}|.$$

- There is ℓ^0 theory.

Frame coefficient images (continued)

- Choose $n \in \{1, ..., s\}$. Take a slice, P_n, of the data cube at n. P_n contains N points m.



(a) Data Cube (b) Top Down Slice (c) $c_{m,n}^{\alpha(m)}$ defined

- The image with N pixels m, associated to the the frame element Ψ_n, is defined by $\{c_{m,n}^{\alpha(m)} \mid m = 1, ..., N\}$.

Hyperspectral image processing

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Urban data set classes

Figure: HYDICE Copperas Cove, TX — http://www.tec.army.mil/Hypercube/

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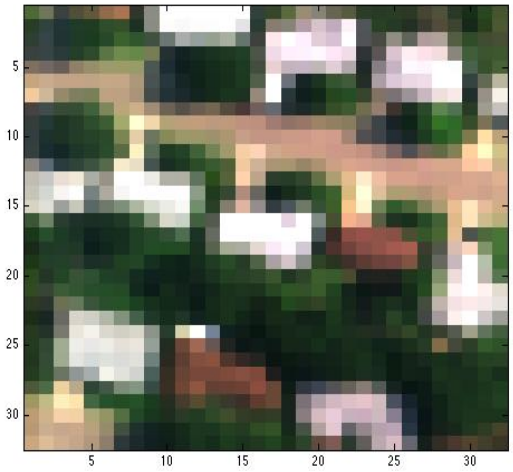
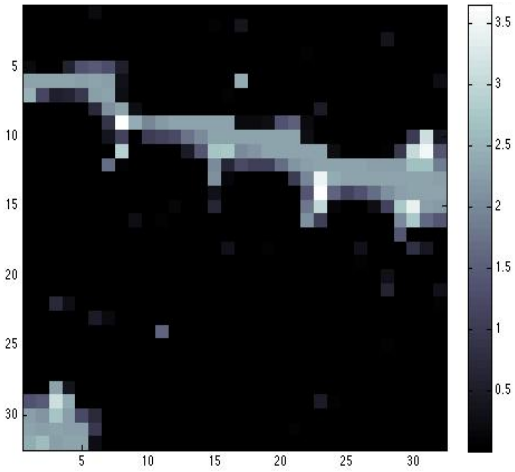
Urban data set classes

- There are 23 classes associated with the different colors in the previous figure.
- In fact, if the 23 classes were to correspond roughly to orthogonal subspaces, then one cannot achieve effective dimension reduction less than dimension $d = 23$.
- However, we could have a frame with 23 elements in a space of reduced dimension $d < 23$.

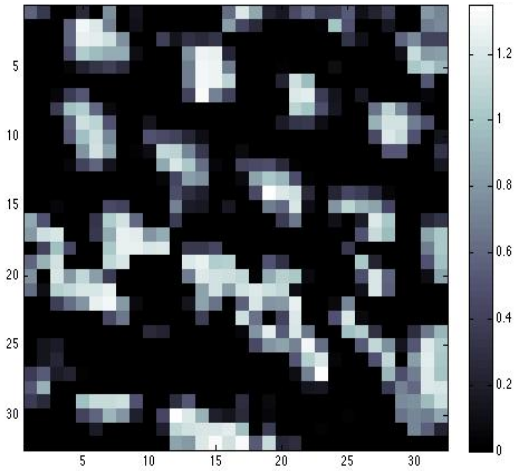
Frame coefficients



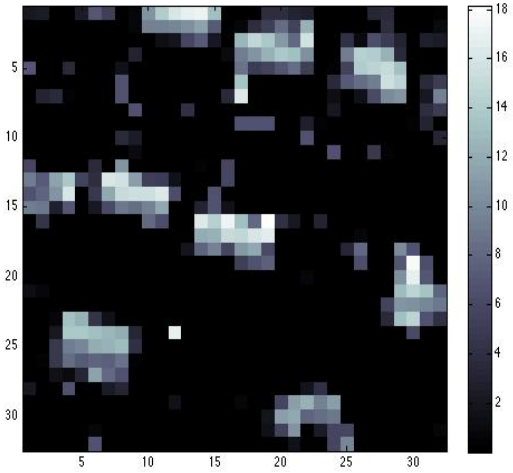
(a) Original



(b) Road coefficients

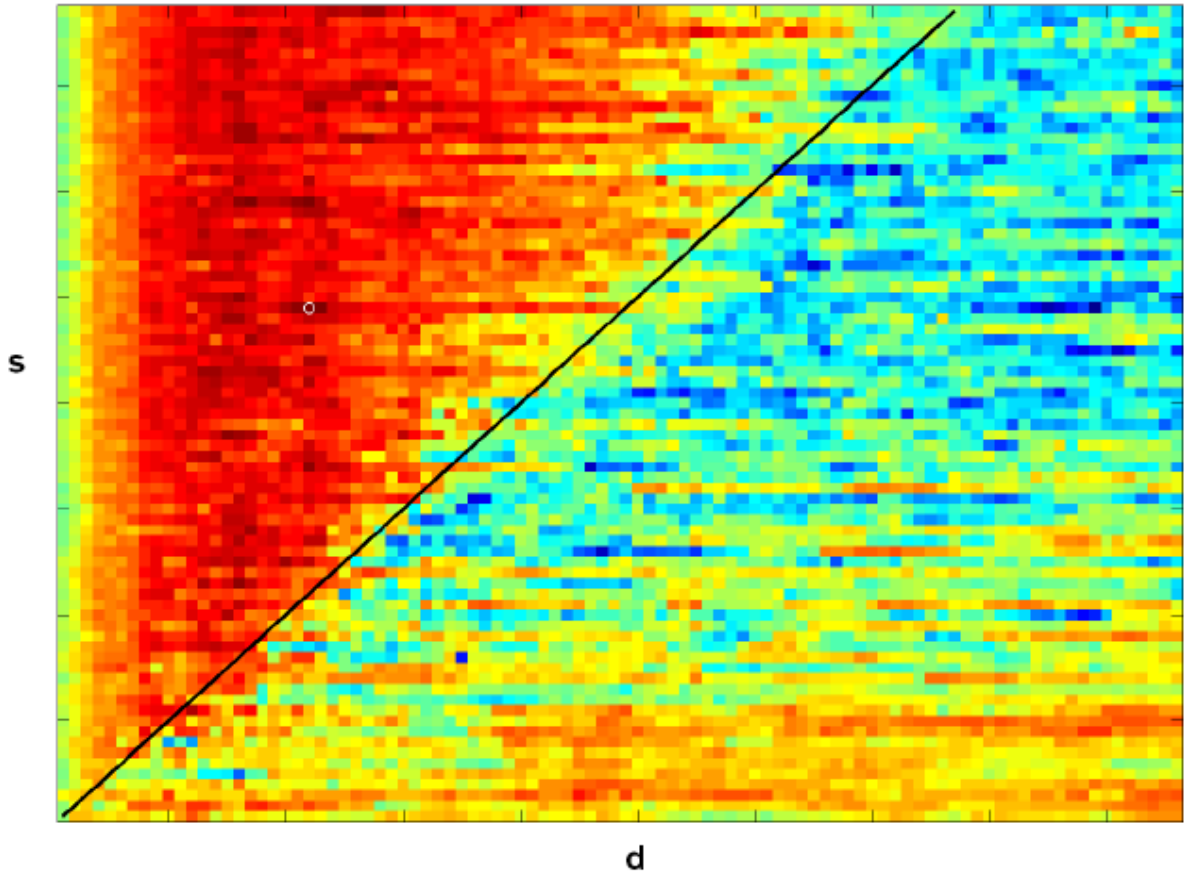


(c) Tree coefficients



(d) White house coefficients

Overview of Classification Results



That's all folks!

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The geometry of finite tight frames

- We saw the vertices of platonic solids are FUNTFs.
- However, points that constitute FUNTFs do not have to be equidistributed, e.g., ONBs and Grassmanian frames.
- FUNTFs can be characterized as minimizers of a frame potential function (with Fickus) analogous to Coulomb's Law.
- Frame potential energy optimization has basic applications dealing with classification problems for hyperspectral and multi-spectral (biomedical) image data.

DFT FUNTFs

- $N \times d$ submatrices of the $N \times N$ DFT matrix are FUNTFs for $\mathbb{C}^d$. These play a major role in finite frame $\Sigma\Delta$-quantization.

$$N = 8,\ d = 5 \qquad \frac{1}{\sqrt{5}} \begin{bmatrix} * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \end{bmatrix}$$

$$x_m = \frac{1}{5}(e^{2\pi i \frac{m}{8}}, e^{2\pi i m \frac{2}{8}}, e^{2\pi i m \frac{5}{8}}, e^{2\pi i m \frac{6}{8}}, e^{2\pi i m \frac{7}{8}})$$

$$m = 1, \ldots, 8.$$

- Sigma-Delta Super Audio CDs - but not all authorities are fans.

CAZACs and FUNTFs

- Let $u = \{u[k]\}_{k=1}^{N}$ be a CAZAC sequence in $\mathbb{C}$. Define

$$\forall\, k = 1, ..., N, \quad v_k = v[k] = \frac{1}{\sqrt{d}}(u[k], u[k+1], ..., u[k+d-1]).$$

  Then $v = \{v[k]\}_{k=1}^{N} \subseteq \mathbb{C}^d$ is a CAZAC sequence in $\mathbb{C}^d$ and $\{v_k\}_{k=1}^{N}$ is a FUNTF for $\mathbb{C}^d$ with frame constant N/d.

- Let $\{x_k\}_{k=1}^{N} \subseteq \mathbb{C}^d$ be a FUNTF for $\mathbb{C}^d$, with frame constant A and with associated Bessel map $L : \mathbb{C}^d \to \ell^2(\mathbb{Z}_N)$; and let $u = \{u[j]\}_{j=1}^{M} \subseteq \mathbb{C}^d$ be a CAZAC sequence in $\mathbb{C}^d$. Then $\{\frac{1}{\sqrt{A}} L(u[j])\}_{j=1}^{M} \subseteq \mathbb{C}^N (= \ell^2(\mathbb{Z}_N)$ is a CAZAC sequence in $\mathbb{C}^N$.

The ambiguity function

- The *complex envelope* w of the *phase coded waveform* $Re(w)$ associated to a unimodular N-periodic sequence $u : \mathbb{Z}_N \to \mathbb{C}$ is

$$w(t) = \frac{1}{\sqrt{\tau}} \sum_{k=0}^{N-1} u[k] \, \mathbb{1}\left(\frac{t - kt_b}{t_b}\right),$$

  where $\mathbb{1}$ is the characteristic function of the interval $[0, 1)$, τ is the pulse duration, and $t_b = \tau/N$.
- For spectral shaping problems, smooth replacements to $\mathbb{1}$ are analyzed.
- The *(aperiodic) ambiguity function* $\mathcal{A}(w)$ of w is

$$\mathcal{A}(w)(t, \gamma) = \int w(s + t)\overline{w(s)} e^{2\pi i s \gamma} ds,$$

  where $t \in \mathbb{R}$ is time delay and $\gamma \in \widehat{\mathbb{R}}(= \mathbb{R})$ is frequency shift.

Lower frame bounds

- Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact S-set, and assume balayage (E, Λ) where E $= \{x_n\}$ is separated.

1. $\forall F \in L^2(\Lambda),\ \Lambda$ convex,

$$\sqrt{A}\ \frac{\int_\Lambda |F(\gamma)+F(2\gamma)+F(3\gamma)|^2\ d\gamma}{(\int_\Lambda |F(\gamma)|^2\ d\gamma)^{1/2}}$$

$$\leq (\sum |\check{F}(x_n)|^2)^{1/2} + \tfrac{1}{2}(\sum |\check{F}(\tfrac{1}{2}x_n)|^2)^{1/2} + \tfrac{1}{3}(\sum |\check{F}(\tfrac{1}{3}x_n)|^2)^{1/2}.$$

2. Given positive $G \in L^2(\Lambda)$. Then $\forall F \in L^2(\Lambda)$,

$$\sqrt{A}\ \frac{\int_\Lambda |F(\gamma)|^2 G(\gamma)\ d\gamma}{(\int_\Lambda |F(\gamma)|^2\ d\gamma)^{\frac{1}{2}}} \leq (\sum |(FG)\check{}(x_n)|^2)^{1/2}.$$

Sampling formulas (1)

- Let $\Lambda \in \widehat{\mathbb{R}}^d$ be a compact S-set, and assume balayage (E, Λ), E $= \{x_n\} \subseteq \mathbb{R}^d$ separated.
- *Theorem* $\exists \epsilon > 0$, balayage (E, Λ_ϵ).
- *Theorem* $\forall x \in \mathbb{R}^d,\ \exists\ \{b_n(x)\} \in l^1(\mathbb{Z})$,

$$\sup_{x\in\mathbb{R}^d} \sum_n |b_n(x)| \leq K(E, \Lambda_\epsilon)$$

and $$e^{-2\pi i x\cdot\gamma} = \sum_n b_n(x) e^{-2\pi i x_n\cdot\gamma} \quad \text{uniformly on } \Lambda_\epsilon.$$

- Let h be entire on $\mathbb{R}^d$ with $e^{-\Omega(|x|)}$ decay,

$$h(0) = 1 \text{ and supp}(\hat{h}) \subseteq \bar{B}(0, \epsilon).$$

Theorem

$\forall f \in C_b(\mathbb{R}), \textit{supp}(\hat{f}) \subseteq \Lambda,$

$$\forall y \in \mathbb{R}^d, \quad f(y) = \sum f(x_n) b_n(y) h(x_n - y)$$

- Weighted sampling function $b_n(y)h(x_n - y)$ independent of $f \in C_b(\mathbb{R}^d), \text{supp}(\hat{f}) \subseteq \Lambda$.

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Sampling formulas (2)

- The Nyquist condition, $2T\Omega \leq 1$, for sampling period T and bandwidth $[-\Omega, \Omega]$, gives way to balayage (E,Λ), where Λ is the bandwidth and the sampling set E is related to Λ by balayage (E,Λ).
- Let $s \in C_b(\mathbb{R}^d)$, supp$(\hat{s}) \subseteq \Lambda$, a compact S-set - *sampling function* s.
- Let $A = \{a(n)\} \subseteq \mathbb{R}^d, n \in \mathbb{Z}$ and distinct points $a(n)$. Define

$$V_A = \{f \in C_b(\mathbb{R}^d) : \forall x \in \mathbb{R}^d, f(x) = \sum_n c_n(f) s(x - a(n)), \sum_n |c_n(f)| < \infty\}.$$

- Assume balayage (E,Λ), E $= \{x_n\} \subseteq \mathbb{R}^d$ separated.
- Define

$$V_E = \{f \in C_b(\mathbb{R}^d) : \forall x \in \mathbb{R}^d, f(x) = \sum_n c_n(f) s(x - x_n), \sum_n |c_n(f)| < \infty\}.$$

Theorem

$V = \cup_A V_A \subseteq V_E \subseteq C_b(\mathbb{R}^d)$. *Thus,*

$$\forall f \in V, \quad f(x) = \textstyle\sum_n c_n(f) s(x - x_n), \text{ uniformly on } \mathbb{R}^d.$$

LLE

- Consider the data points X as the nodes of a graph.
- Define a metric $\rho : X \times X \longrightarrow \mathbb{R}^+$, e.g., $\rho(x_m, x_n) = \|x_m - x_n\|$ is the Euclidean distance.
- Choose $q \in \mathbb{N}$.
- For each x_i choose the q nodes x_n closest to x_i in the metric ρ, and place an edge between x_i and each of these nodes.
- This defines $N'(x_i)$, viz., $N'(x_i) = \{x \in X \,:\, \exists \text{ an edge between } x \text{ and } x_i.\}$.
- To define the weights on the edges, we compute:

$$W = \underset{\widetilde{W}}{\operatorname{argmin}} \left| x_i - \sum_{j \in N'(x_i)} \widetilde{W}(x_i, x_j) x_j \right|^2 .$$

- Set $K = (I - W)(I - W^{\mathsf{T}})$ and diagonalize K.
- K is symmetric and positive semi–definite.

Optimization problem: FUNTF construction

Combine frame potential with "ℓ^1-energy" to construct a FUNTF $\Psi = \{\psi_k\}_{k=1}^s$ by solving a minimization problem of the following type:

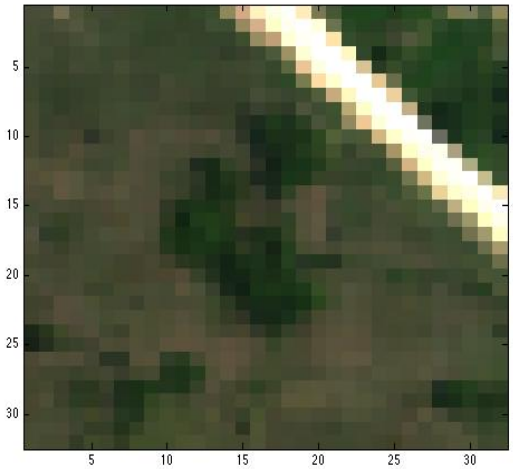
$$\min\{TFP(\Theta) + P(Y, \Theta) : \Theta \in S^{d-1} \times \cdots \times S^{d-1}\},$$

where

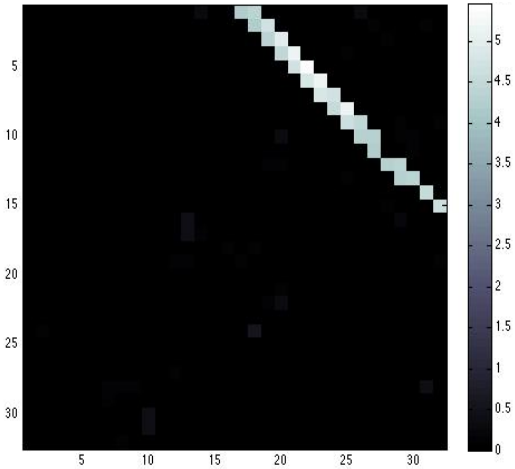
$$P(Y, \Theta) = \sum_{n=1}^{N} \sum_{k=1}^{s} |\langle y_n, \theta_k \rangle| = \sum_{k=1}^{s} p(\theta_k).$$

Remark. a. Minimization of P is convex optimization of ℓ^1-energy of Y for a given frame.
b. By Candes and Tao (2005), under suitable conditions, this can yield a frame Ψ with a sparse set of coefficients $\{\langle y_n, \psi_k \rangle\}$. We do not proceed this way to obtain sparsity.

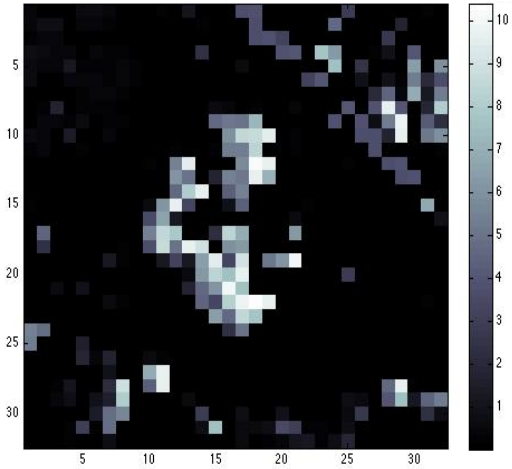
Frame coefficients



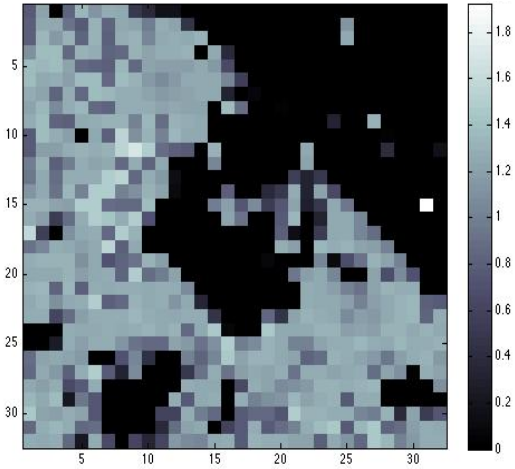
(a) Original



(b) Road coefficients



(c) Tree coefficients



(d) Dirt/grass coefficients

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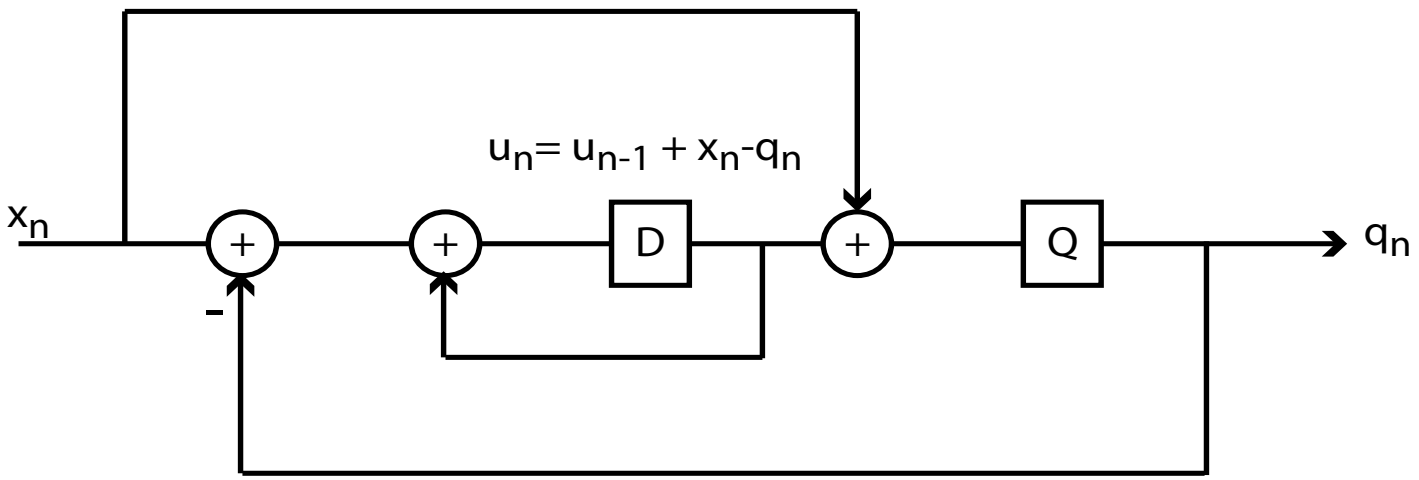
Quantization Methods

SIGMA-DELTA QUANTIZATION

Given u_0 and $\{x_n\}_{n=1}$

$u_n = u_{n-1} + x_n - q_n$
$q_n = Q(u_{n-1} + x_n)$



First Order $\Sigma\Delta$

A quantization problem

Qualitative Problem Obtain *digital* representations for class X, suitable for storage, transmission, recovery.
Quantitative Problem Find dictionary $\{e_n\} \subseteq X$:

1. Sampling [continuous range $\mathbb{K}$ is not digital]

$$\forall x \in X, \quad x = \sum x_n e_n, \quad x_n \in \mathbb{K}.$$

2. Quantization. Construct finite alphabet $\mathcal{A}$ and

$$Q: \ X \to \{\sum q_n e_n : \ q_n \in \mathcal{A} \subseteq \mathbb{K}\}$$

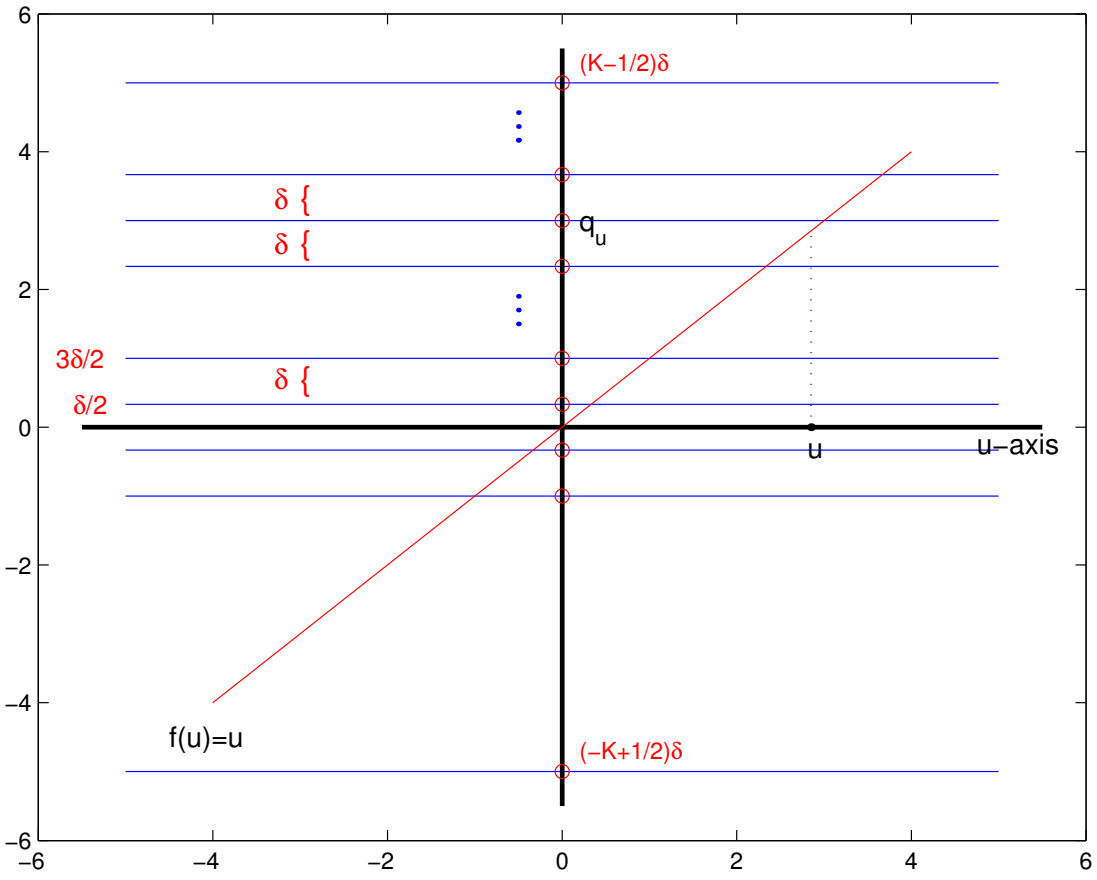
such that $|x_n - q_n|$ and/or $\|x - Qx\|$ small.

Methods

Fine quantization, e.g., PCM. Take $q_n \in \mathcal{A}$ close to given x_n.
Reasonable in 16-bit (65,536 levels)digital audio.
Coarse quantization, e.g., $\Sigma\Delta$. Use fewer bits to exploit redundancy.
SRQP

Quantization

$$\mathcal{A}_K^\delta = \{(-K+1/2)\delta, (-K+3/2)\delta, \ldots, (-1/2)\delta, (1/2)\delta, \ldots, (K-1/2)\delta\}$$



$$Q(u) = \arg\min\{|u-q| : q \in \mathcal{A}_K^\delta\} = q_u$$

PCM and first order Sigma-Delta

Let $x \in \mathbb{C}^d$, $\{e_n\}_{n=1}^N$ be a frame for $\mathbb{C}^d$.

- **PCM:** $\forall n = 1, \ldots, N, \quad q_n = Q_\delta(\langle x, e_n \rangle)$,
- **First Order Sigma-Delta:** Let p be a permutation of $\{1, \ldots, N\}$. First Order Sigma-Delta quantization generates quantized sequence $\{q_n\}_{n=1}^N$ by the iteration

$$\begin{aligned} q_n &= Q_\delta(u_{n-1} + \langle x, e_{p(n)} \rangle), \\ u_n &= u_{n-1} + \langle x, e_{p(n)} \rangle - q_n, \end{aligned}$$

  for $n = 1, \ldots, N$, with an initial condition u_0.

In either case, the quantized estimate is

$$\widetilde{x} = \frac{d}{N} \sum_{n=1}^{N} q_n e_n = \frac{d}{N} L^* q$$

PCM

Replace $x_n \leftrightarrow q_n = \arg\{\min |x_n - q| : q \in \mathcal{A}_K^\delta\}$. Then

$$(PCM) \qquad \tilde{x} = \frac{d}{N} \sum_{n=1}^{N} q_n e_n$$

satisfies

$$\|x - \tilde{x}\| \leq \frac{d}{N} \| \sum_{n=1}^{N} (x_n - q_n) e_n \| \leq \frac{d}{N} \frac{\delta}{2} \sum_{n=1}^{N} \|e_n\| = \frac{d}{2} \delta.$$

Not good!

Bennett's white noise assumption

Assume that $(\eta_n) = (x_n - q_n)$ is a sequence of independent, identically distributed random variables with mean 0 and variance $\frac{\delta^2}{12}$. Then the mean square error (MSE) satisfies

$$\text{MSE} = E\|x - \tilde{x}\|^2 \leq \frac{d}{12A} \delta^2 = \frac{(d\delta)^2}{12N}$$

$\mathcal{A}_1^2 = \{-1, 1\}$ and E_7

Let $x = (\frac{1}{3}, \frac{1}{2})$, $E_7 = \{(\cos(\frac{2n\pi}{7}), \sin(\frac{2n\pi}{7}))\}_{n=1}^{7}$. Consider quantizers with $\mathcal{A} = \{-1, 1\}$.

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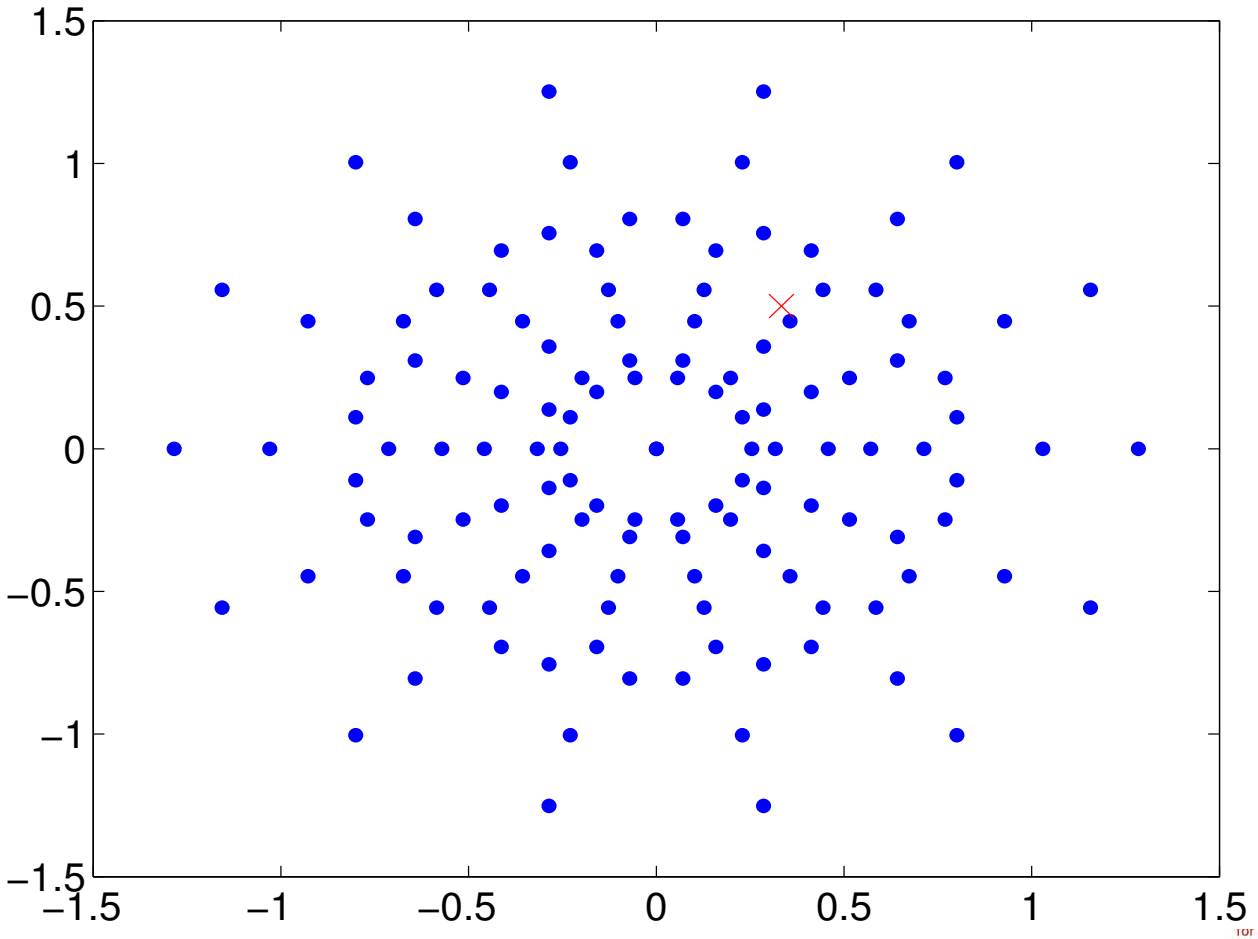

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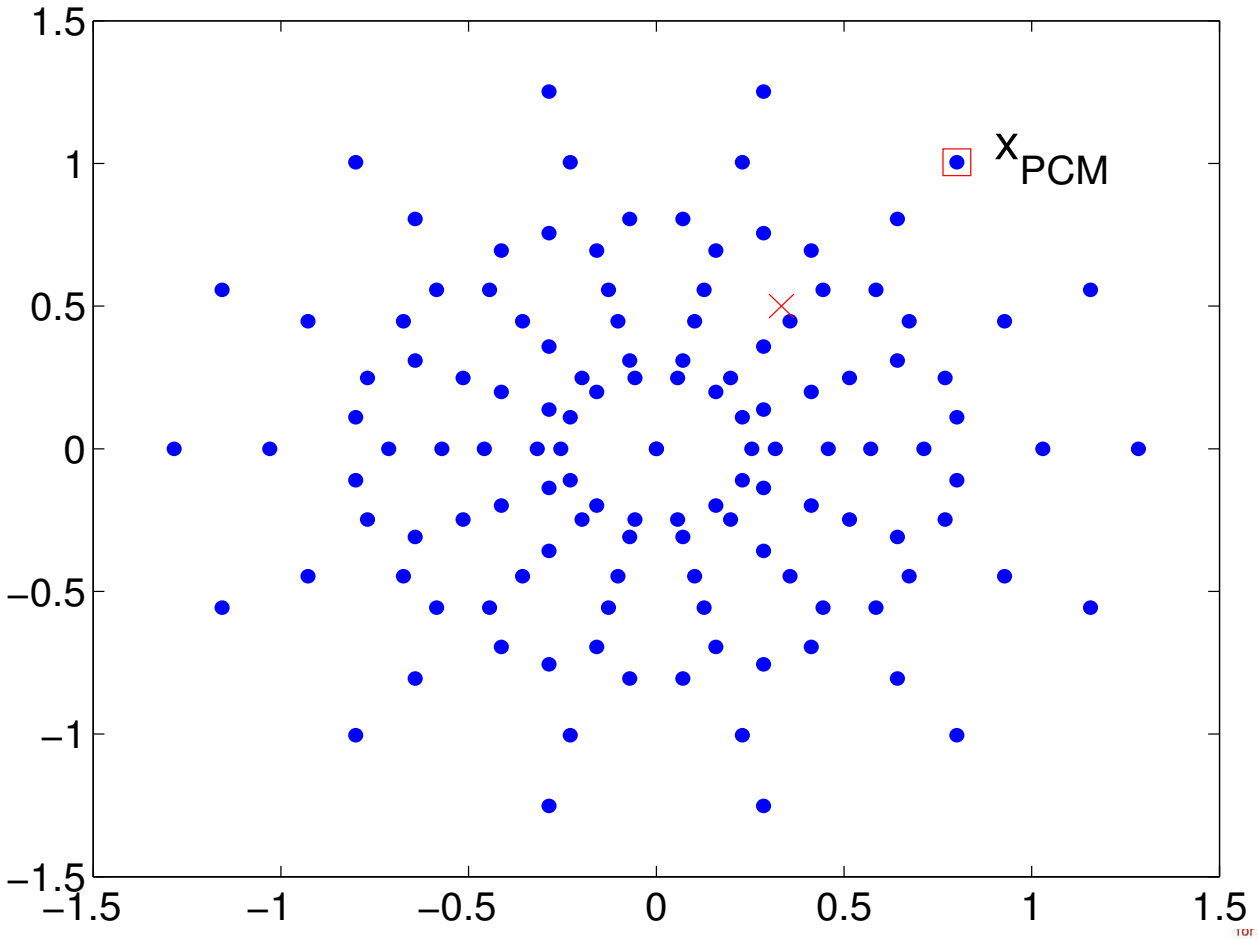

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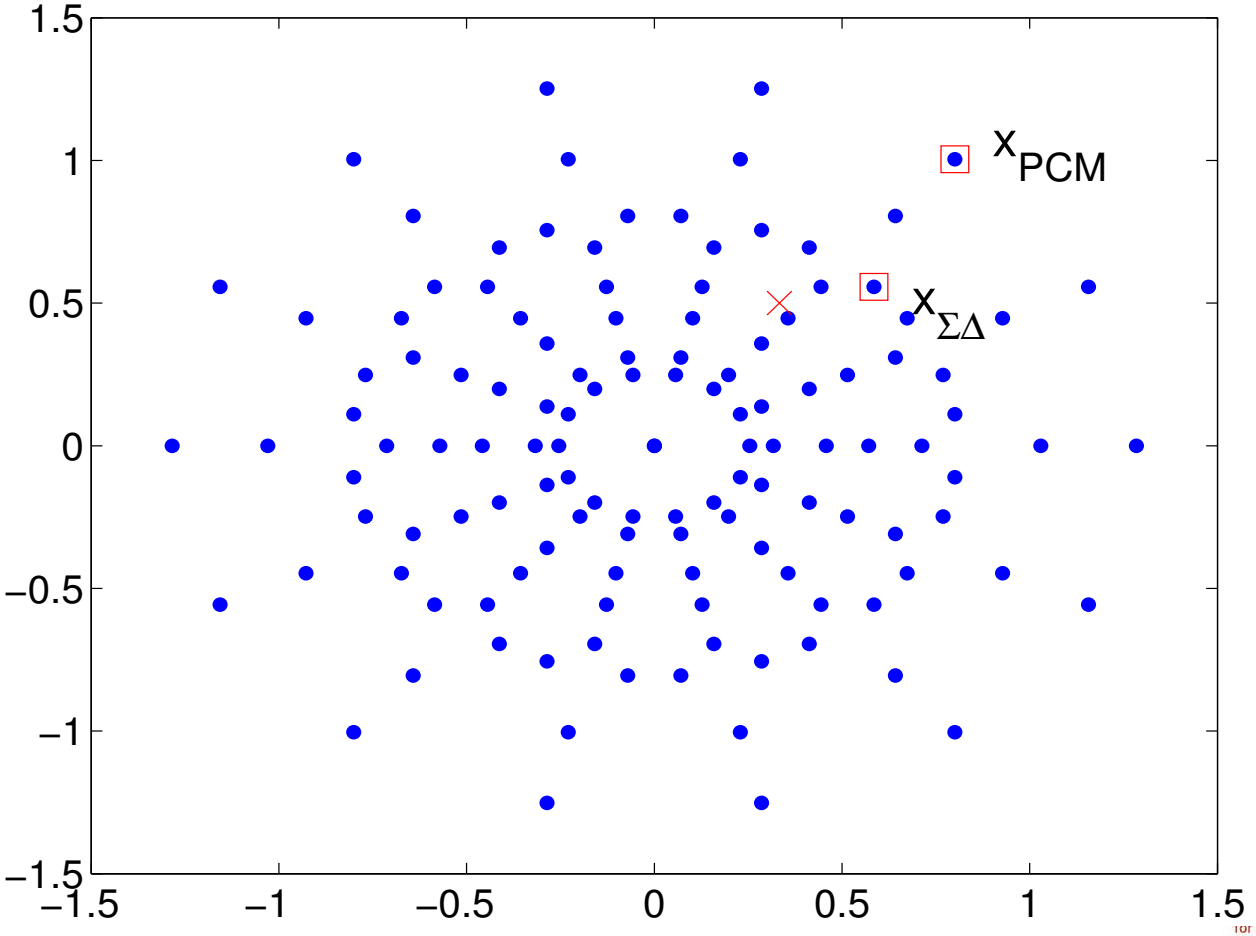

Sigma-Delta quantization – background

- History from 1950s.
- Treatises of Candy, Temes (1992) and Norsworthy, Schreier, Temes (1997).
- PCM for finite frames and $\Sigma\Delta$ for PW_Ω:
  Bölcskei, Daubechies, DeVore, Goyal, Güntürk, Kovačevič, Thao, Vetterli.
- Combination of $\Sigma\Delta$ and finite frames:
  Powell, Yılmaz, and B.
- Subsequent work based on this $\Sigma\Delta$ finite frame theory:
  Bodman and Paulsen; Boufounos and Oppenheim; Jimenez and Yang Wang; Lammers, Powell, and Yılmaz.
- Genuinely apply it.

$\Sigma\Delta$ quantizers for finite frames

Let $F = \{e_n\}_{n=1}^N$ be a frame for $\mathbb{R}^d$, $x \in \mathbb{R}^d$.

Define $x_n = \langle x, e_n \rangle$.

Fix the ordering p, a permutation of $\{1, 2, \ldots, N\}$.

Quantizer alphabet $\mathcal{A}_K^\delta$

Quantizer function $Q(u) = \arg\{\min |u - q| : q \in \mathcal{A}_K^\delta\}$

Define the *first-order* $\Sigma\Delta$ *quantizer* with ordering p and with the quantizer alphabet $\mathcal{A}_K^\delta$ by means of the following recursion.

$$\begin{aligned} u_n - u_{n-1} &= x_{p(n)} - q_n \\ q_n &= Q(u_{n-1} + x_{p(n)}) \end{aligned}$$

where $u_0 = 0$ and $n = 1, 2, \ldots, N$.

Stability

The following stability result is used to prove error estimates.

Proposition

If the frame coefficients $\{x_n\}_{n=1}^N$ satisfy

$$|x_n| \leq (K - 1/2)\delta, \quad n = 1, \cdots, N,$$

then the state sequence $\{u_n\}_{n=0}^N$ generated by the first-order $\Sigma\Delta$ quantizer with alphabet $\mathcal{A}_K^\delta$ satisfies $|u_n| \leq \delta/2, n = 1, \cdots, N.$

- The first-order $\Sigma\Delta$ scheme is equivalent to

$$u_n = \sum_{j=1}^{n} x_{p(j)} - \sum_{j=1}^{n} q_j, \quad n = 1, \cdots, N.$$

- Stability results lead to tiling problems for higher order schemes.

Error estimate

Definition

Let $F = \{e_n\}_{n=1}^N$ be a frame for $\mathbb{R}^d$, and let p be a permutation of $\{1, 2, \ldots, N\}$. The *variation* $\sigma(F, p)$ is

$$\sigma(F, p) = \sum_{n=1}^{N-1} \|e_{p(n)} - e_{p(n+1)}\|.$$

Error estimate

Theorem

Let $F = \{e_n\}_{n=1}^N$ be an A-FUNTF for $\mathbb{R}^d$. The approximation

$$\tilde{x} = \frac{d}{N} \sum_{n=1}^{N} q_n e_{p(n)}$$

generated by the first-order $\Sigma\Delta$ quantizer with ordering p and with the quantizer alphabet $\mathcal{A}_K^\delta$ satisfies

$$\|x - \tilde{x}\| \leq \frac{(\sigma(F,p)+1)d}{N} \frac{\delta}{2}.$$

Harmonic frames

Zimmermann and Goyal, Kelner, Kovačevic, Thao, Vetterli.

Definition

$H = \mathbb{C}^d$. An *harmonic frame* $\{e_n\}_{n=1}^N$ for H is defined by the rows of the Bessel map L which is the complex N-DFT $N \times d$ matrix with $N - d$ columns removed.

$H = \mathbb{R}^d$, d even. The harmonic frame $\{e_n\}_{n=1}^N$ is defined by the Bessel map L which is the $N \times d$ matrix whose nth row is

$$e_n^N = \sqrt{\frac{2}{d}}\left(\cos(\frac{2\pi n}{N}), \sin(\frac{2\pi n}{N}), \ldots, \cos(\frac{2\pi(d/2)n}{N}), \sin(\frac{2\pi(d/2)n}{N})\right).$$

- Harmonic frames are FUNTFs.
- Let E_N be the harmonic frame for $\mathbb{R}^d$ and let p_N be the identity permutation. Then

$$\forall N,\ \sigma(E_N, p_N) \leq \pi d(d+1).$$

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Error estimate for harmonic frames

Theorem

Let E_N be the harmonic frame for $\mathbb{R}^d$ with frame bound N/d. Consider $x \in \mathbb{R}^d$, $\|x\| \leq 1$, and suppose the approximation $\tilde{x}$ of x is generated by a first-order $\Sigma\Delta$ quantizer as before. Then

$$\|x - \tilde{x}\| \leq \frac{d^2(d+1)+d}{N} \frac{\delta}{2}.$$

- Hence, for harmonic frames (and all those with bounded variation),

$$\mathrm{MSE}_{\Sigma\Delta} \leq \frac{C_d}{N^2} \delta^2.$$

- This bound is clearly superior asymptotically to

$$\mathrm{MSE}_{\mathrm{PCM}} = \frac{(d\delta)^2}{12N}.$$

$\Sigma\Delta$ and "optimal" PCM

Theorem

The first order $\Sigma\Delta$ scheme achieves the asymptotically optimal $\mathrm{MSE}_{\mathrm{PCM}}$ for harmonic frames.

The digital encoding

$$\mathrm{MSE}_{\mathrm{PCM}} = \frac{(d\delta)^2}{12N}$$

in PCM format leaves open the possibility that decoding (consistent nonlinear reconstruction, with additional numerical complexity this entails) could lead to

$$\text{“}\mathrm{MSE}_{\mathrm{PCM}}^{\mathrm{opt}}\text{”} \ll O(\frac{1}{N}).$$

Goyal, Vetterli, Thao (1998) proved

$$\text{“}\mathrm{MSE}_{\mathrm{PCM}}^{\mathrm{opt}}\text{”} \sim \frac{\tilde{C}_d}{N^2}\delta^2.$$

A comparison of Σ-Δ and PCM

Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Let $x \in \mathbb{C}^d$, $||x|| \leq 1$.

Definition

- $q_{PCM}(x)$ is the sequence to which x is mapped by PCM.
- $q_{\Sigma\Delta}(x)$ is the sequence to which x is mapped by $\Sigma\Delta$.
- $$\mathtt{err}_{PCM}(x) = ||x - \frac{d}{N} L^* q_{PCM}(x)||$$
  $$\mathtt{err}_{\Sigma\Delta}(x) = ||x - \frac{d}{N} L^* q_{\Sigma\Delta}(x)||$$

Fickus question: We shall analyze to what extent $\mathtt{err}_{\Sigma\Delta}(x) < \mathtt{err}_{PCM}(x)$ beyond our results with Powell and Yilmaz.

PCM and first order Sigma-Delta

Let $x \in \mathbb{C}^d$,
Let $F = \{e_n\}_{n=1}^N$ be a FUNTF for $\mathbb{C}^d$ with the analysis matrix L.

Definition

- $q_{PCM}(x, F, b)$ is the quantized sequence given by b-bit PCM,
- $q_{\Sigma\Delta}(x, F, b)$ is the quantized sequence given by b-bit Sigma-Delta.

$$\mathtt{err}_{PCM}(x, F, b) = \|x - \frac{d}{N} L^* q_{PCM}(x)\|,$$

$$\mathtt{err}_{\Sigma\Delta}(x, F, b) = \|x - \frac{d}{N} L^* q_{\Sigma\Delta}(x)\|.$$

Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Definition

A function $e : [a, b] \to \mathbb{C}^d$ is of *bounded variation (BV)* if there is a $K > 0$ such that for every $a \leq t_1 < t_2 < \cdots < t_N \leq b$,

$$\sum_{n=1}^{N-1} \|e(t_n) - e(t_{n+1})\| \leq K.$$

The smallest such K is denoted by $|e|_{BV}$, and defines a seminorm for the space of BV functions.

Comparison of 1-bit PCM and 1-bit Sigma-Delta

Theorem 1

Let $x \in \mathbb{C}^d$ satisfy $0 < \|x\| \leq 1$, and let $F = \{e_n\}_{n=1}^N$ be a FUNTF for $\mathbb{C}^d$. Then, the 1-bit PCM error satisfies

$$\mathrm{err}_{PCM}(x, F, 1) \geq \alpha_F + 1 - \|x\|$$

where

$$\alpha_F := \inf_{\|x\|=1} \frac{d}{N} \sum_{n=1}^{N} (|Re(\langle x, e_n \rangle)| + |Im(\langle x, e_n \rangle)|) - 1 \geq 0.$$

Comparison of 1-bit PCM and 1-bit Sigma-Delta

Theorem 2

Let $\{F_N = \{e_n^N\}_{n=1}^N\}$ be a family of FUNTFs for $\mathbb{C}^d$. Then,

$$\forall \varepsilon > 0,\ \exists N_0 > 0,\ \ \textit{such that}\ \ \forall N \geq N_0\ \ \textit{and}\ \ \forall 0 < \|x\| \leq 1 - \varepsilon$$

$$\mathtt{err}_{\Sigma\Delta}(x, F_N, 1) \leq \mathtt{err}_{PCM}(x, F_N, 1).$$

Numerical experiments suggest that, we can choose N significantly smaller than $(M/\varepsilon)^{2d}$.

Comparison of 1-bit PCM and 1-bit Sigma-Delta

If $\{\alpha_{F_N}\}$ is bounded below by a positive number, then we can improve Theorem 2:

Theorem 3

Let $\{F_N = \{e_n^N\}_{n=1}^N\}$ be a family of FUNTFs for $\mathbb{C}^d$ such that

$$\exists a > 0,\ \forall N,\ \alpha_{F_N} \geq a.$$

Then,

$$\exists N_0 > 0 \ \textit{such that} \ \forall N \geq N_0 \ \textit{and} \ \forall 0 < \|x\| \leq 1$$

$$\mathtt{err}_{\Sigma\Delta}(x, F_N, 1) \leq \mathtt{err}_{PCM}(x, F_N, 1).$$

Comparison of 1-bit PCM and 1-bit Sigma-Delta

Below is a family $\{F_N\}$ of FUNTFs where $\{\alpha_{F_N}\}$ is bounded below by a positive constant. Harmonic frames are examples to such families.

Theorem 4

Let $e : [0,1] \to \{x \in \mathbb{C}^d : \|x\| = 1\}$ be continuous function of bounded variation such that $F_N = \{e(n/N)\}_{n=1}^N$ is a FUNTF for $\mathbb{C}^d$ for every N. Then,

$$\exists N_0 > 0 \textit{ such that } \forall N \geq N_0 \textit{ and } \forall 0 < \|x\| \leq 1$$

$$\mathtt{err}_{\Sigma\Delta}(x, F_N, 1) \leq \mathtt{err}_{PCM}(x, F_N, 1).$$

One can show that $\alpha := \lim_{N\to\infty} \alpha_{F_N}$ is positive, and that

$$\alpha + 1 = d \inf_{\|x\|=1} \int_0^1 (|Re(\langle x, e(t)\rangle)| + |Im(\langle x, e(t)\rangle)|)\, dt.$$

Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Theorem

Let $e : [0,1] \to \{x \in \mathbb{C}^d : \|x\| = 1\}$ be continuous function of bounded variation such that $F_N = (e(n/N))_{n=1}^N$ is a FUNTF for $\mathbb{C}^d$ for every N. Then,

$$\exists N_0 > 0 \textit{ such that } \forall N \geq N_0 \textit{ and } \forall 0 < \|x\| \leq 1$$

$$\mathtt{err}_{\Sigma\Delta}(x) \leq \mathtt{err}_{PCM}(x).$$

Moreover, a lower bound for N_0 is $d(1 + |e|_{BV})/(\sqrt{d} - 1)$.

Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Example (Roots of unity frames for $\mathbb{R}^2$)

$e_n^N = (\cos(2\pi n/N), \sin(2\pi n/N))$.

Here, $e(t) = (\cos(2\pi t), \sin(2\pi t))$,
$M = |e|_{BV} = 2\pi$, $lim \alpha_{F_N} = 2/\pi$.

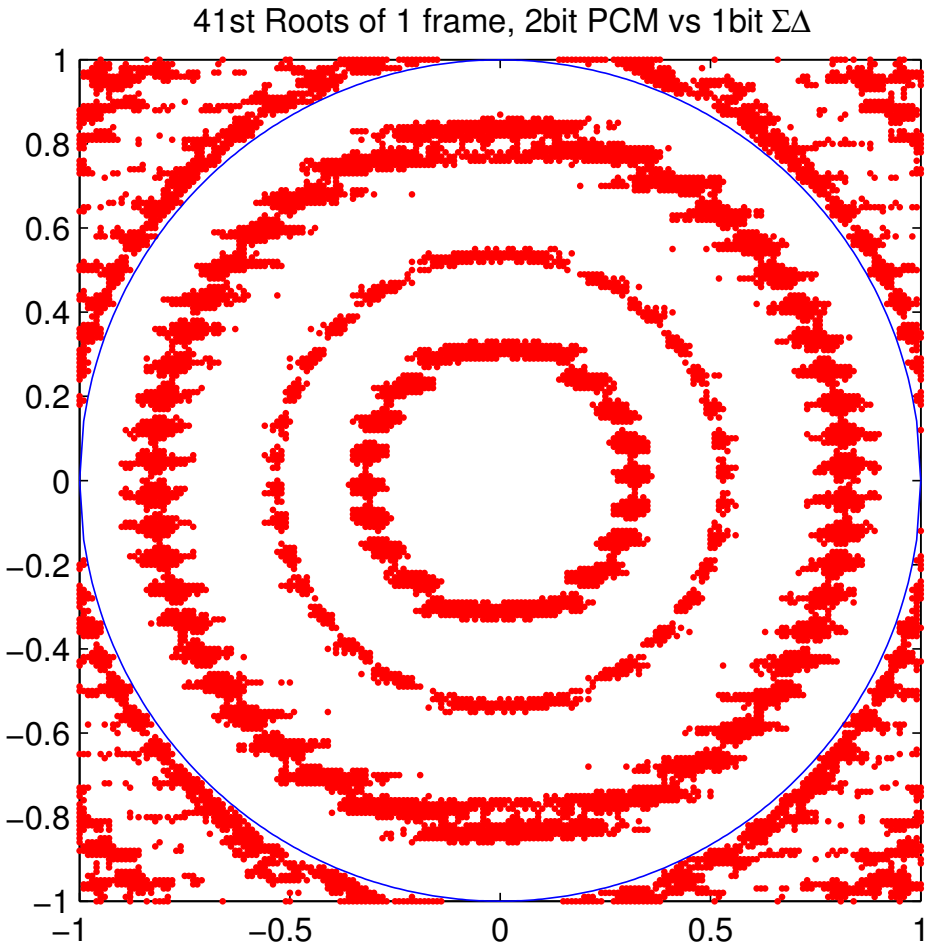
Example (Real Harmonic Frames for $\mathbb{R}^{2k}$)

$e_n^N = \frac{1}{\sqrt{k}}(\cos(2\pi n/N), \sin(2\pi n/N), \ldots, \cos(2\pi kn/N), \sin(2\pi kn/N))$.

In this case, $e(t) = \frac{1}{\sqrt{k}}(\cos(2\pi t), \sin(2\pi t), \ldots, \cos(2\pi kt), \sin(2\pi kt))$,

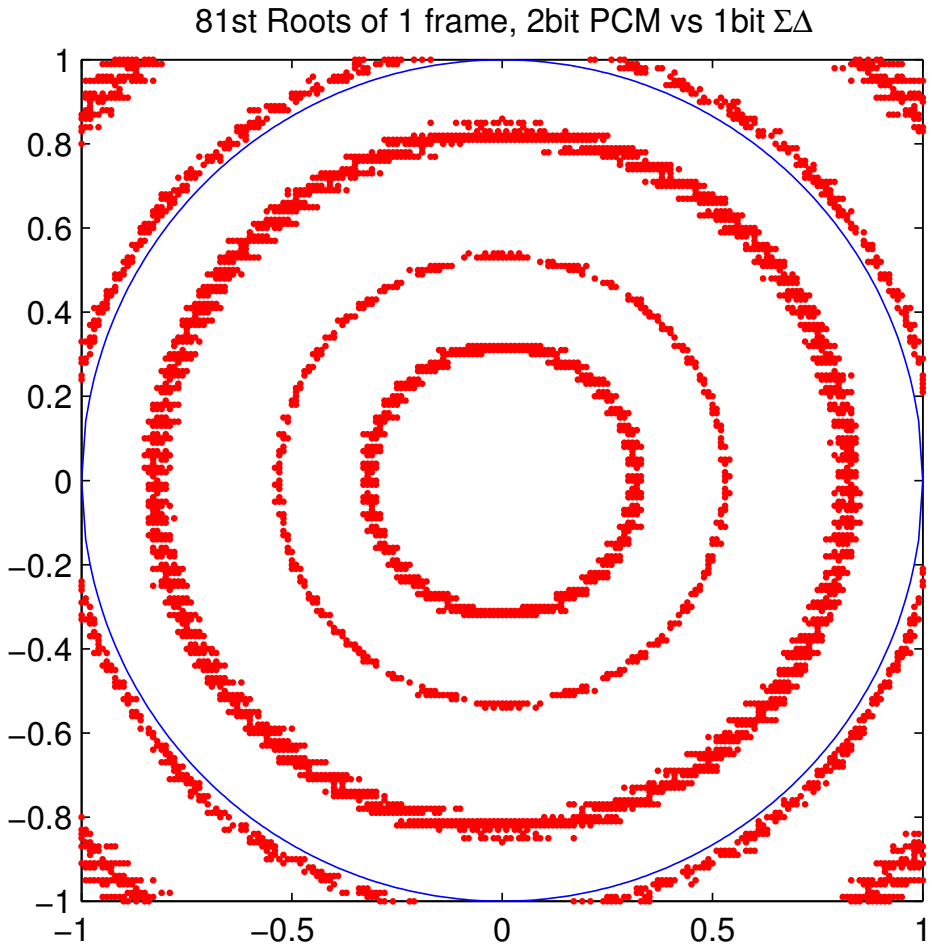
$M = |e|_{BV} = 2\pi\sqrt{\frac{1}{d}\sum_{k=1}^{d} k^2}$.

Comparison of 2-bit PCM and 1-bit $\Sigma\Delta$



Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Comparison of 2-bit PCM and 1-bit $\Sigma\Delta$



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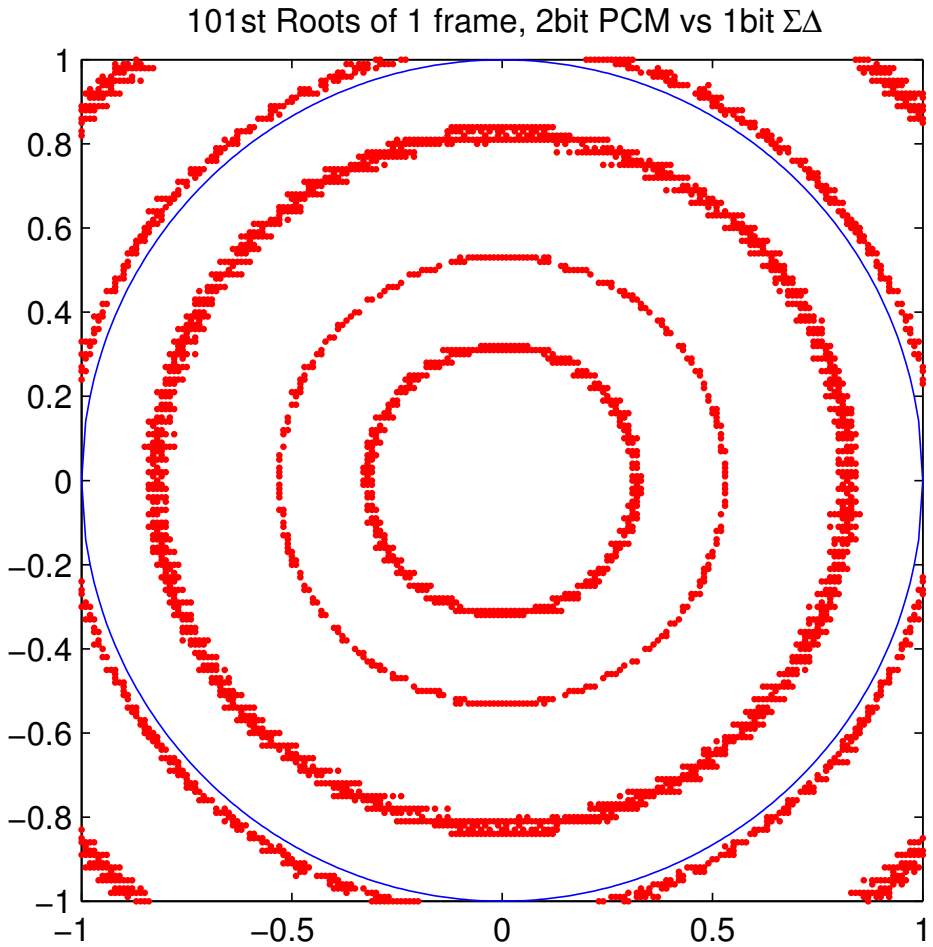
Comparison of 2-bit PCM and 1-bit $\Sigma\Delta$



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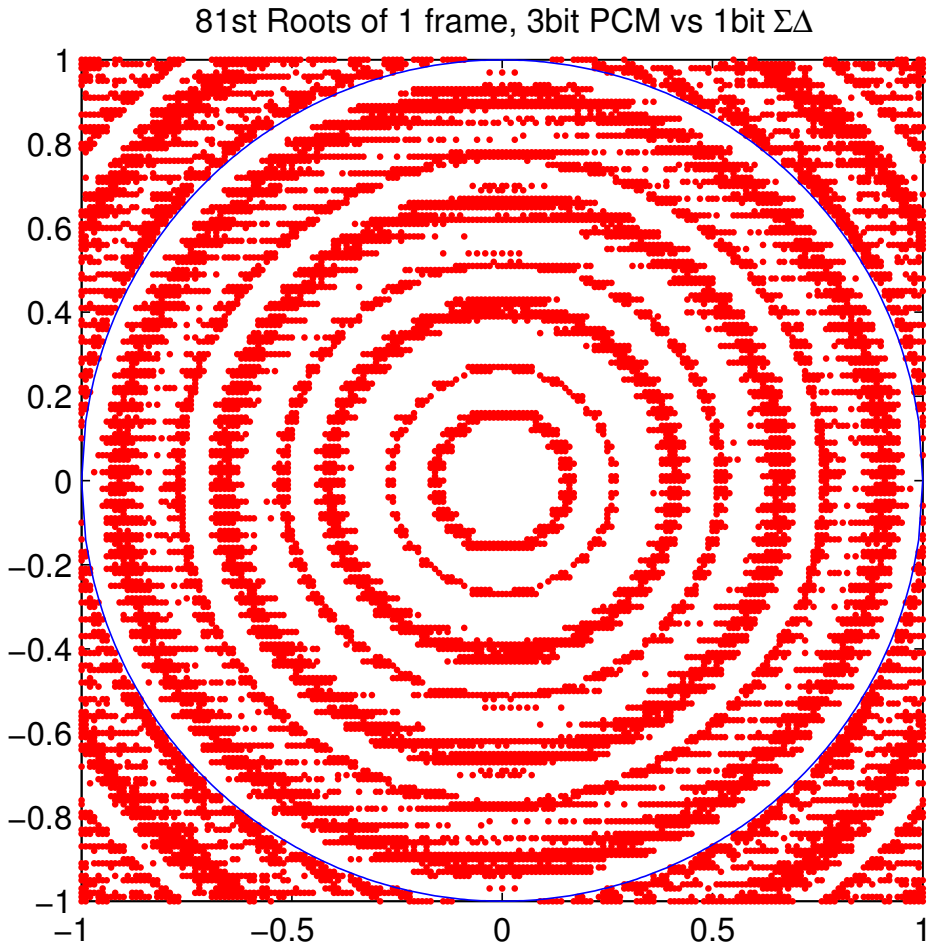
Comparison of 3-bit PCM and 1-bit $\Sigma\Delta$



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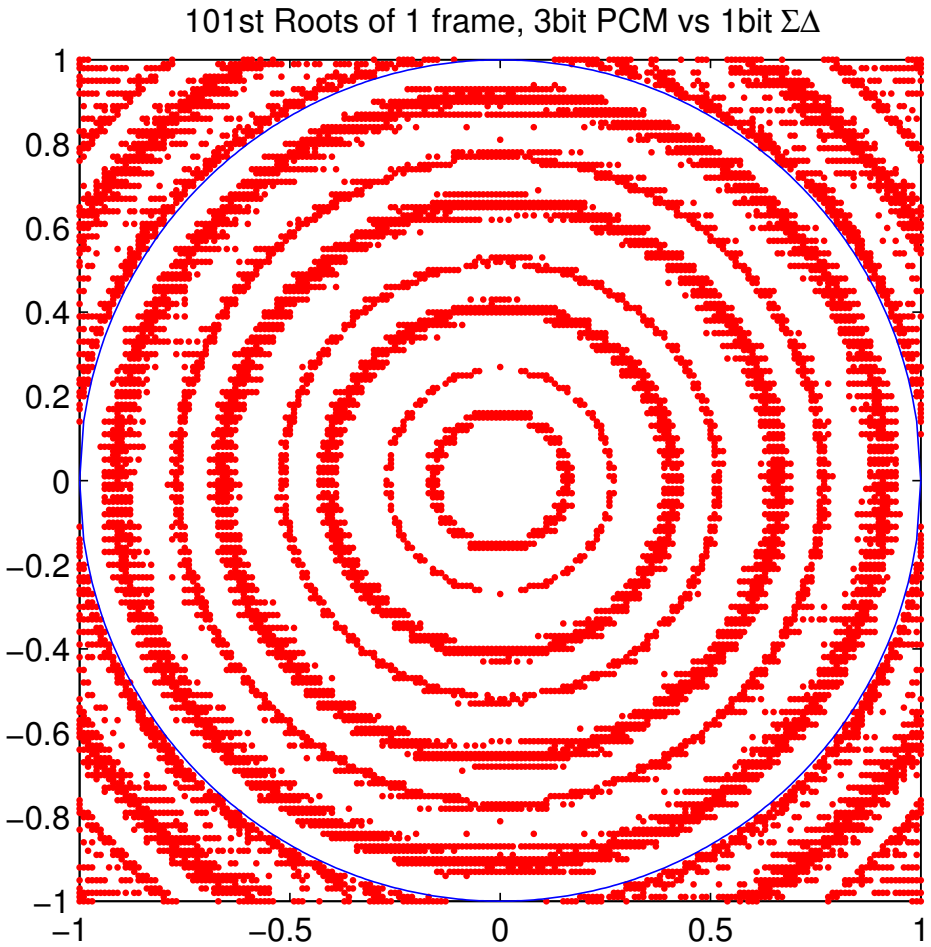
Comparison of 3-bit PCM and 1-bit $\Sigma\Delta$



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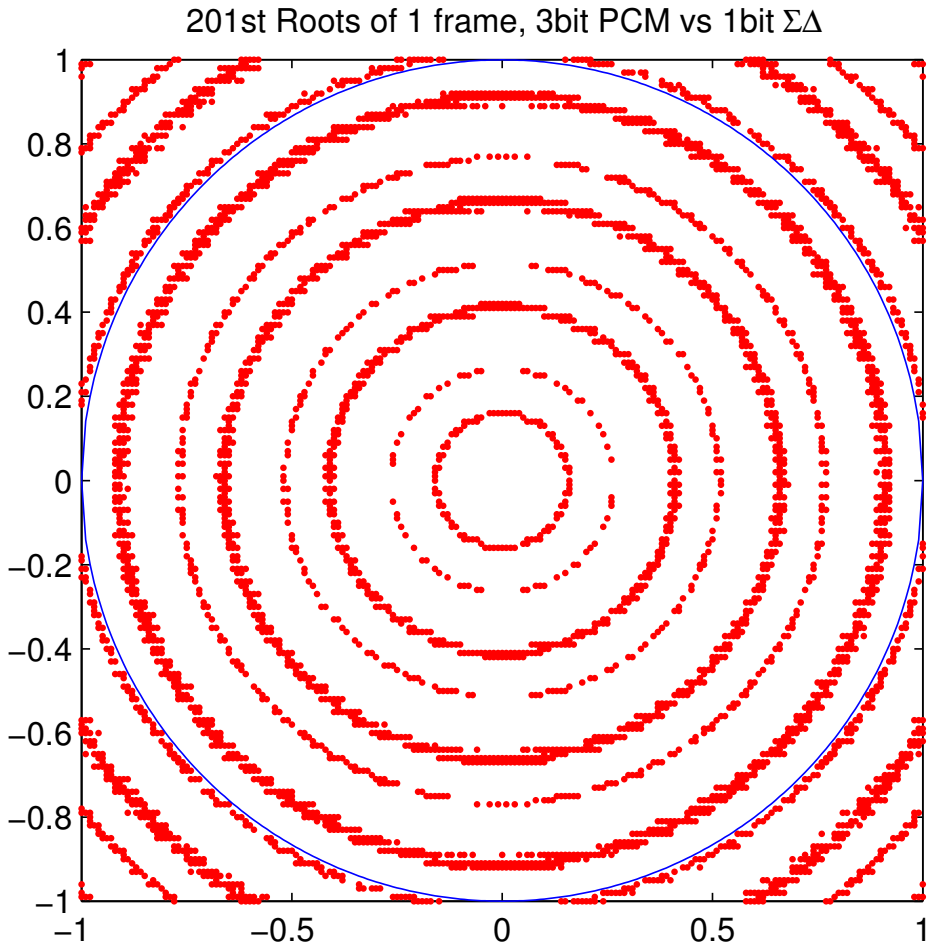
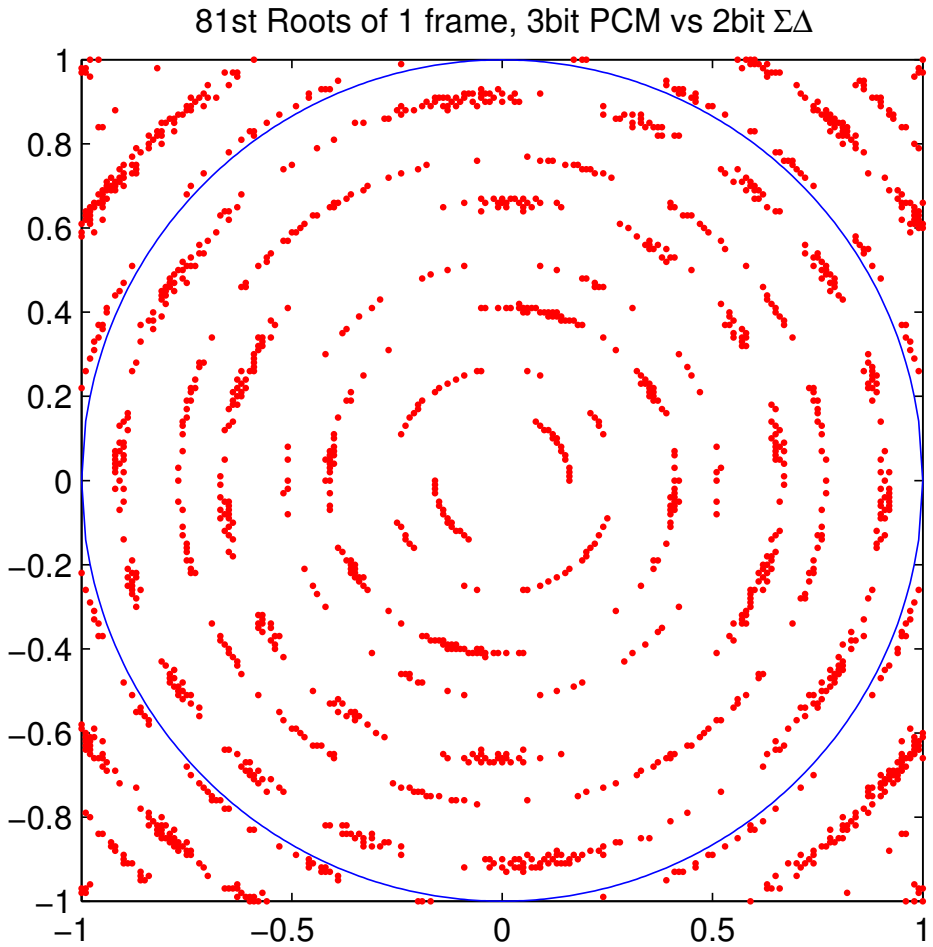
Comparison of 3-bit PCM and 1-bit $\Sigma\Delta$



Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Comparison of 3-bit PCM and 2-bit $\Sigma\Delta$



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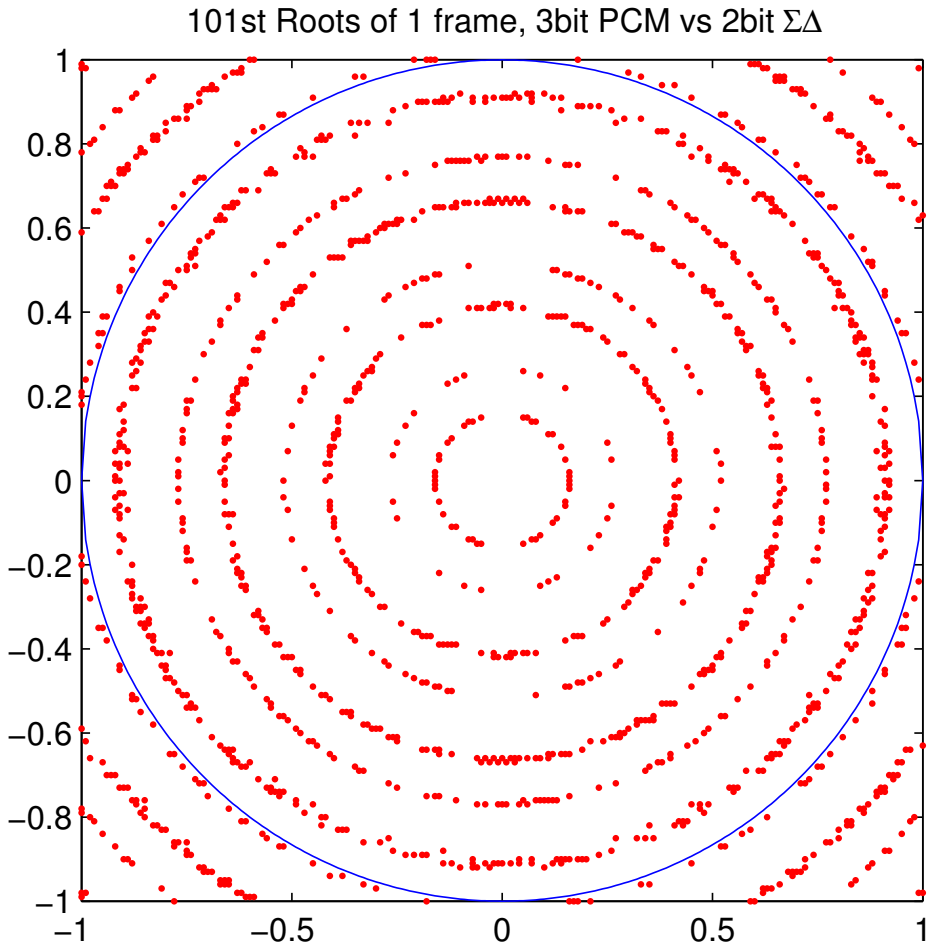
Comparison of 3-bit PCM and 2-bit $\Sigma\Delta$



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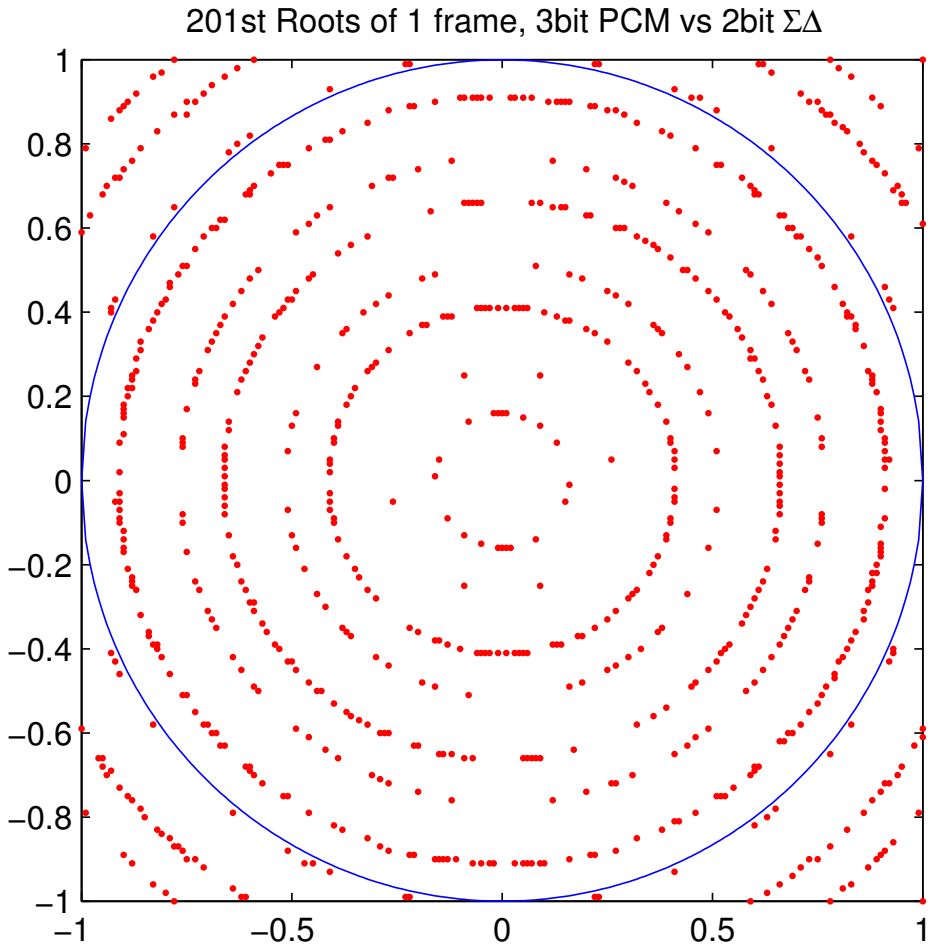
Comparison of 3-bit PCM and 2-bit $\Sigma\Delta$



Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Complex $\Sigma\Delta$ - Alphabet

Let $K \in \mathbb{N}$ and $\delta > 0$. The *midrise* quantization alphabet is

$$\mathcal{A}_K^\delta = \left\{ \left(m + \frac{1}{2} \right) \delta + in\delta : m = -K, \ldots, K-1, \; n = -K, \ldots, K \right\}$$

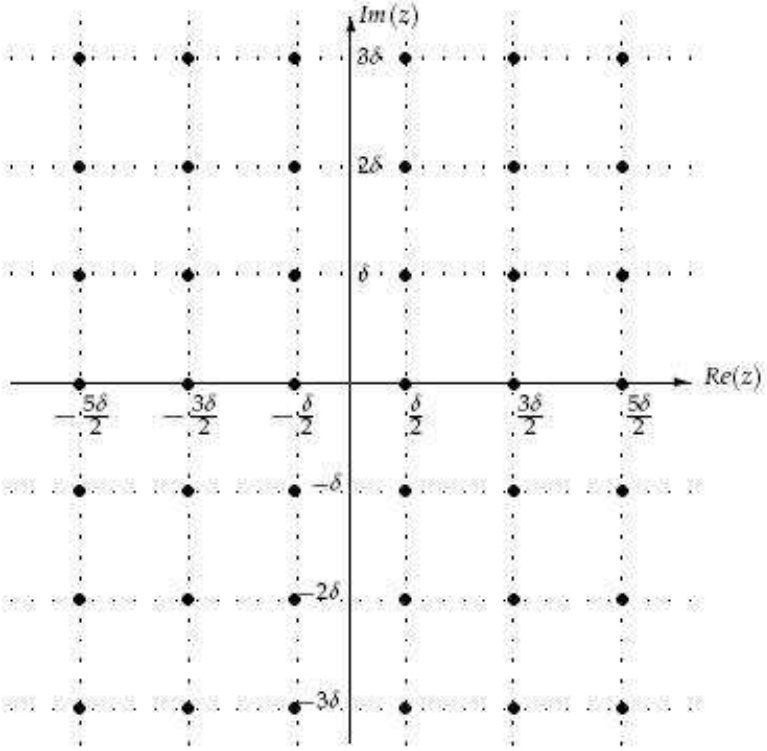


Figure: $\mathcal{A}_K^\delta$ for $K = 3\delta$.

Alphabet

For $K > 0$ (we consider only $K = 1$)
and $b \geq 1$, an integer representing the number of bits,
let $\delta = 2K/(2^b - 1)$.

$$\mathcal{A}_\delta^K = \{(-K + m\delta) + i(-K + n\delta) : m, n = 0, \ldots, 2^b - 1\}.$$

The associated *scalar uniform quantizer* is

$$Q_\delta(u + iv) = \delta\left(\frac{1}{2} + \left\lfloor \frac{u}{\delta} \right\rfloor + i(\frac{1}{2} + \left\lfloor \frac{v}{\delta} \right\rfloor)\right).$$

In particular, for 1-bit case, $Q(u + iv) = sign(u) + isign(v)$

Complex $\Sigma\Delta$

The *scalar uniform quantizer* associated to $\mathcal{A}_K^\delta$ is

$$Q_\delta(a+ib) = \delta\left(\frac{1}{2} + \left\lfloor \frac{a}{\delta} \right\rfloor + i\left\lfloor \frac{b}{\delta} \right\rfloor\right),$$

where $\lfloor x \rfloor$ is the largest integer smaller than x.
For any $z = a + ib$ with $|a| \leq K$ and $|b| \leq K$, Q satisfies

$$|z - Q_\delta(z)| \leq \min_{\zeta \in \mathcal{A}_K^\delta} |z - \zeta|.$$

Let $\{x_n\}_{n=1}^N \subseteq \mathbb{C}$ and let p be a permutation of $\{1, \ldots, N\}$. Analogous to the real case, the first order $\Sigma\Delta$ quantization is defined by the iteration

$$\begin{aligned} u_n &= u_{n-1} + x_{p(n)} - q_n, \\ q_n &= Q_\delta(u_{n-1} + x_{p(n)}). \end{aligned}$$

Complex $\Sigma\Delta$

The following theorem is analogous to BPY

Theorem

Let $F = \{e_n\}_{n=1}^N$ be a finite unit norm frame for $\mathbb{C}^d$, let p be a permutation of $\{1, \ldots, N\}$, let $|u_0| \leq \delta/2$, and let $x \in \mathbb{C}^d$ satisfy $\|x\| \leq (K - 1/2)\delta$. The $\Sigma\Delta$ approximation error $\|x - \widetilde{x}\|$ satisfies

$$\|x - \widetilde{x}\| \leq \sqrt{2}\|S^{-1}\|_{\mathrm{op}} \left(\sigma(F, p)\frac{\delta}{2} + |u_N| + |u_0| \right),$$

where S^{-1} is the inverse frame operator. In particular, if F is a FUNTF, then

$$\|x - \widetilde{x}\| \leq \sqrt{2}\frac{d}{N} \left(\sigma(F, p)\frac{\delta}{2} + |u_N| + |u_0| \right),$$

Complex $\Sigma\Delta$

Let $\{F_N\}$ be a family of FUNTFs, and p_N be a permutation of $\{1, \ldots, N\}$. Then the frame variation $\sigma(F_N, p_N)$ is a function of N. If $\sigma(F_N, p_N)$ is bounded, then

$$\|x - \widetilde{x}\| = \mathcal{O}(N^{-1}) \text{ as } N \to \infty.$$

Wang gives an upper bound for the frame variation of frames for $\mathbb{R}^d$, using the results from the Travelling Salesman Problem.

Theorem YW

Let $S = \{v_j\}_{j=1}^N \subseteq [-\frac{1}{2}, \frac{1}{2}]^d$ with $d \geq 3$. There exists a permutation p of $\{1, \ldots, N\}$ such that

$$\sum_{j=1}^{N-1} \|v_{p(j)} - v_{p(j+1)}\| \leq 2\sqrt{d+3}N^{1-\frac{1}{d}} - 2\sqrt{d+3}.$$

Complex $\Sigma\Delta$

Theorem

Let $F = \{e_n\}_{n=1}^N$ be a FUNTF for $\mathbb{R}^d$, $|u_0| \leq \delta/2$, and let $x \in \mathbb{R}^d$ satisfy $\|x\| \leq (K - 1/2)\delta$. Then, there exists a permutation p of $\{1, 2, \ldots, N\}$ such that the approximation error $\|x - \widetilde{x}\|$ satisfies

$$\|x - \widetilde{x}\| \leq \sqrt{2}\delta d \left((1 - \sqrt{d+3})N^{-1} + \sqrt{d+3}N^{-\frac{1}{d}} \right)$$

This theorem guarantees that

$$\|x - \widetilde{x}\| \leq \mathcal{O}(N^{-\frac{1}{d}}) \text{ as } N \to \infty$$

for FUNTFs for $\mathbb{R}^d$.

Preprocessing for clutter mitigation

- Massive sensor data set → dimension reduction → sparse representation
- False targets caused by clutter inhibit data triage, waste vital resources, and degrade sparse representation algorithms
- View clutter mitigation as preprocessing step for ATR/ATE
- For active sensors, choose waveform to reduce clutter effects by limiting side lobe magnitude
  - improves concise data representation
  - supports dimensionality reduction processing

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Sparse coefficient sets for stable representation

- Opportunistic sensing systems can utilize large networks of diverse sensors
  - sensor quality may vary, e.g., low cost wireless sensors
  - massive amount of noisy sensor data
- Signal representations using sparse coefficient sets
  - compensate for hardware errors
  - ensure numerical stability
  - frame setting → frame dimension reduction

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Frame variation and $\Sigma\Delta$

- $F = \{e_j\}_{j=1}^N$ a FUNTF for $\mathbb{C}^d$
- $x \in \mathbb{C}^d$, p a permutation of $\{1, ..., N\}$, $x_{p(n)} = \langle x, e_{p(n)} \rangle$,

$$x = \frac{d}{N} \sum_{n=1}^{N} x_{p(n)} e_{p(n)} \text{ and } \tilde{x} \equiv \frac{d}{N} \sum_{n=1}^{N} q_n e_{p(n)}$$

- Frame variation,

$$\sigma(F, p) = \sum_{n=1}^{N-1} \left\| e_{p(n)} - e_{p(n+1)} \right\|$$

- Transport $\Sigma\Delta$ FUNTF setting to coefficient sparse representation point of view.

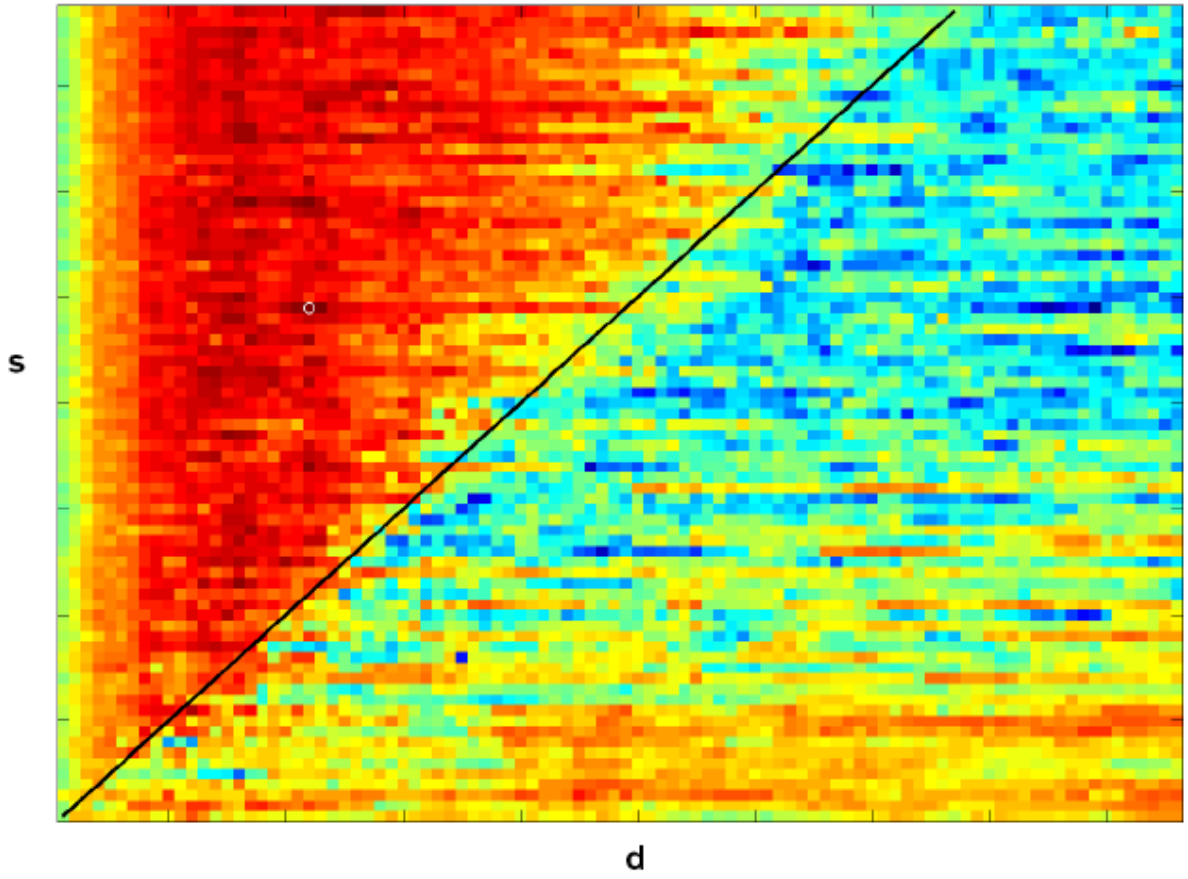
Summary

Given a signal x and a tolerance $r > 0$

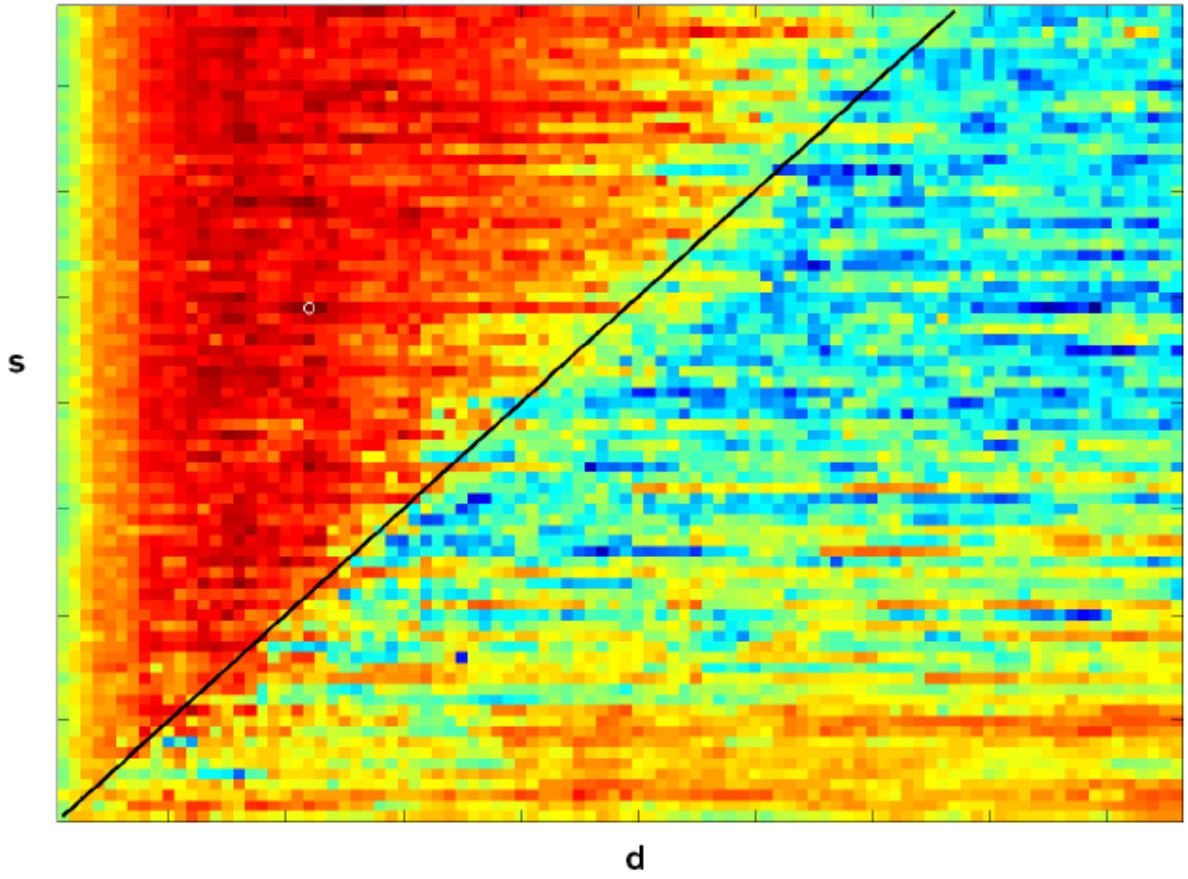
- Define frames using Frame Potential Energy and SQP (or other optimization)
- Analyze Frame Variation in terms of our permutation algorithm
- Compute $\tilde{x}$ having separated coefficients taken from a fixed small and sparse alphabet
- Ensure that $\|x - \tilde{x}\| < r$.

Conclusion: $\tilde{x}$ is a stable sparse coefficient approximant of x

Overview of Classification Results



Overview of Classification Results



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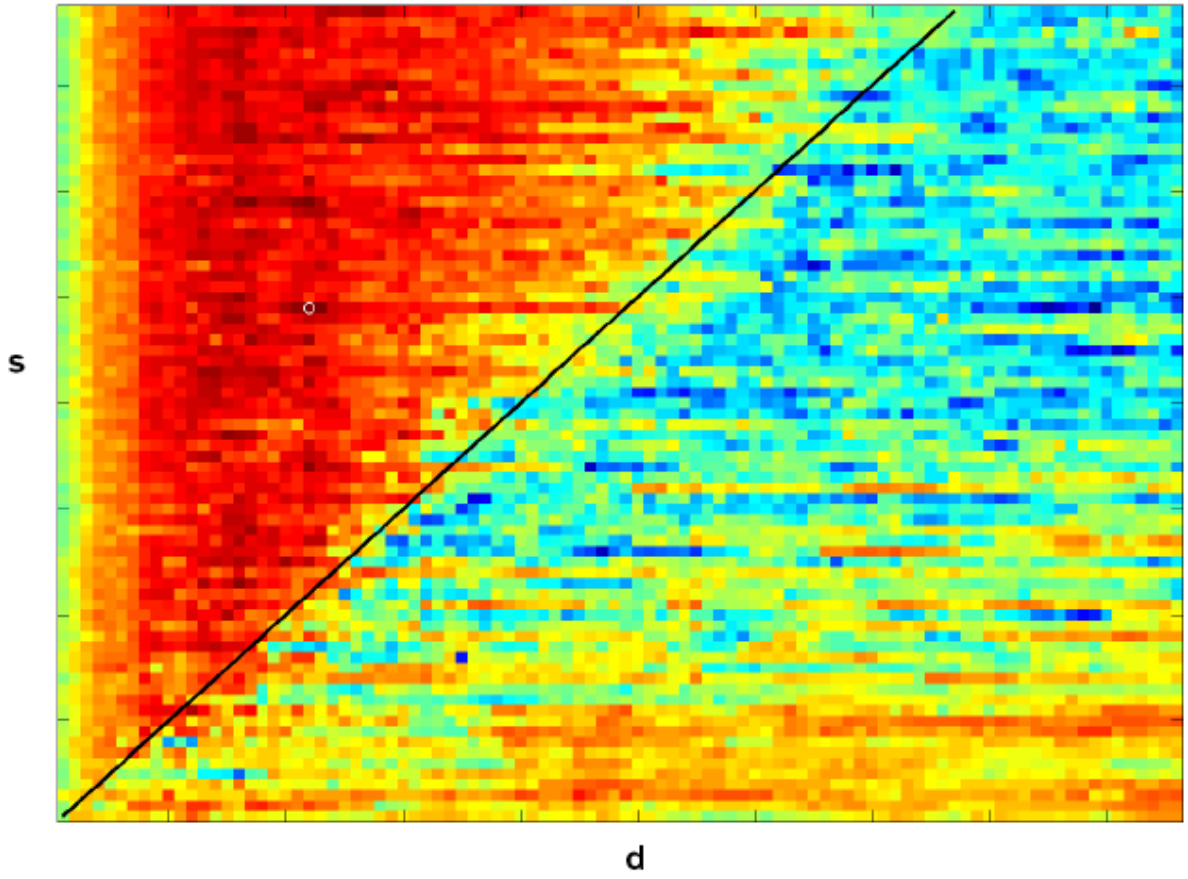

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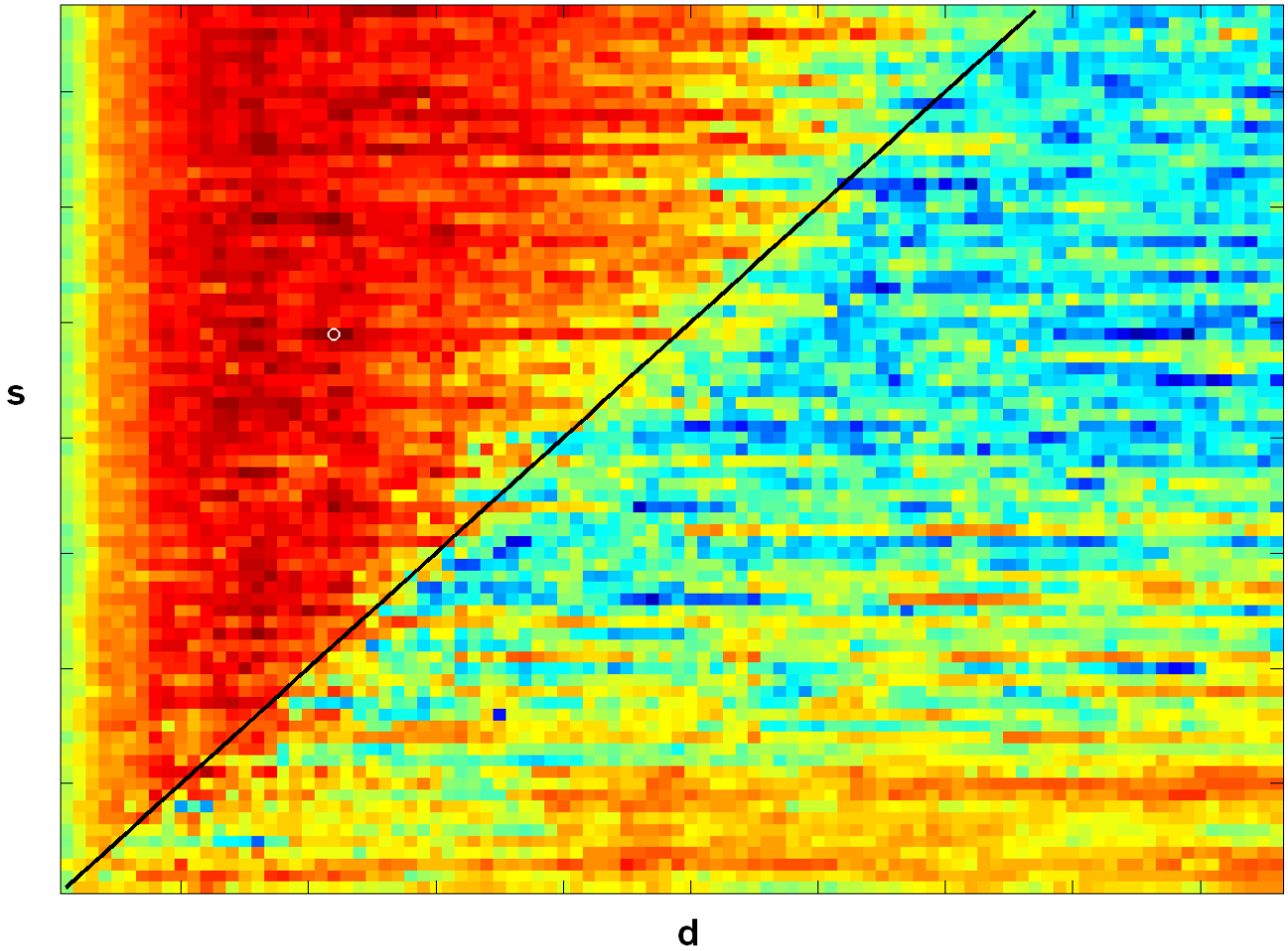

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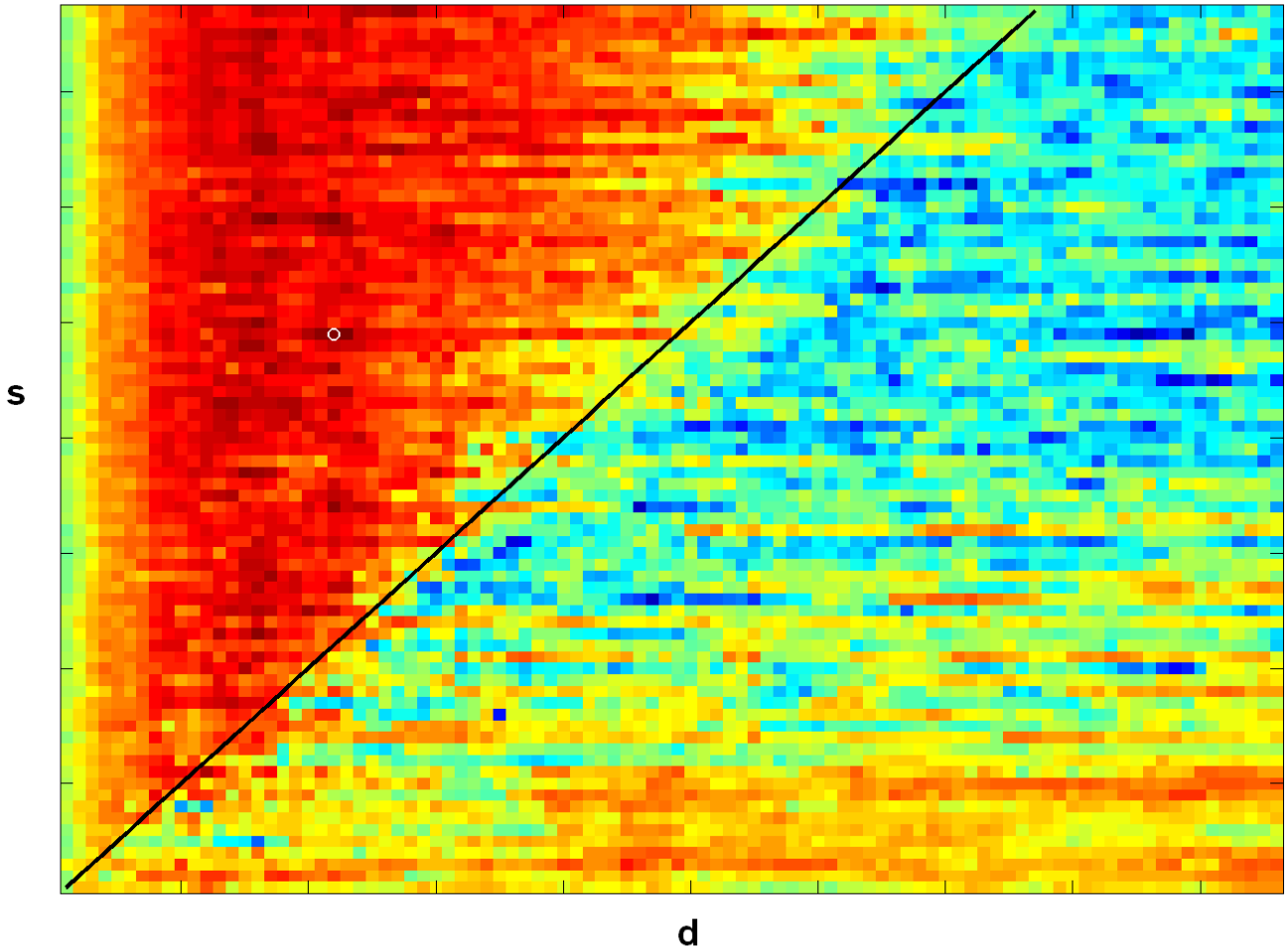

A vector-valued ambiguity function

Outline

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Background

- Originally, our problem was to construct libraries of phase-coded waveforms v parameterized by design variables, for communications and radar.
- A goal was to achieve diverse ambiguity function behavior of v by defining new classes of quadratic phase and number theoretic perfect autocorrelation codes u with which to define v.
- A realistic more general problem was to construct vector-valued waveforms v in terms of vector-valued perfect autocorrelation codes u. Such codes are relevant in light of vector sensor and MIMO capabilities and modeling.
- Example: Discrete time data vector $u(k)$ for a d-element array,

$$k \longmapsto u(k) = (u_0(k), \ldots, u_{d-1}(k)) \in \mathbb{C}^d.$$

We can have $\mathbb{R}^N \to GL(d, \mathbb{C})$, or even more general.

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General problem and STFT theme

- Establish the theory of vector-valued ambiguity functions to estimate v in terms of ambiguity data.
- First, establish this estimation theory by defining the discrete periodic vector-valued ambiguity function in a natural way.
- Mathematically, this natural way is to formulate the discrete periodic vector-valued ambiguity function in terms of the Short Time Fourier Transform (STFT).

STFT and ambiguity function

Short time Fourier transform – STFT

- The narrow band cross-correlation ambiguity function of v, w defined on $\mathbb{R}$ is

$$A(v,w)(t,\gamma) = \int_{\mathbb{R}} v(s+t)\overline{w(s)}e^{-2\pi i s\gamma}ds.$$

- $A(v,w)$ is the STFT of v with window w.
- The *narrow band radar ambiguity function* $A(v)$ of v on $\mathbb{R}$ is

$$A(v)(t,\gamma) = \int_{\mathbb{R}} v(s+t)\overline{v(s)}e^{-2\pi i s\gamma}ds$$

$$= e^{\pi i t\gamma}\int_{\mathbb{R}} v\left(s+\frac{t}{2}\right)\overline{v\left(s-\frac{t}{2}\right)}e^{-2\pi i s\gamma}ds, \text{ for } (t,\gamma)\in\mathbb{R}^2.$$

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Goal

- Let v be a phase coded waveform with N lags defined by the code u.
- Let u be N-periodic, and so $u : \mathbb{Z}_N \longrightarrow \mathbb{C}$, where $\mathbb{Z}_N$ is the additive group of integers modulo N.
- The *discrete periodic ambiguity function* $A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}$ is

$$A_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} u(m+k)\overline{u(k)}e^{-2\pi ikn/N}.$$

Goal

Given a vector valued N-periodic code $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, construct the following in a meaningful, computable way:

- Generalized $\mathbb{C}$-valued periodic ambiguity function $A_p^1(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}$
- $\mathbb{C}^d$-valued periodic ambiguity function $A_p^d(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}^d$

The STFT is the *guide* and the *theory of frames* is the technology to obtain the goal.

Outline

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Multiplication problem

- Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$.
- If $d = 1$ and $e_n = e^{2\pi i n/N}$, then

$$A_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k)e_{nk}\rangle.$$

Multiplication problem

To characterize sequences $\{E_k\} \subseteq \mathbb{C}^d$ and multiplications $*$ so that

$$A_p^1(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * E_{nk}\rangle \in \mathbb{C}$$

is a meaningful and well-defined *ambiguity function.* This formula is clearly motivated by the STFT.

Ambiguity function assumptions

There is a natural way to address the multiplication problem motivated by the fact that $e_m e_n = e_{m+n}$. To this end, we shall make the *ambiguity function assumptions*:

- $\exists \{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ and a multiplication $*$ such that $E_m * E_n = E_{m+n}$ for $m, n \in \mathbb{Z}_N$;
- $\{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ is a tight frame for $\mathbb{C}^d$;
- $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$ is bilinear, in particular,

$$\left(\sum_{j=0}^{N-1} c_j E_j\right) * \left(\sum_{k=0}^{N-1} d_k E_k\right) = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} c_j d_k E_j * E_k.$$

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Calculation

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Given $u, v : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ and $m, n \in \mathbb{Z}_N$.
- Then, one calculates

$$u(m) * v(n) \quad = \quad \frac{d^2}{N^2} \sum_{j=0}^{N-1} \sum_{s=0}^{N-1} \langle u(m), E_j \rangle \langle v(n), E_s \rangle E_{j+s}.$$

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Outline

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$A_p^1(u)$ for DFT frames

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Further, assume that $\{E_j\}_{j=0}^{N-1}$ is a DFT frame, and let r designate a fixed column.
- Without loss of generality, choose the first d columns of the $N \times N$ DFT matrix.
- Then, one calculates

$$\begin{aligned} E_m * E_n(r) &= \frac{d^2}{N^2} \sum_{j=0}^{N-1} \sum_{s=0}^{N-1} \langle E_m, E_j \rangle \langle E_n, E_s \rangle E_{j+s}(r). \\ &= \frac{e_{(m+n)r}}{\sqrt{d}} = E_{m+n}(r). \end{aligned}$$

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$A_p^1(u)$ for DFT frames

- Thus, for DFT frames, $*$ is componentwise multiplication in $\mathbb{C}^d$ with a factor of $\sqrt{d}$.
- In this case $A_p^1(u)$ is well-defined for $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ by

$$\begin{aligned} A_p^1(u)(m,n) &= \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * E_{nk}\rangle \\ &= \frac{d}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\langle E_j, u(k)\rangle\langle u(m+k), E_{j+nk}\rangle. \end{aligned}$$

Remark

- In the previous DFT example, $*$ is intrinsically related to the "addition" defined on the indices of the frame elements, viz., $E_m * E_n = E_{m+n}$.
- Alternatively, we could have $E_m * E_n = E_{m \bullet n}$ for some function $\bullet : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{Z}_N$, and, thereby, we could use frames which are not FUNTFs.
- Given a bilinear multiplication $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$, we can find a frame $\{E_j\}_j$ and an index operation $\bullet$ with the $E_m * E_n = E_{m \bullet n}$ property.
- If $\bullet$ is the multiplication for a group, possibly non-abelian and/or infinite, we may reverse the process and find a FUNTF and bilinear multiplication $*$ with the $E_m * E_n = E_{m \bullet n}$ property.

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$A_p^1(u)$ for cross product frames

- Take $* : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$ and let $\{i, j, k\}$ be the standard basis.
- $i * j = k$, $j * i = -k$, $k * i = j$, $i * k = -j$, $j * k = i$, $k * j = -i$, $i * i = j * j = k * k = 0$. $\{0, i, j, k, -i, -j, -k, \}$ is a tight frame for $\mathbb{C}^3$ with frame constant 2. Let $E_0 = 0, E_1 = i, E_2 = j, E_3 = k, E_4 = -i, E_5 = -j, E_6 = -k$.
- The index operation corresponding to the frame multiplication is the non-abelian operation $\bullet : \mathbb{Z}_7 \times \mathbb{Z}_7 \longrightarrow \mathbb{Z}_7$, where $1 \bullet 2 = 3, 2 \bullet 1 = 6, 3 \bullet 1 = 2, 1 \bullet 3 = 5, 2 \bullet 3 = 1, 3 \bullet 2 = 4, 1 \bullet 1 = 2 \bullet 2 = 3 \bullet 3 = 0, n \bullet 0 = 0 \bullet n = 0, 1 \bullet 4 = 0, 1 \bullet 5 = 6, 1 \bullet 6 = 2, 4 \bullet 1 = 0, 5 \bullet 1 = 3, 6 \bullet 1 = 5, 2 \bullet 4 = 3, 2 \bullet 5 = 0$, etc.
- The three ambiguity function assumptions are valid and so we can write the cross product as

$$u \times v = u * v = \frac{1}{2^2} \sum_{s=1}^{6} \sum_{t=1}^{6} \langle u, E_s \rangle \langle v, E_t \rangle E_{s \bullet t}.$$

- Consequently, $A_p^1(u)$ can be well-defined.

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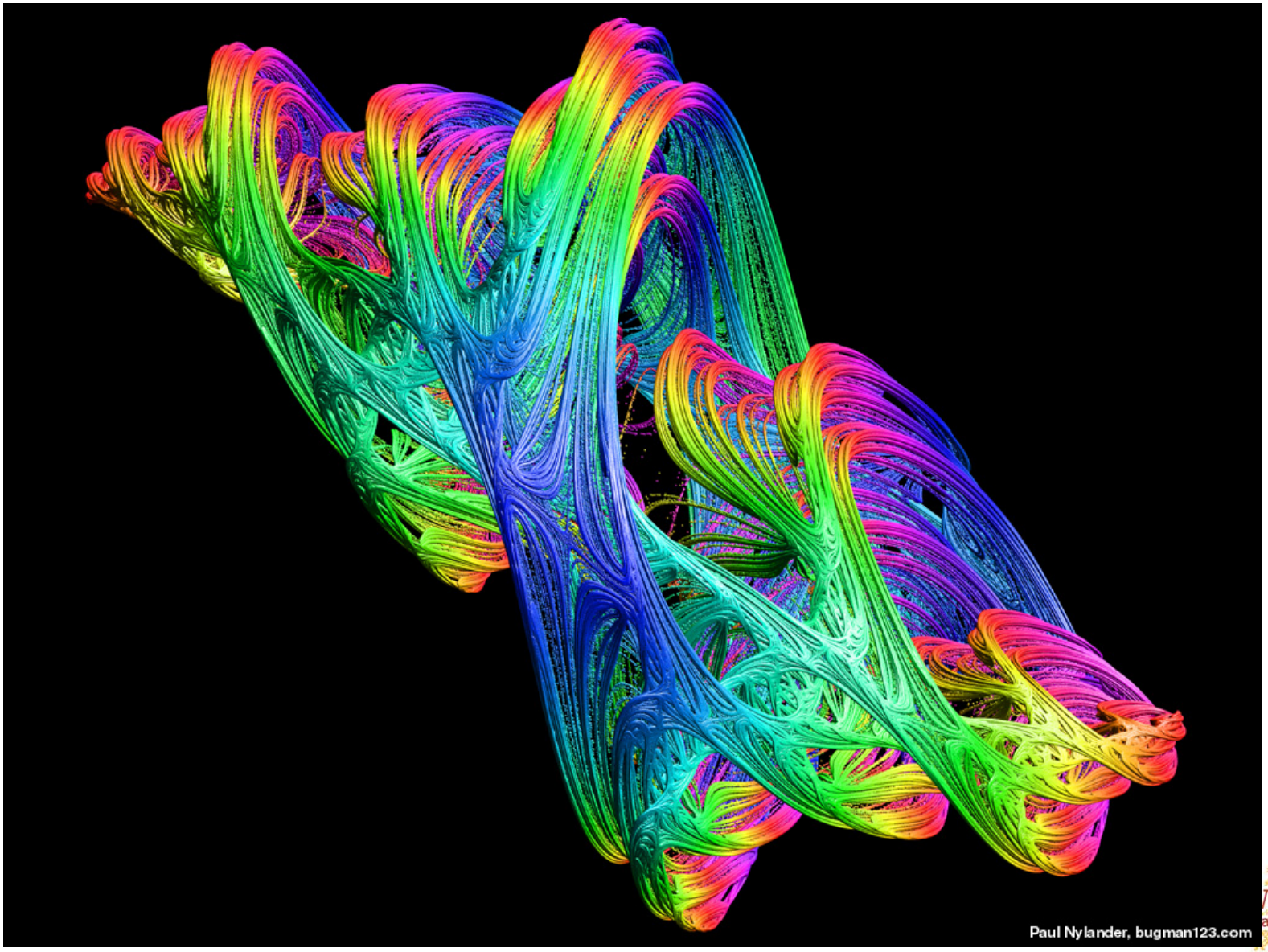
Inverse 4D Quaternion Julia Set



Vector-valued ambiguity function $A_p^d(u)$

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$.
- The following definition is clearly *motivated* by the STFT.

Definition

$A_p^d(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ is defined by

$$A_p^d(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m+k) * \overline{u(k)} * \overline{E_{nk}}.$$

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Outline

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STFT formulation of $A_p(u)$

- The discrete periodic ambiguity function of $u : \mathbb{Z}_N \longrightarrow \mathbb{C}$ can be written as

$$A_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} \langle \tau_m u(k), F^{-1}(\tau_n \hat{u})(k) \rangle,$$

  where $\tau_(m)u(k) = u(m+k)$ is translation by m and $F^{-1}(u)(k)) = \check{u}(k)$ is Fourier inversion.
- As such we see that $A_p(u)$ has the form of a STFT.
- We shall develop a vector-valued DFT theory to *verify* (not just *motivate*) that $A_p^d(u)$ is an STFT in the case $\{E_k\}_{k=0}^{N-1}$ is a DFT frame for $\mathbb{C}^d$.

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DFT frames and the vector-valued DFT

Definition

Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, and let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$. The *vector-valued discrete Fourier transform* of u is

$$\forall n \in \mathbb{Z}_N,\ F(u)(n) = \hat{u}(n) = \sum_{m=0}^{N-1} u(m) * E_{mn},$$

where $*$ is pointwise (coordinatewise) multiplication.

- The vector-valued DFT inversion formula is valid if N is prime.
- Vector-valued DFT uncertainty principle inequalities are valid, similar to Tao-Candes in compressive sensing.

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Vector-valued Fourier inversion theorem

- Inversion process for the vector-valued case is analogous to the 1-dimensional case.
- We must define a new multiplication in the frequency domain to avoid divisibility problems.
- Define the weighted multiplication $(*) : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$ by $u(*)v = u * v * \omega$ where $\omega = (\omega_1, \ldots, \omega_d)$ has the property that each $\omega_n = \frac{1}{\#\{m \in \mathbb{Z}_N : mn = 0\}}$.
- For the following theorem assume $d << N$ or N prime.

Theorem - Vector-valued Fourier inversion

The vector valued Fourier transform F is an isomorphism from $\ell^2(\mathbb{Z}_N)$ to $\ell^2(\mathbb{Z}_N, \omega)$ with inverse

$$\forall\ m \in \mathbb{Z}_N, \quad F^{-1}(m) = u(m) = \frac{d}{N} \sum_{n=0}^{N-1} \hat{u}(n) * E_{-mn} * \omega.$$

N prime implies F is unitary.

$A_p^d(u)$ as an STFT

- Given $u, v : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, and let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$.
- $u * \overline{v}$ denotes pointwise (coordinatewise) multiplication with a factor of $\sqrt{d}$.
- We compute

$$A_p^d(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}(\tau_m u(k)) * \overline{F^{-1}(\tau_n \hat{u})(k)}.$$

- Thus, $A_p^d(u)$ is compatible with point of view of defining a vector-valued ambiguity function in the context of the STFT.

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Outline

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Epilogue

- If $(G, \bullet)$ is a finite group with representation $\rho : G \longrightarrow GL(\mathbb{C}^d)$, then there is a frame $\{E_n\}_{n \in G}$ and bilinear multiplication, $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$, such that $E_m * E_n = E_{m \bullet n}$. Thus, we can develop $A_p^d(u)$ theory in this setting.
- Analyze ambiguity function behavior for (phase-coded) vector-valued waveforms $v : \mathbb{R} \longrightarrow \mathbb{C}^d$, defined by $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ as

$$v = \sum_{k=0}^{N-1} u(k) \mathbb{1}_{[kT,(k+1)T)},$$

in terms of $A_p^d(u)$. (See Figure)

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Computation of $u : \mathbb{Z}_N \to \mathbb{C}^d$ from ambiguity

- CAZAC and waveform computation of $u : \mathbb{Z}_N \to \mathbb{C}^d$ from A(u): Let A_u be the $N \times N$ matix, $(A(u)(m,n))$. Define the $N \times N$ matrix $U = (U_{i,j})$, where $U_{i,j} = \langle u(i+j), u(j) \rangle$. Then

$$U = A_u D_N, \quad \text{where} \quad D_N = DFT \text{ matrix.}$$

- Let $d = 1$. Note that $U_{k,0} = u(k)\overline{u(0)}$. Hence, if we know the values of the ambiguity function, and, thus, the ambiguity function matrix A_u, then the sequence u, which generates it, can be computed as long as $u(0) \neq 0$. In fact, if $u(0) = 1$ then $u(k) = (A_u D_N)(k, 0)$.
- Similar result for $A_V(u)$ using our vector-valued Fourier analysis.
- Now we can address the classical *radar ambiguity problem*: Find the structure of all $z : \mathbb{Z}_N \to \mathbb{C}^d$ for which $|A(u)| = |A(z)|$ on $X \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$.

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Computation of $u : \mathbb{Z}_N \to \mathbb{C}^d$ from ambiguity

- CAZAC and waveform computation of $u : \mathbb{Z}_N \to \mathbb{C}^d$ from A(u): Let A_u be the $N \times N$ matix, $(A(u)(m,n))$. Define the $N \times N$ matrix $U = (U_{i,j})$, where $U_{i,j} = \langle u(i+j), u(j) \rangle$. Then

$$U = A_u D_N, \quad \text{where} \quad D_N = DFT \text{ matrix.}$$

- Let $d = 1$. If $u(0) = 1$ then $u(k) = (A_u D_N)(k, 0)$.
- Similar result for $A_V(u)$ using our vector-valued Fourier analysis.
- We are addressing the classical *radar ambiguity problem*: Find the structure of all $z : \mathbb{Z}_N \to \mathbb{C}^d$ for which $|A(u)| = |A(z)|$ on $X \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$. This is not even resolved for the narrow-band case.
- The radar ambiguity problem is closely related to our approach of achieving diverse ambiguity function behavior.

That's all folks!

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