Constructive digital harmonic analysis

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Narrow band ambiguity functions and CAZAC codes

Discrete ambiguity functions

Let $u : \{0, 1, \ldots, N-1\} \to \mathbb{C}$.

- $u_p : \mathbb{Z}_N \to \mathbb{C}$ is the N-periodic extension of u.
- $u_a : \mathbb{Z} \to \mathbb{C}$ is an aperiodic extension of u:

$$u_a[m] = \begin{cases} u[m], & m = 0, 1, \ldots, N-1 \\ 0, & \text{otherwise.} \end{cases}$$

- The *discrete periodic ambiguity function* $A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{C}$ of u is

$$A_p(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u_p[m+k]\overline{u_p[k]} e^{2\pi i k n/N}.$$

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CAZAC sequences

- $u : \mathbb{Z}_N \rightarrow \mathbb{C}$ is *Constant Amplitude Zero Autocorrelation (CAZAC)*:

$$\forall m \in \mathbb{Z}_N, \quad |u[m]| = 1, \text{ (CA)}$$

and

$$\forall m \in \mathbb{Z}_N \setminus \{0\}, \quad A_p(u)(m, 0) = 0. \text{ (ZAC)}$$

- Empirically, the (ZAC) property of CAZAC sequences u leads to phase coded waveforms w with low *aperiodic autocorrelation* $\mathcal{A}(w)(t, 0)$.
- Are there only finitely many non-equivalent CAZAC sequences?
  - "Yes" for N prime and "No" for $N = MK^2$,
  - Generally unknown for N square free and not prime.

Björck CAZAC codes and ambiguity function comparisons

Legendre symbol

Let N be a prime and $(k, N) = 1$.

- k is a quadratic residue mod N if $x^2 = k \pmod N$ has a solution.
- k is a quadratic non–residue mod N if $x^2 = k \pmod N$ has no solution.
- The Legendre symbol:

$$\left(\frac{k}{N}\right) = \begin{cases} 1, & \text{if } k \text{ is a quadratic residue mod } N\,, \\ -1, & \text{if } k \text{ is a quadratic non–residue mod } N. \end{cases}$$

The diagonal of the product table of $\mathbb{Z}_N$ gives values $k \in \mathbb{Z}$ which are squares. As such we can program Legendre symbol computation.

Example: $N = 7$. $(\frac{k}{N}) = 1$ if $k = 1, 2, 4$.

Definition

Let N be a prime number. A *Björck CAZAC sequence* of length N is

$$u[k] = e^{i\theta_N(k)}, \quad k = 0, 1, \dots, N-1,$$

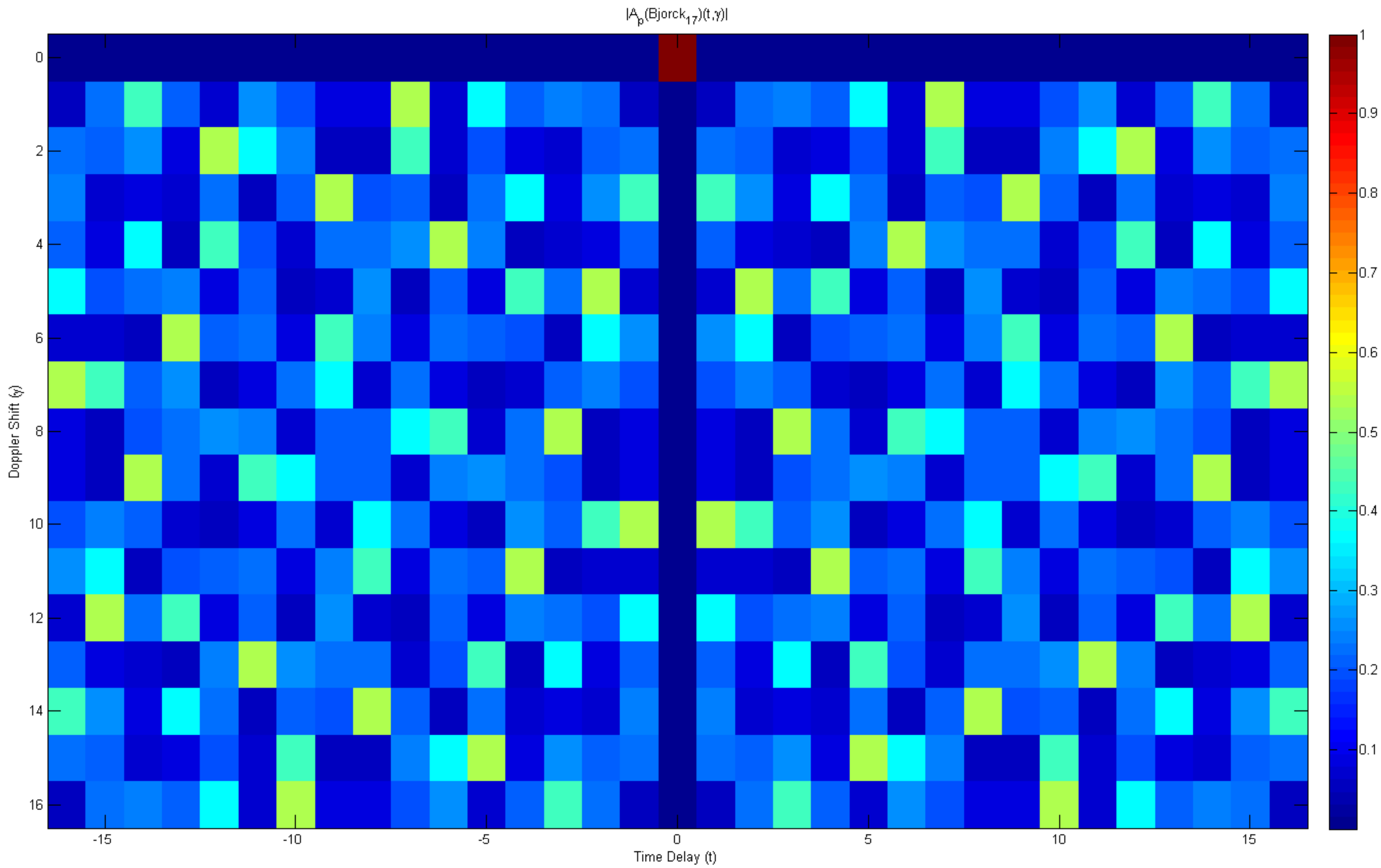
where, for $N = 1 \pmod 4$,

$$\theta_N(k) = \arccos\left(\frac{1}{1+\sqrt{N}}\right)\left(\frac{k}{N}\right),$$

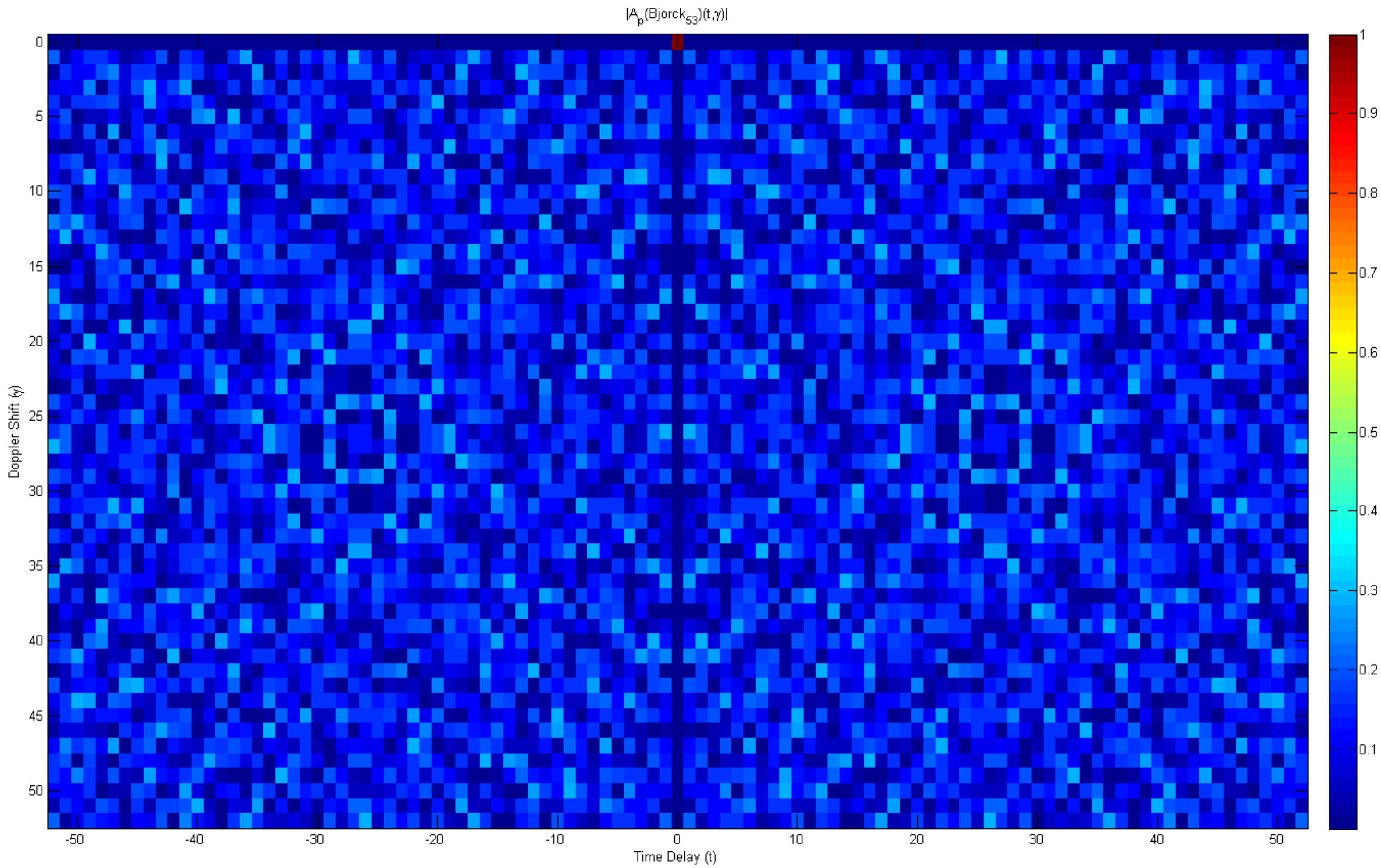
and, for $N = 3 \pmod 4$,

$$\theta_N(k) = \frac{1}{2}\arccos\left(\frac{1-N}{1+N}\right)\left[(1-\delta_k)\left(\frac{k}{N}\right) + \delta_k\right].$$

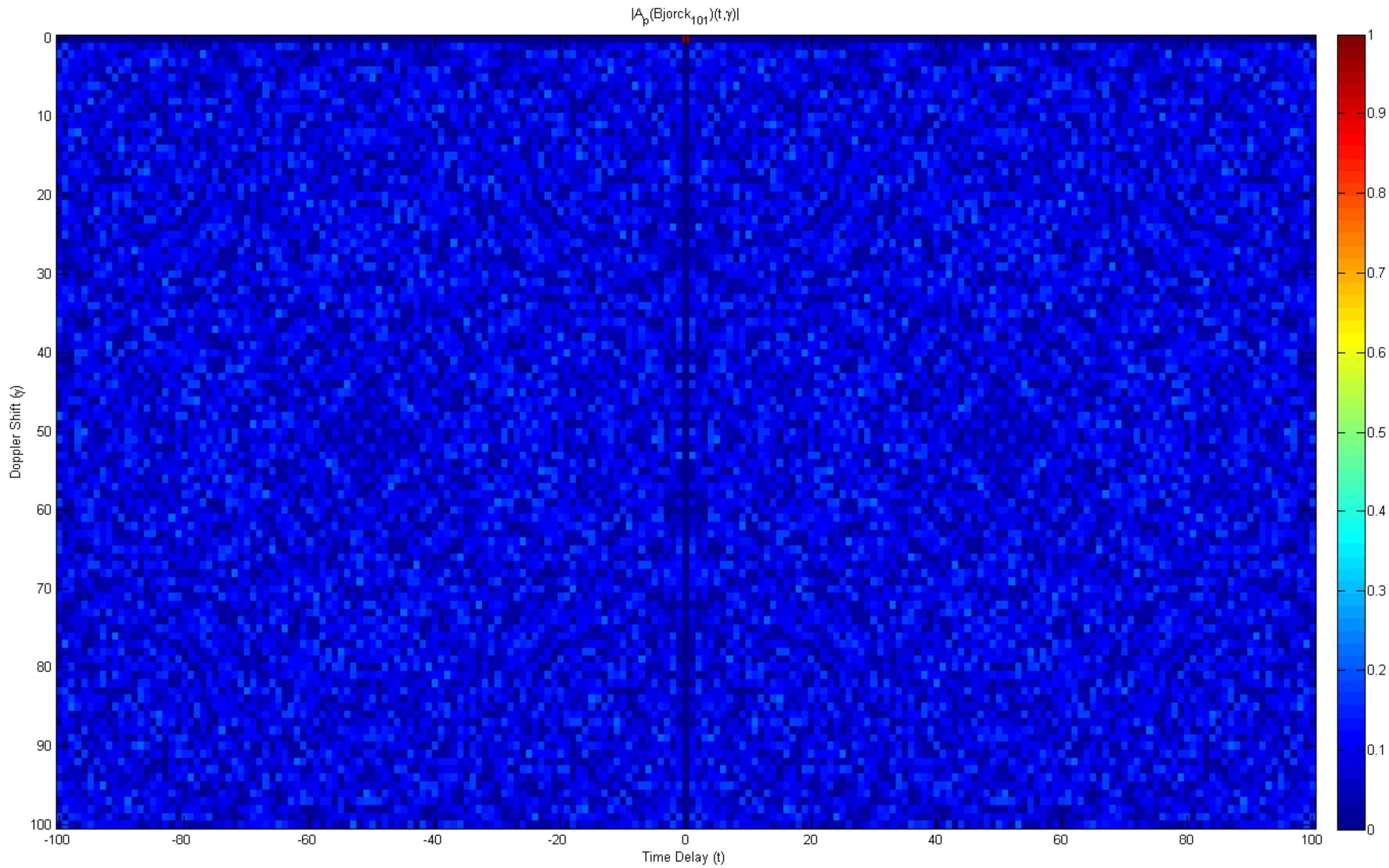
δ_k is Kronecker delta and $\left(\frac{k}{N}\right)$ is Legendre symbol.



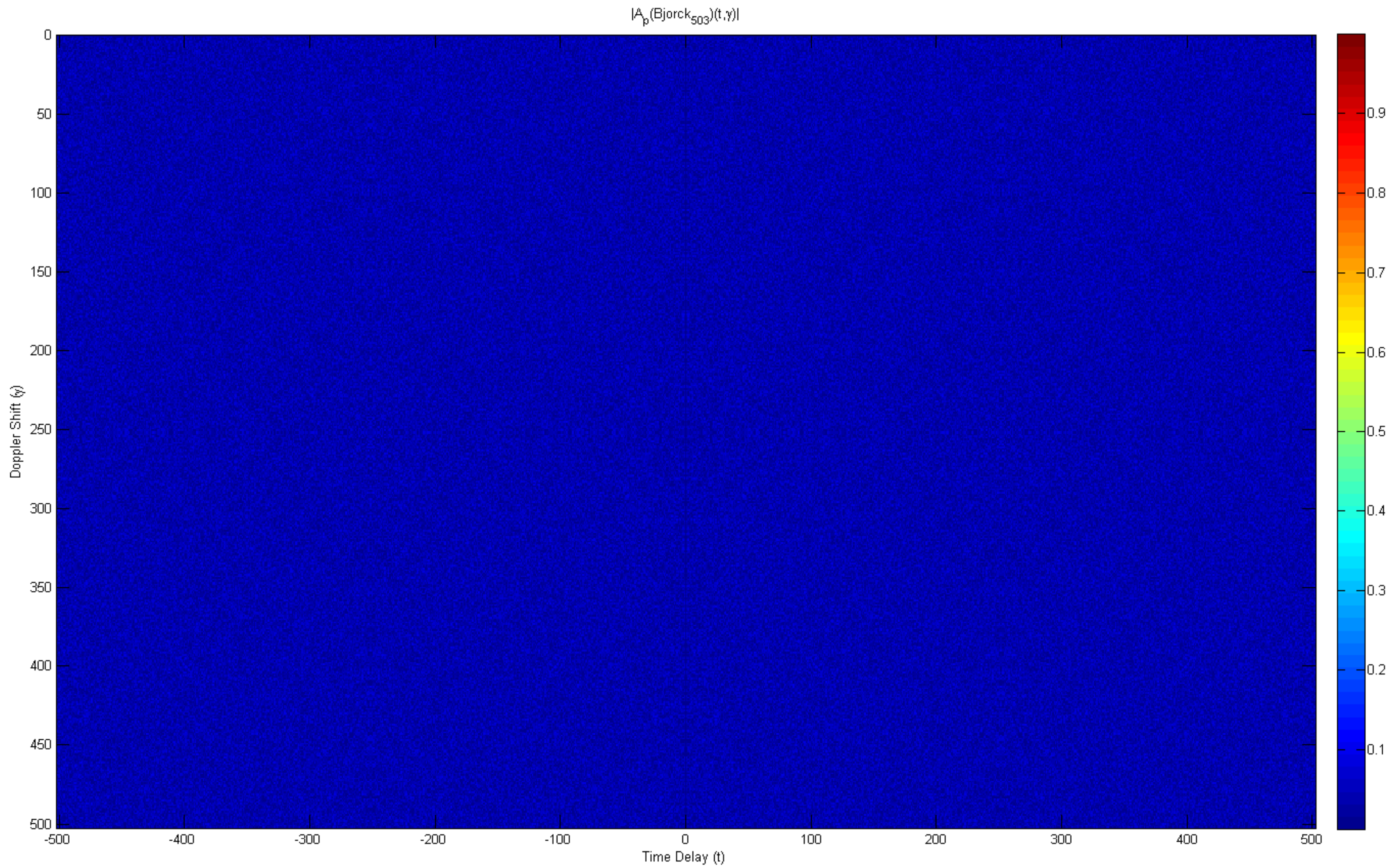
Absolute value of Bjorck code of length 17



Absolute value of Bjorck code of length 53



Absolute value of Bjorck code of length 101



Absolute value of Bjorck code of length 503

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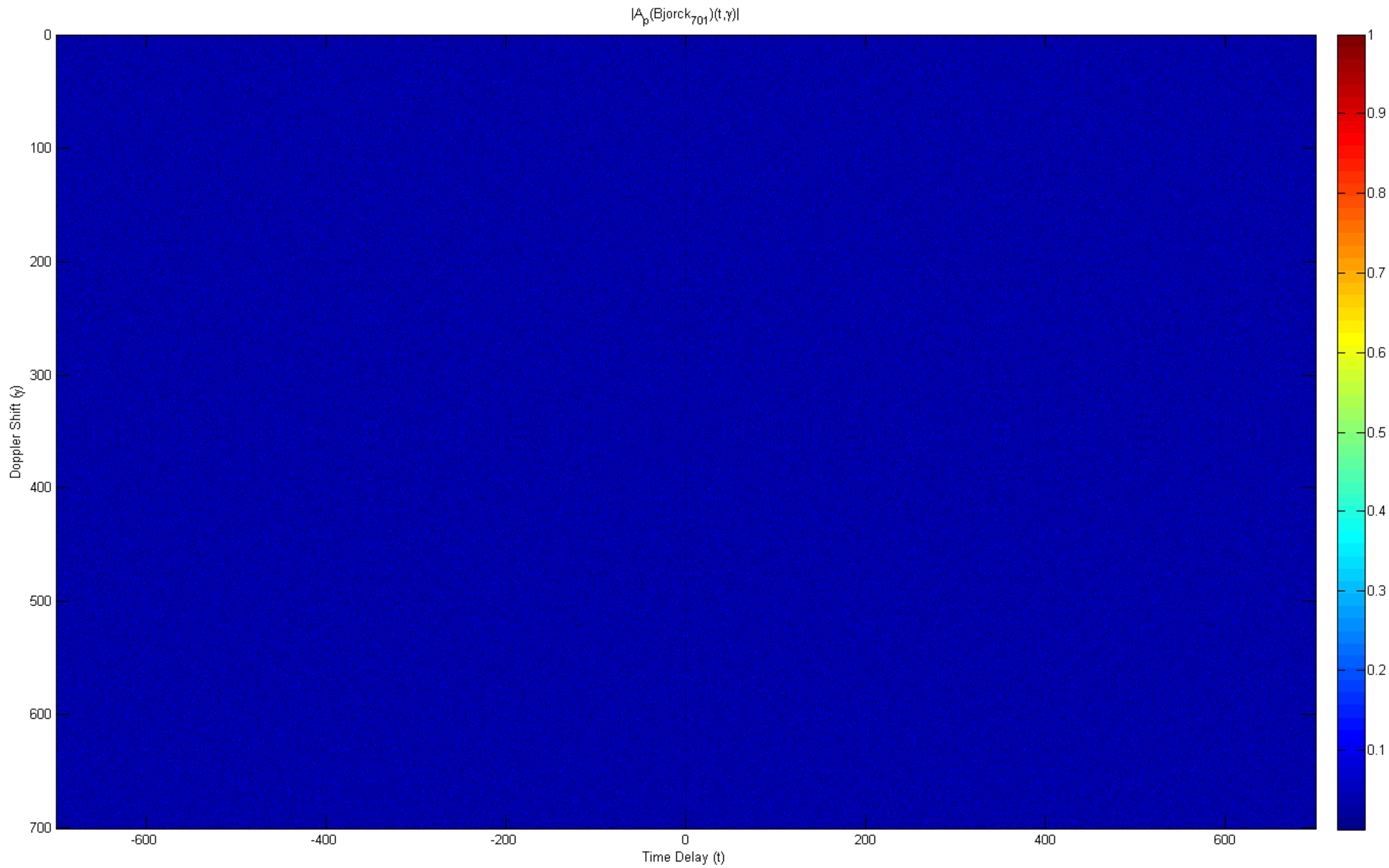

Absolute value of Bjorck code of length 701

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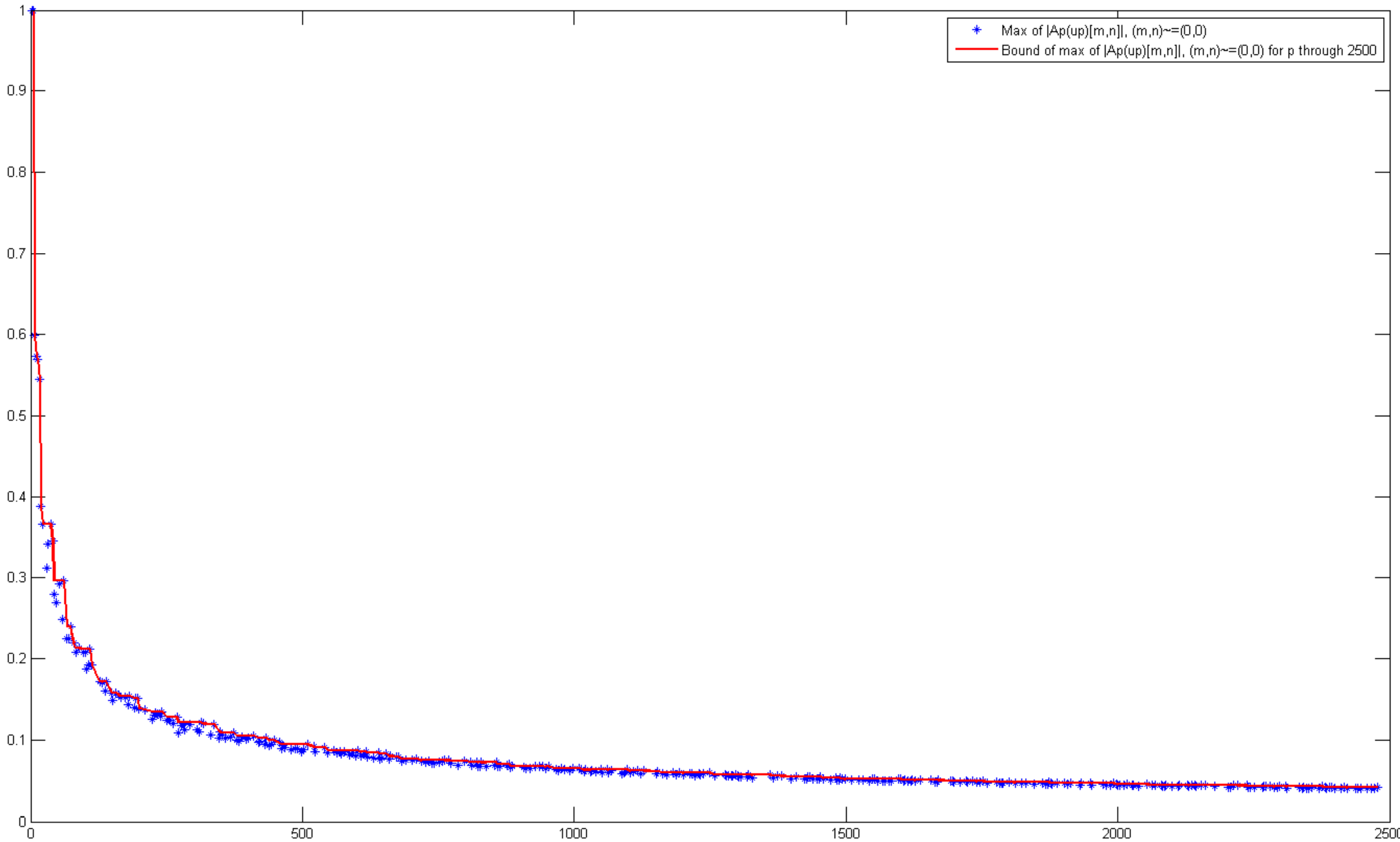

Max of |Ap(up)[m,n]|, (m,n)~=(0,0)
Bound of max of |Ap(up)[m,n]|, (m,n)~=(0,0) for p through 2500
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
500
1000
1500
2000
2500

Björck CAZAC Discrete Narrow-band Ambiguity Function

Let u_p denote the Björck CAZAC sequence for prime p, and let $A_p(u_p)$ be the discrete narrow band ambiguity function defined on $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Theorem (J. and R. Benedetto and J. Woodworth)

$$|A_p(u_p)(m,n)| \leq \frac{2}{\sqrt{p}} + \frac{4}{p}$$

for all $(m,n) \in (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \setminus (0,0)$.

- The bound is more precise but not better than $\frac{2}{\sqrt{p}}$ depending on whether $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$.
- The proof is at the level of Weil's proof of the Riemann hypothesis for finite fields and depends on Weil's exponential sum bound.
- Elementary construction/coding and intricate combinatorial/geometrical patterns.

The ambiguity function

- The *complex envelope* w of the *phase coded waveform* $Re(w)$ associated to a unimodular N-periodic sequence $u : \mathbb{Z}_N \to \mathbb{C}$ is

$$w(t) = \frac{1}{\sqrt{\tau}} \sum_{k=0}^{N-1} u[k] \, \mathbb{1}\left(\frac{t - kt_b}{t_b}\right),$$

  where $\mathbb{1}$ is the characteristic function of the interval $[0, 1)$, τ is the pulse duration, and $t_b = \tau/N$.
- For spectral shaping problems, smooth replacements to $\mathbb{1}$ are analyzed.
- The *(aperiodic) ambiguity function* $\mathcal{A}(w)$ of w is

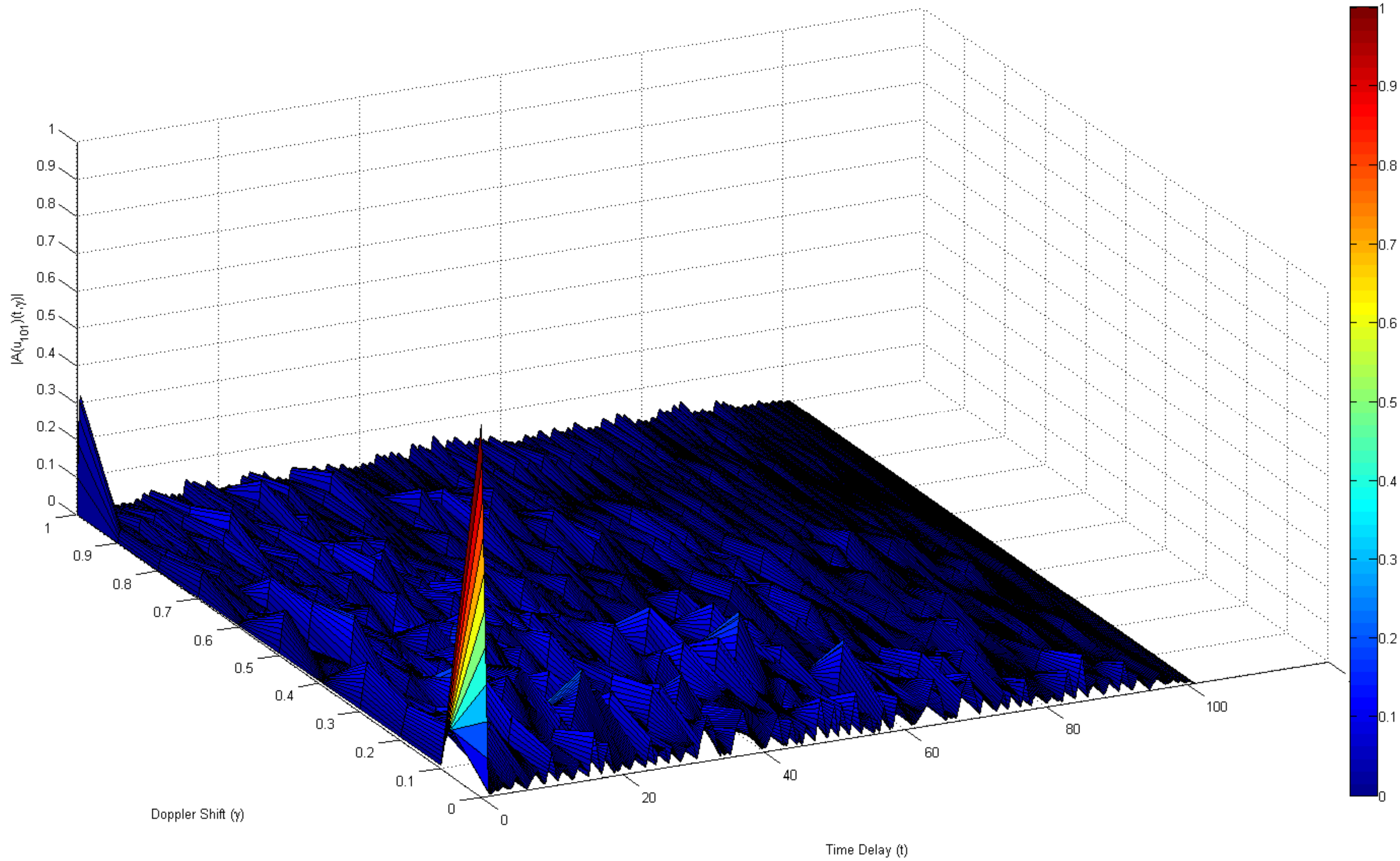
$$\mathcal{A}(w)(t,\gamma) = \int w(s+t)\overline{w(s)}e^{2\pi i s\gamma}\,ds,$$

  where $t \in \mathbb{R}$ is time delay and $\gamma \in \widehat{\mathbb{R}}(= \mathbb{R})$ is frequency shift.

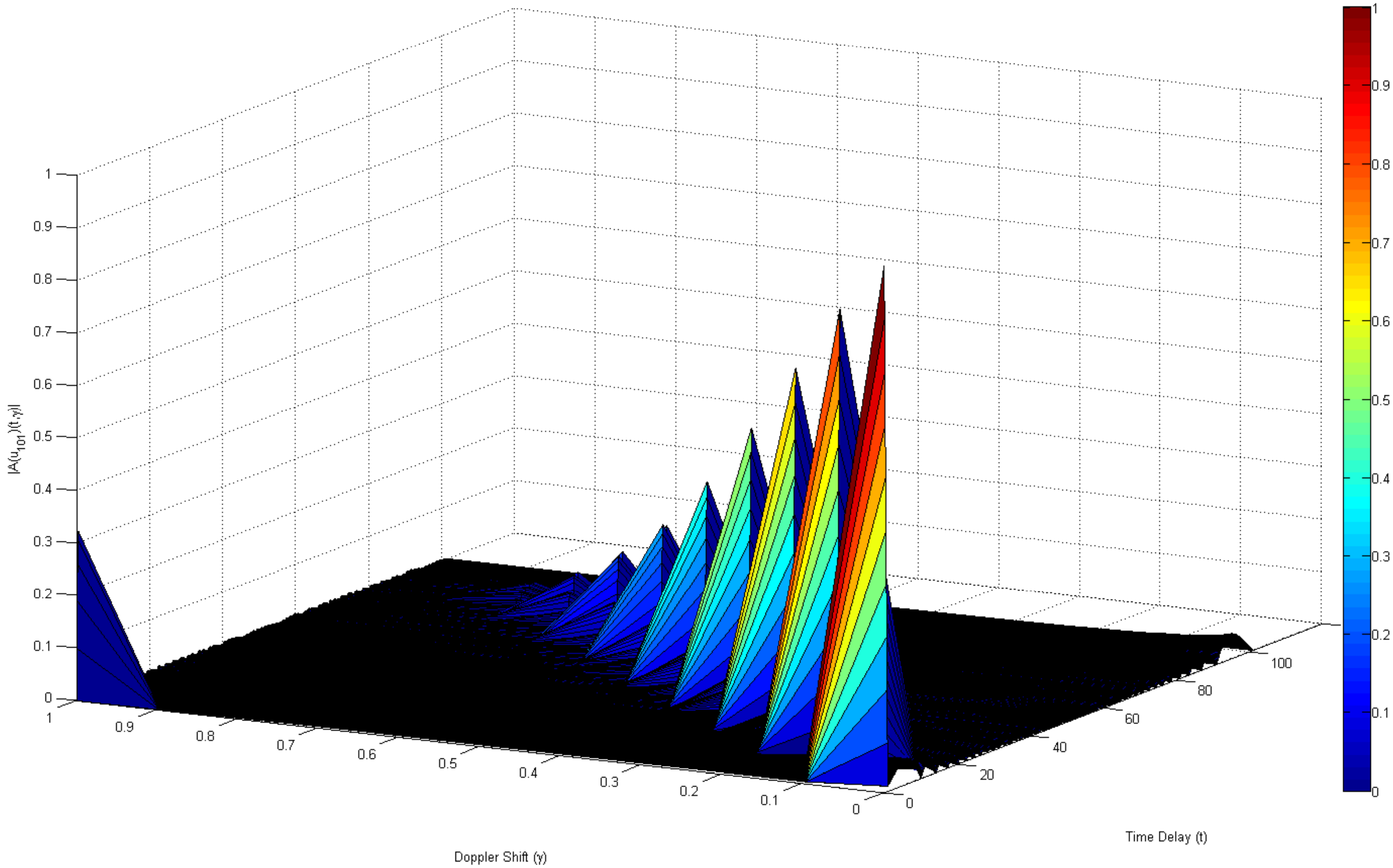
Caveat emptor

Waveform diversity is a government program for disadvantaged waveforms

– G. Linde, (a real) radar engineer



|A(u101)(t,γ)|
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Doppler Shift (γ)
Time Delay (t)
0
20
40
60
80
100



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Perspective

Sequences for coding theory, cryptography, phase-coded waveforms, and communications (synchronization, fast start-up equalization, frequency hopping) include the following in the periodic case:

- Gauss, Wiener (1927), Zadoff (1963), Schroeder (1969), Chu (1972), Zhang and Golomb (1993)
- Frank (1953), Zadoff and Abourezk (1961), Heimiller (1961)
- Milewski (1983)
- Björck (1985) and Golomb (1992),

and their generalizations, both periodic and aperiodic.

The general problem of using codes to generate signals leads to frames.

Balayage, Fourier frames, and sampling theory

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Fourier frames, goal, and a litany of names

Definition

E $= \{x_n\} \subseteq \mathbb{R}^d, \Lambda \subseteq \widehat{\mathbb{R}}^d$. E is a *Fourier frame* for $L^2(\Lambda)$ if

$\exists A, B > 0, \forall F \in L^2(\Lambda)$,

$$A \; ||F||^2_{L^2(\Lambda)} \leq \sum_n |< F(\gamma), e^{-2\pi i x_n \cdot \gamma} > |^2 \leq B \; ||F||^2_{L^2(\Lambda)}.$$

- *Goal* Formulate a general theory of Fourier frames and non-uniform sampling formulas parametrized by the space $M(\mathbb{R}^d)$ of bounded Radon measures.
- *Motivation* Beurling theory (1959-1960).
- *Names* Riemann-Weber, Dini, G.D. Birkhoff, Paley-Wiener, Levinson, Duffin-Schaeffer, Beurling-Malliavin, Beurling, H.J. Landau, Jaffard, Seip, Ortega-Certà–Seip.

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Balayage

- Let $M(G)$ be the algebra of bounded Radon measures on the LCAG G.
- Balayage in potential theory was introduced by Christoffel (early 1870s) and Poincaré (1890).

Definition

(Beurling) Balayage is possible for $(E, \Lambda) \subseteq G \times \widehat{G}$, a LCAG pair, if

$$\forall \mu \in M(G), \exists \nu \in M(E) \text{ such that } \hat{\mu} = \hat{\nu} \text{ on } \Lambda.$$

We write balayage (E, Λ).

- The set, Λ, of group characters is the analogue of the original role of Λ in balayage as a collection of potential theoretic kernels.
- Kahane formulated balayage for the harmonic analysis of restriction algebras.

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Spectral synthesis

Definition

(Wiener, Beurling) Closed $\Lambda \subseteq \widehat{G}$ is a set of *spectral synthesis* (S-set) if

$\forall \mu \in M(G), \forall f \in C_b(G),$

$\text{supp}(\widehat{f}) \subseteq \Lambda$ and $\hat{\mu} = 0$ on $\Lambda \Longrightarrow \int_G f \, d\mu = 0.$

$(\forall T \in A'(\widehat{G}), \forall \phi \in A(\widehat{G}), \quad \text{supp}(T) \subseteq \Lambda \text{ and } \phi = 0 \text{ on } \Lambda \Rightarrow T(\phi) = 0.)$

- Ideal structure of $L^1(G)$ - the Nullstellensatz of harmonic analysis
- $T \in D'(\widehat{\mathbb{R}}^d), \phi \in C_c^\infty(\widehat{\mathbb{R}}^d)$, and $\phi = 0$ on $\text{supp}(T) \Rightarrow T(\phi) = 0$, with same result for $M(\widehat{\mathbb{R}}^d)$ and $C_0(\widehat{\mathbb{R}}^d)$.
- $S^2 \subseteq \widehat{\mathbb{R}}^3$ is not an S-set (L. Schwartz), and every non-discrete $\widehat{G}$ has non-S-sets (Malliavin).
- Polyhedra are S-sets. The $\frac{1}{3}$-Cantor set is an S-set with non-S-subsets.

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Strict multiplicity

Definition

$\Gamma \subseteq \widehat{G}$ is a set of *strict multiplicity* if

$$\exists\ \mu \in M(\Gamma)\backslash\{0\} \text{ such that } \check{\mu} \text{ vanishes at infinity in } G.$$

- Riemann and sets of uniqueness in the wide sense.
- Menchov (1916): $\exists$ closed $\Gamma \subseteq \widehat{\mathbb{R}/\mathbb{Z}}$ and $\mu \in M(\Gamma)\backslash\{0\}$,
  $$|\Gamma| = 0 \text{ and } \check{\mu}(n) = O((\log|n|)^{-1/2}), |n| \to \infty.$$
- 20th century history to study rate of decrease: Bary (1927), Littlewood (1936), Salem (1942, 1950), Ivašev-Mucatov (1957), Beurling.

Assumption

$\forall\ \gamma \in \Lambda$ and $\forall\ N(\gamma)$, compact neighborhood, $\Lambda \cap N(\gamma)$ is a set of strict multiplicity.

A theorem of Beurling

Definition

E $= \{x_n\} \subseteq \mathbb{R}^d$ is *separated* if

$$\exists\ r > 0,\ \forall m, n,\ m \neq n \Rightarrow ||x_m - x_n|| \geq r.$$

Theorem

Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact S-set, symmetric about $0 \in \widehat{\mathbb{R}}^d$, and let E $\subseteq \mathbb{R}^d$ be separated. If balayage (E,Λ), then

$$\textit{E is a Fourier frame for } L^2(\Lambda).$$

- Equivalent formulation in terms of

$$PW_\Lambda = \{f \in L^2(\mathbb{R}^d) : \text{supp}(\hat{f}) \subseteq \Lambda\}.$$

- $\forall F \in L^2(\Lambda), \qquad F = \sum_{x \in \mathsf{E}} < F, S^{-1}(e_x) >_\Lambda e_x$ in $L^2(\Lambda)$.
- For $\mathbb{R}^d$ and other generality beyond Beurling's theorem in $\mathbb{R}$, the result above was formulated by Hui-Chuan Wu and JB (1998), see Landau (1967).

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Let $G \in L^2(\widehat{\mathbb{R}}^d)$ satisfy $||G||_{L^2(\widehat{\mathbb{R}}^d)} = 1$; let $\Lambda \subset \widehat{\mathbb{R}}^d$ be an S-set, symmetric about 0; and let $E \subset \mathbb{R}^d$ be separated. Define

(STFT) $\quad \forall F \in L^2(\Lambda), \quad V_G F(x,\gamma) = \int_\Lambda F(\lambda)G(\lambda-\gamma)e^{2\pi i x\cdot\lambda}\, d\lambda.$

Theorem

If balayage (E,Λ), then

$$\exists A, B > 0, \quad \forall F \in L^2(\Lambda),$$

$$A\ ||F||^2_{L^2(\Lambda)} \leq \int_{\widehat{\mathbb{R}}^d} \sum_{x\in E} |V_G F(x,\gamma)|^2\, d\gamma \leq B\ ||F||^2_{L^2(\Lambda)}.$$

Remark There are basic problems to be resolved and there have been fundamental recent advances.

Non-uniform Gabor frames

Let $g \in L^2(\mathbb{R}^d)$ satisfy $\|g\|_{L^2(\mathbb{R}^d)} = 1$; let $E = \{(s_m, t_n)\} \subseteq \mathbb{R}^{2d}$ be separated; and let $\Lambda \subseteq \hat{\mathbb{R}}^{2d}$ be an S-set, symmetric about the origin.

Theorem

If balayage (E, Λ), *then*

$$\exists A, B > 0, \forall f \in M^1(\mathbb{R}^d) \text{ such that } supp(V_g f)\hat{} \subseteq \Lambda,$$

$$A\|f\|_{L^2(\mathbb{R}^d)} \leq \sum_m \sum_n |V_g f(s_m, t_n)|^2 \leq B\|f\|_{L^2(\mathbb{R}^d)}.$$

- Feichtinger theory of modulation spaces $M_m^{p,q}$. $f \in M^1$ means $V_g f \in L^1(\mathbb{R}^{2d})$.
- Gröchenig theorem for non-uniform Gabor frames involves an analysis of convolution operators on the Heisenberg group.

Examples of balayage

1. Let E $\subseteq \mathbb{R}^d$ be separated. Define
$$r = r(E) = \sup_{x\in\mathbb{R}^d} \text{dist}(x, E).$$
If $r\rho < \frac{1}{4}$, then balayage (E, $\bar{B}(0,\rho)$). $\frac{1}{4}$ is the best possible.
2. If balayage (E, Λ) and $\Lambda_0 \subseteq \Lambda$, then balayage (E, Λ_0).
3. Let E $= \{x_n\}$ be a Fourier frame for PW$_\Lambda$. Then for all $\Lambda_0 \subseteq \Lambda$ with dist$(\Lambda_0, \Lambda^c\ ) > 0$, we have balayage (E, Λ_0).
4. In $\mathbb{R}^1$, for a separated set E, Beurling lower density $> \rho$ is necessary and sufficient for balayage (E, $[\frac{-\rho}{2}, \frac{\rho}{2}]$).

Remark In $\mathbb{R}^1$, if E is uniformly dense in the sense of Duffin-Schaeffer, then $D^-(E), D^+(E)$, and $D_u(E)$ coincide.
So Beurling's result $\Rightarrow$ Duffin-Schaeffer's result on Fourier frames.

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ΦDOs and the Kohn-Nirenberg correspondence

Definition/notation for $\Lambda \subseteq \widehat{\mathbb{R}}^d$

- $\forall \gamma \in \Lambda$, $g_\gamma \in C_b(\mathbb{R}^d)$ and $supp(\widehat{g_\gamma}) \subseteq \Lambda$
- $s(x, \gamma) = e^{2\pi i x \cdot \gamma} g_\gamma(x)$

The *Kohn-Nirenberg correspondence*

$$s \longmapsto H_s$$

with *symbol* H_s is defined by the *Hörmander operator*

$$H_s : L^2(\widehat{\mathbb{R}}^d) \to L^2(\Lambda) \subseteq L^2(\widehat{\mathbb{R}}^d)$$

$$H_s(\hat{f})(\gamma) = \int_{\mathbb{R}^d} s(x, \gamma) f(x) e^{-2\pi i x \cdot \gamma}\, dx$$

Remark

Classically, the symbol is σ and integration is over $\widehat{\mathbb{R}}^d$.

ΦDOs and generalized Fourier frames for non-uniform sampling

Theorem

Assume balayage (E, Λ) where $\Lambda \subseteq \widehat{\mathbb{R}}^d$ is a compact, symmetric S-set. Assume $E = \{x_n\}$ is separated. Let $s(x, \gamma) = e^{2\pi i x \cdot \gamma} g_\gamma(x)$, where

$$\{g_\gamma : \gamma \in \Lambda\} \subseteq C_b(\mathbb{R}^d)$$

and

$$\forall \gamma \in \Lambda, \quad supp(\widehat{g_\gamma}) \subseteq \Lambda$$

Let $f \in X_s \subseteq L^2(\mathbb{R}^d)$ if $H_s(\widehat{f}) = F \in L^2(\Lambda)$ and $suppF \subseteq \Lambda$, then

$$\exists A > 0 \quad \text{such that} \quad \forall f \in X_s$$

$$A \frac{\int_\Lambda |F(\gamma)|^2 \, d\gamma}{\|f\|_{L^2(\mathbb{R}^d)}} \leq \left(\sum_{n \in \mathbb{Z}} \left| \int_\Lambda \overline{F(\gamma)} s(x_n, \gamma) e^{2\pi i x_n \cdot \gamma} \, d\gamma \right|^2 \right)^{1/2}.$$

Classification

- Dimension reduction
- Finite frames and frame potential energy
- Frame potential energy classification algorithm
- Hyperspectral image processing

Dimension reduction

Kernel dimension reduction

Given data space X of N vectors in $\mathbb{R}^D$. (N is the number of pixels in the hypercube, D is the number of spectral bands.)

Two Steps:

1. Construction of an $N \times N$ symmetric, positive semi–definite kernel, K, from these N data points in $\mathbb{R}^D$.
2. Diagonalization of K, and then choosing $d \leq D$ *significant orthogonal* eigenmaps of K.

Motivation

- Different classes of interest may not be orthogonal to each other; however, they may be captured by different frame elements. It is plausible that classes may correspond to elements in a frame but not elements in a basis.
- A *frame* generalizes the concept of an orthonormal basis. Frame elements are non–orthogonal.

Dimension reduction paradigm

- Given data space X of N vectors $x_m \in \mathbb{R}^D$, and let

$$K : X \times X \to \mathbb{R}$$

be a symmetric ($K(x,y) = K(y,x)$), positive semi–definite kernel.

- We map X to a low dimensional space via the following mapping:

$$X \longrightarrow K \longrightarrow \mathbb{R}^d(K), \quad d < D$$

$$x_m \mapsto y_m = (y[m,n_1], y[m,n_2], \ldots, y[m,n_d]) \in \mathbb{R}^d(K),$$

where $y[\cdot,n] \in \mathbb{R}^N$ is an eigenvector of K.

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Laplacian Eigenmaps

- Consider the data points X as the nodes of a graph.
- Define a metric $\rho : X \times X \longrightarrow \mathbb{R}^+$, e.g., $\rho(x_m, x_n) = \|x_m - x_n\|$ is the Euclidean distance.
- Choose $q \in \mathbb{N}$.
- For each x_i choose the q nodes x_n closest to x_i in the metric ρ, and place an edge between x_i and each of these nodes.
- This defines $N'(x_i)$, viz.,
  $N'(x_i) = \{x \in X \ : \ \exists$ an edge between x and $x_i.\}$.
- To define the weights on the edges, we compute:

$$W_{ij} = \begin{cases} \exp(-\|x_i - x_j\|^2/\sigma) & \text{if } x_j \in N'(x_i) \text{ or } x_i \in N'(x_j) \\ 0 & \text{otherwise} \end{cases}$$

- Set $K = D - W$, where $D_{ii} = \sum_j W_{ij}$ and $D_{ij} = 0$ for $i \neq j$;
- Diagonalize K.
- K is symmetric and positive semi–definite.

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Finite frames and frame potential energy

FUNTF

- A set $F = \{e_j\}_{j \in J} \subseteq \mathbb{F}^d$ is a *frame* for $\mathbb{F}^d$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, if

$$\exists\, A, B > 0 \quad \text{such that} \quad \forall\, x \in \mathbb{F}^d, \quad A\|x\|^2 \leq \sum_{j \in J} |\langle x, e_j \rangle|^2 \leq B\|x\|^2.$$

- F *tight* if $A = B$. A finite unit-norm tight frame F is a FUNTF.
- N row vectors from any fixed $N \times d$ submatrix of the $N \times N$ DFT matrix, $\frac{1}{\sqrt{d}}(e^{2\pi imn/N})$, is a FUNTF for $\mathbb{C}^d$.
- If F is a FUNTF for $\mathbb{F}^d$, then

$$\forall x \in \mathbb{F}^d, \quad x = \frac{d}{N} \sum_{j=1}^{N} \langle x, e_j \rangle e_j.$$

- Frames: redundant representation, compensate for hardware errors, inexpensive, numerical stability, minimize effects of noise.

DFT FUNTFs

- $N \times d$ submatrices of the $N \times N$ DFT matrix are FUNTFs for $\mathbb{C}^d$. These play a major role in finite frame $\Sigma\Delta$-quantization.

$$N = 8,\ d = 5 \qquad \frac{1}{\sqrt{5}}\begin{bmatrix} * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \end{bmatrix}$$

$$x_m = \frac{1}{5}\left(e^{2\pi i\frac{m}{8}}, e^{2\pi i m\frac{2}{8}}, e^{2\pi i m\frac{5}{8}}, e^{2\pi i m\frac{6}{8}}, e^{2\pi i m\frac{7}{8}}\right)$$

$$m = 1, \ldots, 8.$$

- Sigma-Delta Super Audio CDs - but not all authorities are fans.

CAZACs and FUNTFs

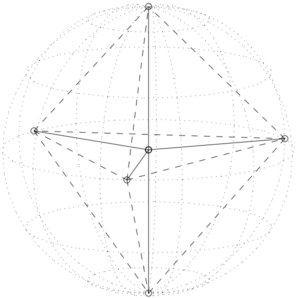
- Let $u = \{u[k]\}_{k=1}^{N}$ be a CAZAC sequence in $\mathbb{C}$. Define

$$\forall\, k = 1, ..., N, \quad v_k = v[k] = \frac{1}{\sqrt{d}}(u[k], u[k+1], ..., u[k+d-1]).$$

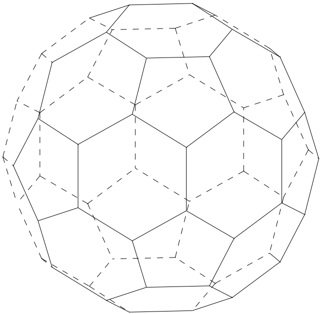
Then $v = \{v[k]\}_{k=1}^{N} \subseteq \mathbb{C}^d$ is a CAZAC sequence in $\mathbb{C}^d$ and $\{v_k\}_{k=1}^{N}$ is a FUNTF for $\mathbb{C}^d$ with frame constant N/d.

- Let $\{x_k\}_{k=1}^{N} \subseteq \mathbb{C}^d$ be a FUNTF for $\mathbb{C}^d$, with frame constant A and with associated Bessel map $L : \mathbb{C}^d \rightarrow \ell^2(\mathbb{Z}_N)$; and let $u = \{u[j]\}_{j=1}^{M} \subseteq \mathbb{C}^d$ be a CAZAC sequence in $\mathbb{C}^d$. Then $\{\frac{1}{\sqrt{A}}L(u[j])\}_{j=1}^{M} \subseteq \mathbb{C}^N (= \ell^2(\mathbb{Z}_N))$ is a CAZAC sequence in $\mathbb{C}^N$.

Examples of frames



(a) Non–FUNTF



(b) FUNTF

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The geometry of finite tight frames

- We saw the vertices of platonic solids are FUNTFs.
- However, points that constitute FUNTFs do not have to be equidistributed, e.g., ONBs and Grassmanian frames.
- FUNTFs can be characterized as minimizers of a frame potential function (with Fickus) analogous to Coulomb's Law.
- Frame potential energy optimization has basic applications dealing with classification problems for hyperspectral and multi-spectral (biomedical) image data.

Frame force and potential energy

$$F : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R}^d$$

$$P : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R},$$

where $P(a,b) = p(\|a-b\|), \qquad p'(x) = -xf(x)$

- Coulomb force

$$CF(a,b) = (a-b)/\|a-b\|^3, \quad f(x) = 1/x^3$$

- Frame force

$$FF(a,b) = \langle a,b \rangle (a-b), \quad f(x) = 1 - x^2/2$$

- Total potential energy for the frame force

$$TFP(\{x_n\}) = \sum_{m=1}^{N} \sum_{n=1}^{N} |\langle x_m, x_n \rangle|^2$$

Characterization of FUNTFs

Theorem

Let $N \leq d$. The minimum value of *TFP*, for the frame force and N variables, is N; and the *minimizers* are precisely the orthonormal sets of N elements for $\mathbb{R}^d$.

Let $N \geq d$. The minimum value of *TFP*, for the frame force and N variables, is N^2/d; and the *minimizers* are precisely the FUNTFs of N elements for $\mathbb{R}^d$.

Problem

Find FUNTFs analytically, effectively, computationally.

Frame potential energy classification algorithm

Optimization problem: maximal separation

Goal: Construct a FUNTF $\{\Psi_k\}_{k=1}^s$ such that each Ψ_k is associated to only one classifiable material.

For $\{\theta_k\}_{k=1}^s \in S^{d-1} \times \cdots \times S^{d-1}$ and $n = 1, \ldots, s$, set

$$p(\theta_n) = \sum_{m=1}^{N} |\langle y_m, \theta_n \rangle|$$

and consider the maximal separation

$$\sup_{\{\theta_j\}_{j=1}^s} \min\{|p(\theta_k) - p(\theta_n)| : k \neq n\}.$$

Optimization problem: ideal class definition

- $Y = \{y_m\}_{m=1}^N \subseteq \mathbb{R}^d(K) \subseteq \mathbb{R}^N$
- Given s classes $C_j, j = 1, \ldots, s$, defined in terms of a tolerance $\epsilon > 0$ and partition $\{P_j\}_{j=0}^s$ of Y, as $C_j = P_j \cap Y \subseteq B(z_j, \epsilon)$ $j = 1, \ldots, s$ for some $z_j \in \mathbb{R}^d(K)$.

Optimization problem: FUNTF construction

- Point of view: Combine frame potential energy theorem, maximal separation criteria (M_δ), and ideal class definition (C_ϵ).

- Paradigm: Given Y, s, M_δ, and C_ϵ. Construct a FUNTF $\{\Psi_j\}_{j=1}^{s}$ such that

$$\forall j = 1, \ldots, s, \quad |\langle \Psi_j, Y \cap P_j \rangle| \geq R(\epsilon, \delta)$$

and

$$\forall j \neq k, \quad |\langle \Psi_k, Y \cap P_j \rangle| \leq r(\epsilon, \delta),$$

where $r(\epsilon, \delta) < R(\epsilon, \delta)$.

Frame coefficient images

- Given $\Psi = \{\Psi_n\}_{n=1}^s \subseteq \mathbb{R}^d = \mathbb{R}^d(K) \subseteq \mathbb{R}^N$ and $m \in \{1, \ldots, N\}$. Consider the set of frame decompositions

$$\forall y_m \in \mathbb{R}^d, m = 1, \ldots, N, \quad y_m = \sum_{n=1}^{s} c_{m,n}^{\alpha} \Psi_n, \quad \text{indexed by } \alpha \in \mathbb{R}.$$

- For each $m \in \{1, \ldots, N\}$ choose an ℓ^1 sparse decomposition

$$y_m = \sum_{n=1}^{s} c_{m,n}^{\alpha(m)} \Psi_n$$

defined by the inequality,

$$\forall \alpha, \quad \sum_{n=1}^{s} |c_{m,n}^{\alpha(m)}| \leq \sum_{n=1}^{s} |c_{m,n}^{\alpha}|.$$

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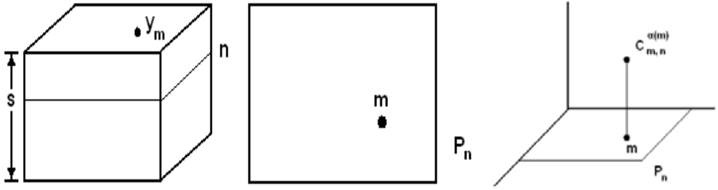
Frame coefficient images (continued)

- Choose $n \in \{1, ..., s\}$. Take a slice, P_n, of the data cube at n. P_n contains N points m.



(a) Data Cube (b) Top Down Slice (c) $c_{m,n}^{\alpha(m)}$ defined

- The image with N pixels m, associated to the the frame element Ψ_n, is defined by $\{c_{m,n}^{\alpha(m)} \mid m = 1, ..., N\}$.

Hyperspectral image processing

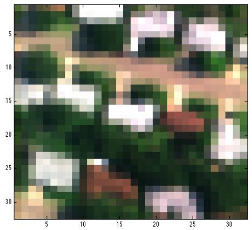
Urban data set classes

Figure: HYDICE Copperas Cove, TX — http://www.tec.army.mil/Hypercube/

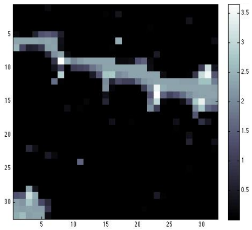
Urban data set classes

- There are 23 classes associated with the different colors in the previous figure.
- In fact, if the 23 classes were to correspond roughly to orthogonal subspaces, then one cannot achieve effective dimension reduction less than dimension $d = 23$.
- However, we could have a frame with 23 elements in a space of reduced dimension $d < 23$.

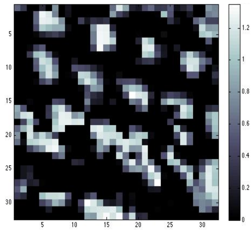
Frame coefficients



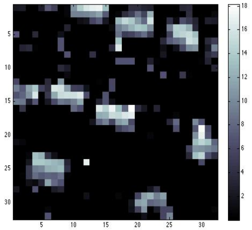
(a) Original



(b) Road coefficients



(c) Tree coefficients



(d) White house coefficients

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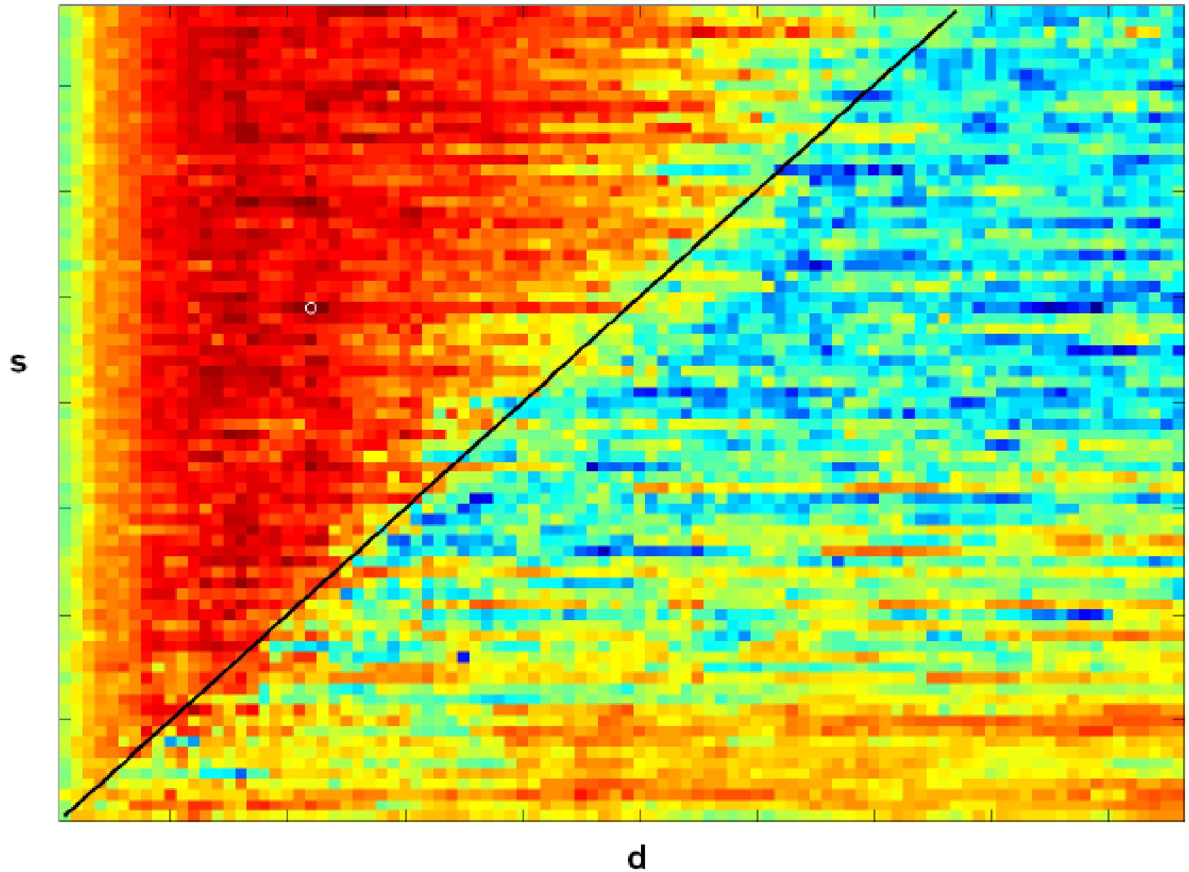
Overview of Classification Results



Thats all folks!

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