The construction of perfect autocorrelation codes

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Abstract

A significant underlying component for the effective design of phase-coded waveforms is the construction of finite unimodular codes whose autocorrelations are zero everywhere except at the dc-component. We refer to such codes as CAZACs, Constant Amplitude Zero AutoCorrelation codes.

We begin by describing some known results in the long history of this subject. Then we construct new CAZACs and show that there is an infinitude of distinct CAZACs. This is important in the realm of waveform diversity, especially as regards a fine local analysis of the ambiguity function and the solutions of both the narrow band and wide band radar ambiguity problems.

We also present the vector-valued theory as well as constructions of infinite CAZAC codes.

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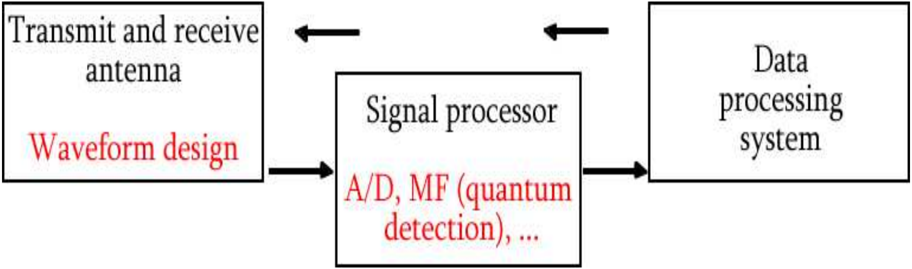
Processing


Transmit and receive antenna
Waveform design
Signal processor
A/D, MF (quantum detection), ...
Data processing system

Topics

- Narrow band ambiguity functions and CAZAC codes
- Wiener CAZAC codes
- Frames
- Björck CAZAC codes and ambiguity function comparisons
- Shapiro-Rudin polynomials
- A vector-valued ambiguity function

Topics (continued)

- Quantization methods
- A comparison of Σ-Δ and PCM
- Complex Σ-Δ and Yang Wang's idea and algorithm
- Σ-Δ and analytic number theory
- Hadamard matrices and infinite CAZAC codes

Narrow band ambiguity functions and CAZAC codes

Discrete ambiguity functions

Let $u : \{0, 1, \dots, N-1\} \to \mathbb{C}$.

- $u_p : \mathbb{Z}_N \to \mathbb{C}$ is the N-periodic extension of u.
- $u_a : \mathbb{Z} \to \mathbb{C}$ is an aperiodic extension of u:

$$u_a[m] = \begin{cases} u[m], & m = 0, 1, \dots, N-1 \\ 0, & \text{otherwise.} \end{cases}$$

- The *discrete periodic ambiguity function* $A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{C}$ of u is

$$A_p(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u_p[m+k]\overline{u_p[k]} e^{2\pi i k n / N}.$$

- The *discrete aperiodic ambiguity* function $A_a(u) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ of u is

$$A_a(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u_a[m+k]\overline{u_a[k]} e^{2\pi i k n / N}.$$

The ambiguity function

- The *complex envelope* w of the *phase coded waveform* $Re(w)$ associated to a unimodular N-periodic sequence $u : \mathbb{Z}_N \to \mathbb{C}$ is

$$w(t) = \frac{1}{\sqrt{\tau}} \sum_{k=0}^{N-1} u[k] \, \mathbb{1}\left(\frac{t - kt_b}{t_b}\right),$$

  where $\mathbb{1}$ is the characteristic function of the interval $[0, 1)$, τ is the pulse duration, and $t_b = \tau / N$.
- For spectral shaping problems, smooth replacements to $\mathbb{1}$ are analyzed.
- The *(aperiodic) ambiguity function* $\mathcal{A}(w)$ of w is

$$\mathcal{A}(w)(t, \gamma) = \int w(s + t)\overline{w(s)} e^{2\pi i s\gamma} ds,$$

  where $t \in \mathbb{R}$ is time delay and $\gamma \in \widehat{\mathbb{R}}(= \mathbb{R})$ is frequency shift.

CAZAC sequences

- $u : \mathbb{Z}_N \to \mathbb{C}$ is *Constant Amplitude Zero Autocorrelation (CAZAC)*:

$$\forall m \in \mathbb{Z}_N, \quad |u[m]| = 1, \text{ (CA)}$$

  and

$$\forall m \in \mathbb{Z}_N \setminus \{0\}, \quad A_p(u)(m, 0) = 0. \text{ (ZAC)}$$

- Empirically, the (ZAC) property of CAZAC sequences u leads to phase coded waveforms w with low *aperiodic autocorrelation* $\mathcal{A}(w)(t, 0)$.
- Are there only finitely many non-equivalent CAZAC sequences?
  - "Yes" for N prime and "No" for $N = MK^2$,
  - Generally unknown for N square free and not prime.

Properties of CAZAC codes

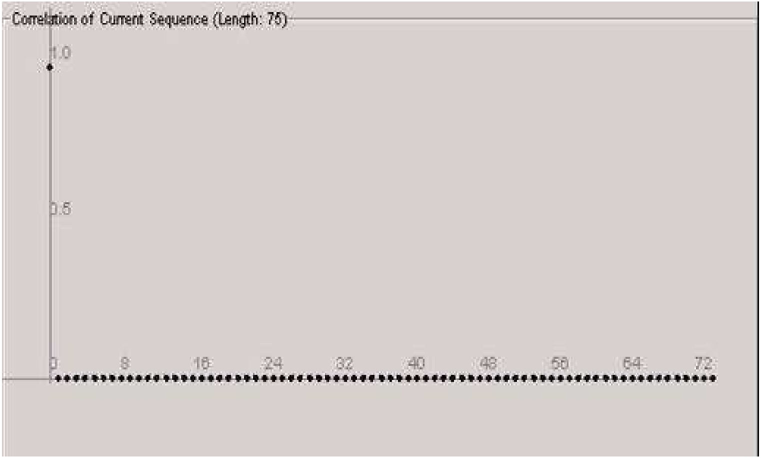
- u CAZAC $\Rightarrow$ u is broadband (full bandwidth).
- There are different constructions of different CAZAC codes resulting in different behavior vis à vis Doppler, additive noises, and approximation by bandlimited waveforms.
- u CA $\Leftrightarrow$ DFT of u is ZAC off dc. (DFT of u can have zeros)
- u CAZAC $\Leftrightarrow$ DFT of u is CAZAC.
- User friendly software: http://www.math.umd.edu/~jjb/cazac

Examples of CAZAC codes

$K = 75 : u(x) =$

$(1, 1, 1, 1, 1, 1, e^{2\pi i \frac{1}{15}}, e^{2\pi i \frac{2}{15}}, e^{2\pi i \frac{1}{5}}, e^{2\pi i \frac{4}{15}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{7}{15}}, e^{2\pi i \frac{3}{5}},$
$e^{2\pi i \frac{11}{15}}, e^{2\pi i \frac{13}{15}}, 1, e^{2\pi i \frac{1}{5}}, e^{2\pi i \frac{2}{5}}, e^{2\pi i \frac{3}{5}}, e^{2\pi i \frac{4}{5}}, 1, e^{2\pi i \frac{4}{15}}, e^{2\pi i \frac{8}{15}}, e^{2\pi i \frac{4}{5}},$
$e^{2\pi i \frac{16}{15}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{2}{3}}, e^{2\pi i}, e^{2\pi i \frac{4}{3}}, e^{2\pi i \frac{5}{3}}, 1, e^{2\pi i \frac{2}{5}}, e^{2\pi i \frac{4}{5}}, e^{2\pi i \frac{6}{5}},$
$e^{2\pi i \frac{8}{5}}, 1, e^{2\pi i \frac{7}{15}}, e^{2\pi i \frac{14}{15}}, e^{2\pi i \frac{7}{5}}, e^{2\pi i \frac{28}{15}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{13}{15}}, e^{2\pi i \frac{7}{5}}, e^{2\pi i \frac{29}{15}},$
$e^{2\pi i \frac{37}{15}}, 1, e^{2\pi i \frac{3}{5}}, e^{2\pi i \frac{6}{5}}, e^{2\pi i \frac{9}{5}}, e^{2\pi i \frac{12}{5}}, 1, e^{2\pi i \frac{2}{3}}, e^{2\pi i \frac{4}{3}}, e^{2\pi i \cdot 2}, e^{2\pi i \frac{8}{3}},$
$e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{16}{15}}, e^{2\pi i \frac{9}{5}}, e^{2\pi i \frac{38}{15}}, e^{2\pi i \frac{49}{15}}, 1, e^{2\pi i \frac{4}{5}}, e^{2\pi i \frac{8}{5}}, e^{2\pi i \frac{12}{5}}, e^{2\pi i \frac{16}{5}},$
$1, e^{2\pi i \frac{13}{15}}, e^{2\pi i \frac{26}{15}}, e^{2\pi i \frac{13}{5}}, e^{2\pi i \frac{52}{15}}, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{19}{15}}, e^{2\pi i \frac{11}{5}}, e^{2\pi i \frac{47}{15}}, e^{2\pi i \frac{61}{15}})$

Autocorrelation of CAZAC $K = 75$



Definition

A *quadratic phase* CAZAC $u : \mathbb{Z}_N \to \mathbb{C}$ is given by

$$u[k] = e^{\pi i P(k)/N}, \quad k = 0, 1, \ldots, N-1,$$

where $P(k)$ is a quadratic polynomial.

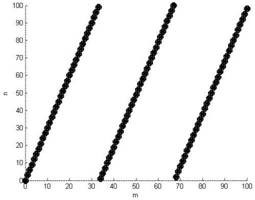
Examples:

- *Chu* sequences: $P(k) = k(k-1)$, N odd,
- *P4* sequences: $P(k) = k(k-N)$,
- *Wiener* CAZAC sequences: $P(k) = k^2$, N odd.

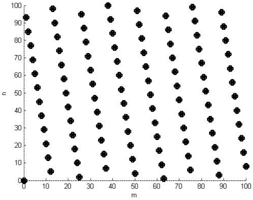
Wiener CAZAC codes

Wiener CAZAC sequences

- Elementary number theoretic techniques, dealing with primitive roots of unity, are used to analyze Wiener CAZAC sequences.
- Peaks in the discrete ambiguity function $A_p(u)$ of a Wiener CAZAC u are not stable under small perturbations in its domain, see [BD2007].



101–49



101–4

Rationale and theorem

Different CAZACs exhibit different behavior in their ambiguity plots, according to their construction method. Thus, the ambiguity function reveals localization properties of different constructions.

Theorem

Given K odd, $\zeta = e^{\frac{2\pi i}{K}}$, and $u[k] = \zeta^{k^2},\ k \in \mathbb{Z}_K$

- $1 \leq k \leq K-2$ odd implies

$$A[m,k] = e^{\pi i (K-k)^2/K} \text{ for } m = \frac{1}{2}(K-k), \text{ and 0 elsewhere}$$

- $2 \leq k \leq K-1$ even implies

$$A[m,k] = e^{\pi i (2K-k)^2/K} \text{ for } m = \frac{1}{2}(2K-k), \text{ and 0 elsewhere}$$

Rationale and theorem

Theorem 1

Given $N \geq 1$. Let

$$M = \begin{cases} N, & \text{N odd,} \\ 2N, & \text{N even,} \end{cases}$$

and let ω be a primitive Mth root of unity. Define the Wiener waveform $u : \mathbb{Z}_N \to \mathbb{C}$ by $u(k) = \omega^{k^2}$, $0 \leq k \leq N-1$. Then u is a CAZAC waveform.

Rationale and theorem

Theorem 2

Let $j \in \mathbb{Z}$. Define $u_j : \mathbb{Z}_N \to \mathbb{C}$ by $u_j(k) = e^{2\pi ijk^2/M}$, where $M = 2N$ if N is even and $M = N$ if N is odd. If N is even, then

$$A_{u_j}(m,n) = \begin{cases} e^{2\pi ijm^2/(2N)}, & jm + n \equiv 0 \ mod\ N, \\ 0, & \text{otherwise.} \end{cases}$$

If N is odd

$$A_{u_j}(m,n) = \begin{cases} e^{2\pi ijm^2/N}, & 2jm + n \equiv 0 \ mod\ N, \\ 0, & \text{otherwise.} \end{cases}$$

Rationale and theorem

Proof. Let N be even, and set $u_j(k) = e^{\pi i j k^2/N}$. We calculate

$$\begin{aligned} A_{u_j}(m,n) &= \frac{1}{N}\sum_{k=0}^{N-1} u_j(m+k)\overline{u_j(k)}e^{2\pi ikn/N} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} e^{(\pi i/N)(jm^2+2jkm+2kn)} = e^{\pi ijm^2/N}\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi ik(jm+n)/N}. \end{aligned}$$

If $jm+n \equiv 0 \bmod N$, then

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi ik(jm+n)/N} = 1.$$

Otherwise, we have

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi ik(jm+n)/N} = \frac{e^{(2\pi i(jm+n)/N)N}-1}{e^{2\pi i(jm+n)/N}-1} = 0.$$

Rationale and theorem

Proof.(Continued) Let N be odd, and set $u(k) = e^{2\pi i k^2/N}$. We calculate

$$\begin{aligned} A_{u_j}(m,n) &= \frac{1}{N}\sum_{k=0}^{N-1} u_j(m+k)\overline{u_j(k)}e^{2\pi ikn/N} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} e^{(2\pi i/N)(jm^2+2jkm+kn)} = e^{2\pi ijm^2/N}\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi ik(2jm+n)/N}. \end{aligned}$$

If $2jm + n \equiv 0 \bmod N$, then

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi ik(2jm+n)/N} = 1.$$

Otherwise, we have

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi ik(2jm+n)/N} = \frac{e^{2\pi i(2m+n)/N)N} - 1}{e^{(2\pi i(2m+n)/N} - 1} = 0.$$

Rationale and theorem

Corollary

Let $\{u(k)\}_{k=0}^{N-1}$ be a Wiener CAZAC waveform as given in Theorem 1. (In particular, ω is a primitive M-th root of unity.)

If N is even, then

$$A_u(m,n) = \begin{cases} \omega^{m^2}, & m \equiv -n \; mod \; N, \\ 0, & \text{otherwise.} \end{cases}$$

If N is odd, then

$$A_u(m,n) = \begin{cases} \omega^{m^2}, & m \equiv -n(N+1)/2 \; mod \; N, \\ 0, & \text{otherwise.} \end{cases}$$

Rationale and theorem

Example

a. Let N be odd and let $\omega = e^{2\pi i/N}$. Then, $u(k) = \omega^{k^2}$, $0 \leq k \leq N-1$, is a CAZAC waveform. By the Corollary, $|A_u(m,n)| = |\omega^{m^2}| = 1$ if $2m + n = l_{m,n}N$ for some $l_{m,n} \in \mathbb{Z}$ and $|A_u(m,n)| = 0$ otherwise, i.e., $A_u(m,n) = 0$ on $\mathbb{Z}_N \times \mathbb{Z}_N$ unless $2m + n \equiv 0 \bmod N$. In the case $2m + n = l_{m,n}N$ for some $l_{m,n} \in \mathbb{Z}$, we have the following phenomenon.

Rationale and theorem

Example (Continued)

If $0 \leq m \leq \frac{N-1}{2}$ and $2m + n = l_{m,n}N$ for some $l_{m,n} \in \mathbb{Z}$, then n is odd; and if $\frac{N+1}{2} \leq m \leq N-1$ and $2m + n = l_{m,n}N$ for some $l_{m,n} \in \mathbb{Z}$, then n is even. Thus, the values (m, n) in the domain of the discrete periodic ambiguity function A_u, for which $A_u(m, n) = 0$, appear as two parallel discrete lines. The line whose domain is $0 \leq m \leq \frac{N-1}{2}$ has odd function values n; and the line whose domain is $\frac{N+1}{2} \leq m \leq N-1$ has even function values n.

Rationale and theorem

Example

b. The behavior observed in (a) has extensions for primitive and non-primitive roots of unity.
Let $u : \mathbb{Z}_N \to \mathbb{C}$ be a Wiener waveform. Thus, $u(k) = \omega^{k^2}$, $0 \leq k \leq N-1$, and $\omega = e^{2\pi ij/M}$, $(j, M) = 1$, where M is defined in terms of N in Theorem 1. By the Corollary, for each fixed $n \in \mathbb{Z}_N$, the function $A_u(\bullet, n)$ of m vanishes everywhere except for a *unique* value $m_n \in \mathbb{Z}_N$ for which $|A_u(m_n, n)| = 1$.

Rationale and theorem

Example (Continued)

The hypotheses of Theorem 2 do not assume that $e^{2\pi ij/M}$ is a primitive Mth root of unity. In fact, in the case that $e^{2\pi ij/M}$ is *not* primitive, then, for certain values of n, $A_u(\bullet, n)$ will be identically 0 and, for certain values of n, $|A_u(\bullet, n)| = 1$ will have several solutions. For example, if $N = 100$ and $j = 2$, then, for each odd n, $A_u(\bullet, n) = 0$ as a function of m. If $N = 100$ and $j = 3$, then $(100, 3) = 1$ so that $e^{2\pi i3/100}$ is a primitive 100th root of unity; and, in this case, for each $n \in \mathbb{Z}_N$ there is a *unique* $m_n \in \mathbb{Z}_N$ such that $|A_u(m_n, n)| = 1$ and $A_u(m, n) = 0$ for each $m \neq m_n$.

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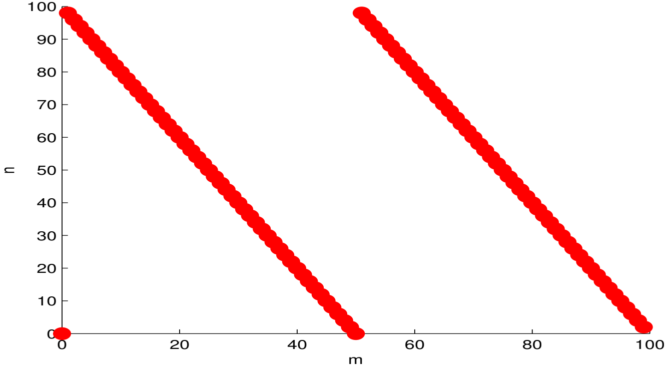
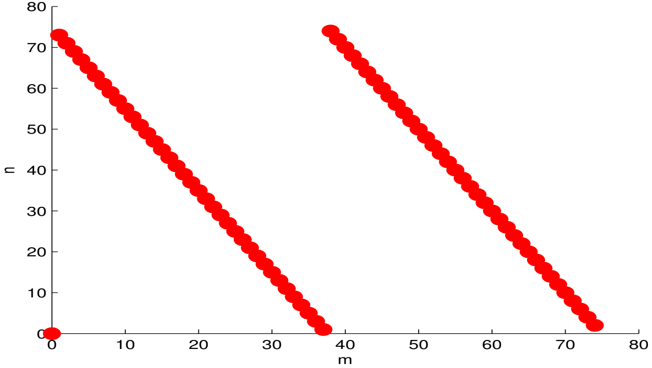
Wiener CAZAC ambiguity domain

$K = 100, j = 2$



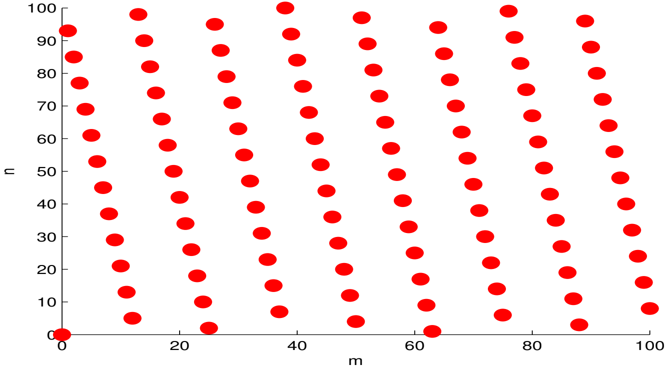
Wiener CAZAC ambiguity domain

$K = 75, j = 1$



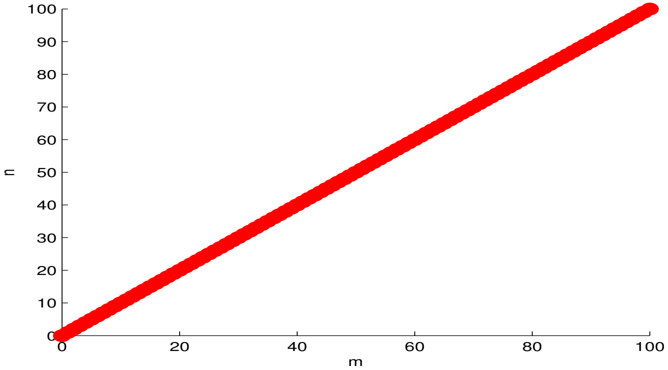
Wiener CAZAC ambiguity domain

$K = 101, j = 4$



Wiener CAZAC ambiguity domain

$K = 101, j = 50$



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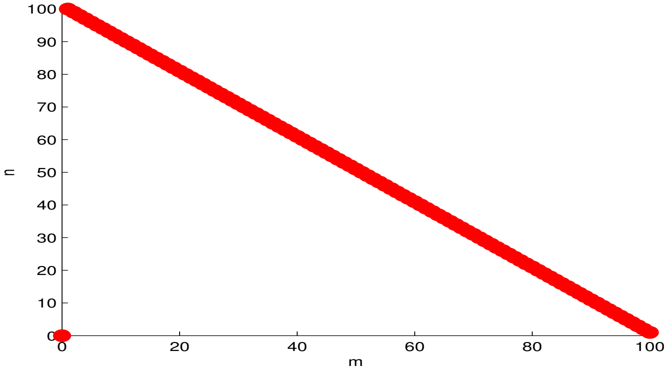
Wiener CAZAC ambiguity domain

$K = 101, j = 51$



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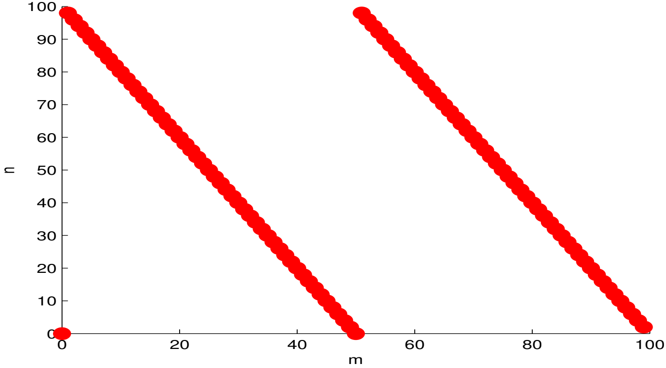
Wiener CAZAC ambiguity domain

$K = 100, j = 2$



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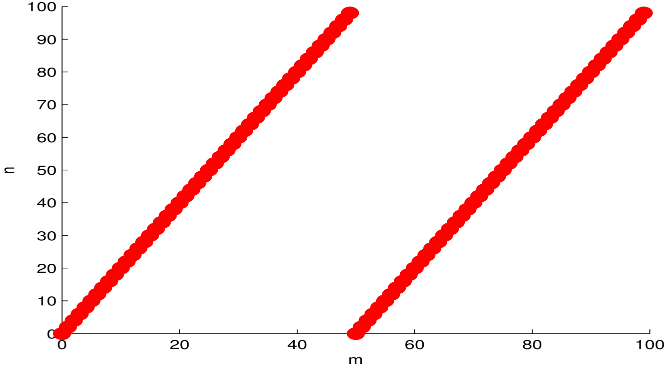
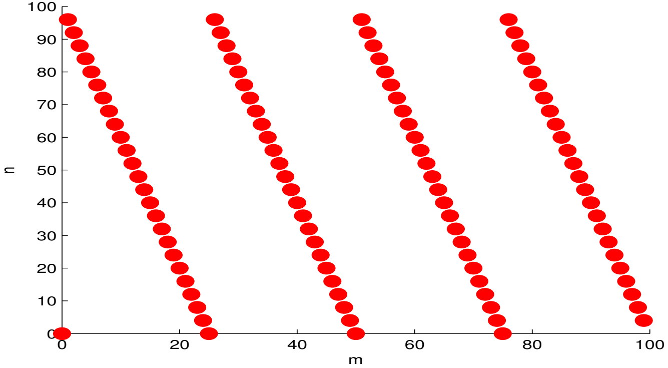
Wiener CAZAC ambiguity domain

$K = 100, j = 98$



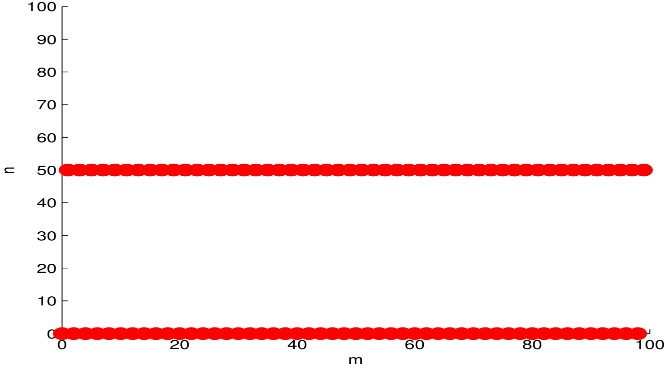
Wiener CAZAC ambiguity domain

$K = 100, j = 4$



Wiener CAZAC ambiguity domain

$K = 100, j = 50$



Perspective

Sequences for coding theory, cryptography, phase-coded waveforms, and communications (synchronization, fast start-up equalization, frequency hopping) include the following in the periodic case:

- Gauss, Wiener (1927), Zadoff (1963), Schroeder (1969), Chu (1972), Zhang and Golomb (1993)
- Frank (1953), Zadoff and Abourezk (1961), Heimiller (1961)
- Milewski (1983)
- Björck (1985) and Golomb (1992),

and their generalizations, both periodic and aperiodic.

The general problem of using codes to generate signals leads to frames.

Frames

FRAMES

Definition

A collection $(e_n)_{n\in\Lambda}$ in a Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist $0 < A \leq B < \infty$ such that

$$\forall x \in \mathcal{H},\ A\|x\|^2 \leq \sum_{n\in\Lambda} |\langle x, e_n\rangle|^2 \leq B\|x\|^2.$$

The constants A and B are the *frame bounds*. If $A = B$, the frame is an *A-tight* frame.

Frames

Definition

- Bessel (analysis) operator L: $\mathcal{H} \to \ell^2(\Lambda)$

$$Lx = (\langle x, e_n \rangle)$$

- Synthesis operator L^*, the Hilbert space adjoint of L
- Frame operator $S = L^*L : \ \mathcal{H} \to \mathcal{H}$,

$$Sx = \sum \langle x, e_n \rangle e_n.$$

  By the definition of frames, S satisfies $AI \leq S \leq BI$.
- Grammian operator $G = LL^* : \ell^2(\Lambda) \to \ell^2(\Lambda)$.

Frames

$AI \leq S \leq BI$ implies that S is invertible and that $B^{-1}I \leq S^{-1} \leq A^{-1}I$.

Definition

Let $F = \{e_n\}$ be a frame, and let $\tilde{e}_n = S^{-1}e_n$. $\tilde{F} = \{\tilde{e}_n\}$ is the *dual frame* of F.

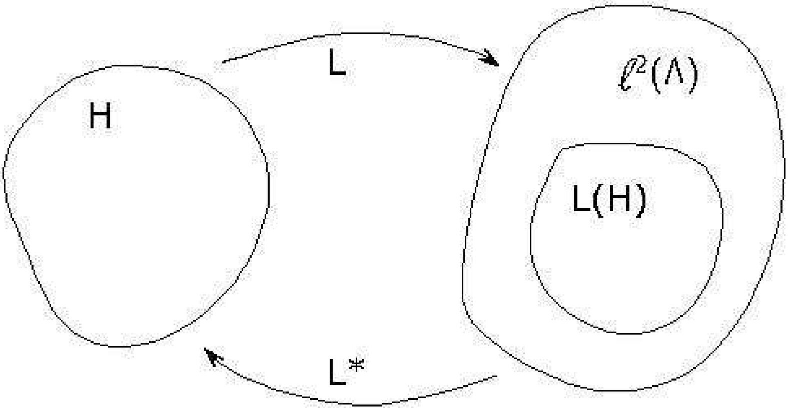
- $\sum \langle x, e_n \rangle \tilde{e}_n = S^{-1} \sum \langle x, e_n \rangle e_n = S^{-1}Sx = x$.
- $\sum \langle x, \tilde{e}_n \rangle e_n = \sum \langle S^{-1}x, e_n \rangle e_n = SS^{-1}x = x$.
- The frame operator of $\tilde{F}$ is S^{-1} since

$$\sum \langle x, \tilde{e}_n \rangle \tilde{e}_n = S^{-1} \sum \langle S^{-1}x, e_n \rangle e_n = S^{-1}SS^{-1}x = S^{-1}x.$$

- $\sum |\langle x, \tilde{e}_n \rangle|^2 = \langle S^{-1}x, x \rangle$. Then,

$$B^{-1}\|x\|^2 \leq \sum |\langle x, \tilde{e}_n \rangle|^2 \leq A^{-1}\|x\|^2.$$

Frames



Frames

Theorem

Let H be a Hilbert space.

$$\{e_n\}_{n\in\Lambda} \subseteq H \text{ is } A\text{-tight} \Leftrightarrow S = AI,$$

where I is the identity operator.

Proof. ($\Rightarrow$) If $S = L^*L = AI$, then $\forall x \in H$

$$\begin{aligned} A\|x\|^2 &= A\langle x, x\rangle = \langle Ax, x\rangle = \langle Sx, x\rangle \\ &= \langle L^*Lx, x\rangle = \langle Lx, Lx\rangle \\ &= \|Ly\|^2_{l^2(\Lambda)} \\ &= \sum_{i\in\Lambda} |\langle x, e_i\rangle|^2. \end{aligned}$$

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Frames

Proof. ($\Leftarrow$) If $\{e_i\}_{i\in\Lambda}$ is A-tight, then $\forall x \in H$, $A\langle x, x\rangle$ is

$$A\|x\|^2 = \sum_{i\in K} |\langle x, e_i\rangle|^2 = \sum_{i\in K} \langle x, e_i\rangle\langle e_i, x\rangle = \left\langle \sum_{i\in K} \langle x, e_i\rangle e_i, x \right\rangle = \langle Sx, x\rangle.$$

Therefore,

$$\forall x \in H, \quad \langle (S - AI)x, x\rangle = 0.$$

In particular, $S - AI$ is Hermitian and positive semi-definite, so

$$\forall x, y \in H, \quad |\langle (S - AI)x, y\rangle| \leq \sqrt{\langle (S - AI)x, x\rangle\langle (S - AI)y, y\rangle} = 0.$$

Thus, $(S - AI) = 0$, so, $S = AI$.

Theorem (Vitali, 1921)

Let H be a Hilbert space, $\{e_n\} \subseteq H$, $\|e_n\| = 1$.

$$\{e_n\} \text{ is 1-tight} \Leftrightarrow \{e_n\} \text{ is an ONB.}$$

Proof. If $\{e_n\}$ is 1-tight, then $\forall y \in H$

$$\|y\|^2 = \sum_n |\langle y, e_n\rangle|^2.$$

Since each $\|e_n\| = 1$, we have

$$1 = \|e_n\|^2 = \sum_k |\langle e_n, e_k\rangle|^2 = 1 + \sum_{k, k\neq n} |\langle e_n, e_k\rangle|^2$$

$$\Rightarrow \sum_{k\neq n} |\langle e_n, e_k\rangle|^2 = 0 \Rightarrow \forall n \neq k,\ \langle e_n, e_k\rangle = 0$$

Finite frames

Frames $F = \{e_n\}_{n=1}^N$ for d-dimensional Hilbert space H, e.g., $H = \mathbb{K}^d$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$.

- Any spanning set of vectors in $\mathbb{K}^d$ is a *frame* for $\mathbb{K}^d$.
- If $\{e_n\}_{n=1}^N$ is a finite unit norm tight frame (FUNTF) for $\mathbb{K}^d$, with frame constant A, then $A = N/d$.
- $\{e_n\}_{n=1}^d$ is a A-tight frame for $\mathbb{K}^d$, then it is a $\sqrt{A}$-normed orthogonal set.

Properties and examples of FUNTFs

- Frames give redundant signal representation to compensate for hardware errors, to ensure numerical stability, and to minimize the effects of noise.
- Thus, if certain types of noises are known to exist, then the FUNTFs are constructed using this information.
- Orthonormal bases, vertices of Platonic solids, kissing numbers (sphere packing and error correcting codes) are FUNTFs.
- The vector-valued CAZAC – FUNTF problem: Characterize $u : \mathbb{Z}_K \longrightarrow \mathbb{C}^d$ which are CAZAC FUNTFs.

FUNTF

- A set $F = \{e_j\}_{j \in J} \subseteq \mathbb{F}^d$ is a *frame* for $\mathbb{F}^d$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, if

$$\exists\, A, B > 0 \quad \text{such that} \quad \forall\, x \in \mathbb{F}^d, \quad A\|x\|^2 \leq \sum_{j \in J} |\langle x, e_j \rangle|^2 \leq B\|x\|^2.$$

- F *tight* if $A = B$. A finite unit-norm tight frame F is a FUNTF.
- N row vectors from any fixed $N \times d$ submatrix of the $N \times N$ DFT matrix, $\frac{1}{\sqrt{d}}(e^{2\pi imn/N})$, is a FUNTF for $\mathbb{C}^d$.
- If F is a FUNTF for $\mathbb{F}^d$, then

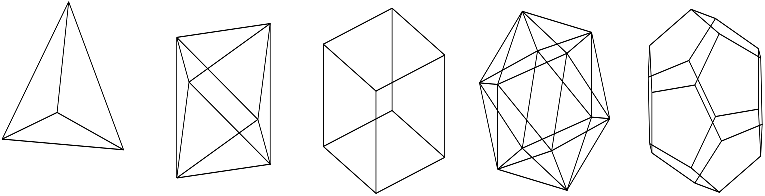
$$\forall x \in \mathbb{F}^d, \quad x = \frac{d}{N} \sum_{j=1}^{N} \langle x, e_j \rangle e_j.$$

- Frames: redundant representation, compensate for hardware errors, inexpensive, numerical stability, minimize effects of noise.

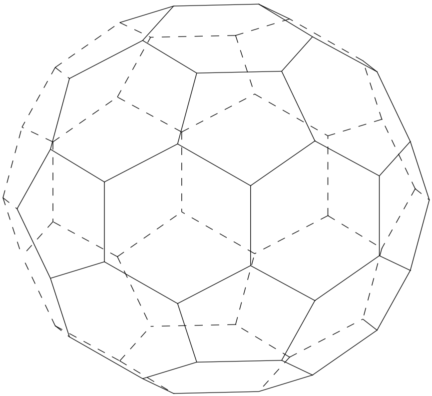
Recent applications of FUNTFs

- Robust transmission of data over erasure channels such as the internet [Casazza, Goyal, Kelner, Kovačević]
- Multiple antenna code design for wireless communications [Hochwald, Marzetta,T. Richardson, Sweldens, Urbanke]
- Multiple description coding [Goyal, Heath, Kovačević, Strohmer,Vetterli]
- Quantum detection [Bölcskei, Eldar, Forney, Oppenheim, Kebo, B]
- Grassmannian ”min-max” waveforms [Calderbank, Conway, Sloane, et al., Kolesar, B]

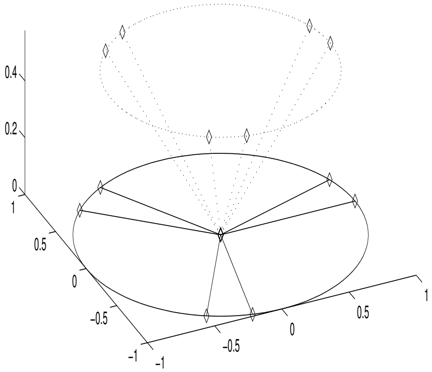
Recent applications of FUNTFs



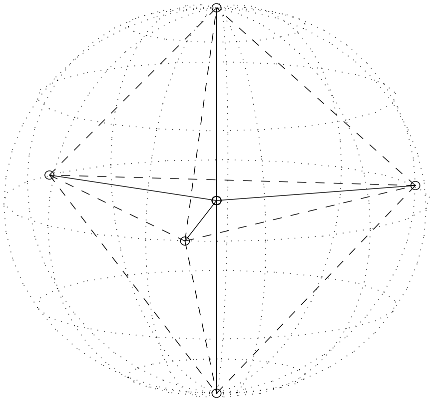
Recent applications of FUNTFs



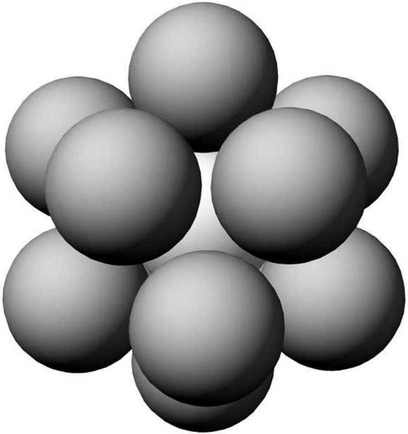
Recent applications of FUNTFs



Recent applications of FUNTFs



Recent applications of FUNTFs



DFT FUNTFs

- $N \times d$ submatrices of the $N \times N$ DFT matrix are FUNTFs for $\mathbb{C}^d$. These play a major role in finite frame $\Sigma\Delta$-quantization.

$$N = 8,\ d = 5 \qquad \frac{1}{\sqrt{5}} \begin{bmatrix} * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \end{bmatrix}$$

$$x_m = \frac{1}{5}(e^{2\pi i \frac{m}{8}}, e^{2\pi i m \frac{2}{8}}, e^{2\pi i m \frac{5}{8}}, e^{2\pi i m \frac{6}{8}}, e^{2\pi i m \frac{7}{8}})$$

$$m = 1, \ldots, 8.$$

- Sigma-Delta Super Audio CDs - but not all authorities are fans.

Naimark Theorem

Definition

Let H be a Hilbert space, $V \subseteq H$ a closed subspace, and

$$V^{\perp} = \{z \in H : \forall y \in V,\ \langle z, y \rangle = 0\}$$

be its orthogonal complement. Then, for every $x \in H$, there is a unique $y \in V$ satisfying

$$\|x - y\| = \min\{\|x - y'\| : y' \in V\},$$

and a unique $z \in V^{\perp}$ such that $x = y + z$.
The map $P_V : H \to V$, $P_V x = y$ is the *orthogonal projection* on V.

If $\{v_n\}$ is an orthonormal basis for V, then P_V can be expressed as

$$\forall x \in H, \quad P_V x = \sum_n \langle x, v_n \rangle v_n.$$

Naimark Theorem

Can we make tight frames for $H = \mathbb{F}^d$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) with prescribed redundancy?

Yes. Take an $N \times N$ unitary matrix U, and choose any d columns of it to form an $N \times d$ matrix L. Then, $L^* L = I$, which means, the rows of L form a 1-tight frame for $\mathbb{F}^d$.

How about FUNTFs?

Yes, we shall explain how to generate FUNTFs by using the frame potential.

Naimark Theorem

If $\{e_n\}_{n=1}^N$ is an A-tight frame for $\mathbb{F}^d$, and L is its Bessel map, then $L^*L = AI$, i.e., the set of the columns of L, $\{c_1, \ldots, c_d\}$ is a $\sqrt{A}$-normed orthogonal set in $\mathbb{F}^N$. Let $V = span\{c_1, \ldots, c_d\}$, and let $\{c_{d+1}, \ldots, c_N\}$ be a $\sqrt{A}$-normed orthogonal basis for $V^\perp$. Then, the matrix

$$U = A^{-1/2}[c_1 \ldots c_N]$$

is a unitary matrix, since its columns give an ONB for $\mathbb{F}^d$. Then, the rows of U also give an ONB for $\mathbb{F}^d$. Let $\widetilde{e}_k$ be the kth row of $A^{1/2}U$. Then,

1. $\{\widetilde{e}_k\}$ is a $\sqrt{A}$-normed orthogonal basis for $\mathbb{F}^N$,
2. $e_k = P\widetilde{e}_k$, where $P : \mathbb{F}^N \to \mathbb{F}^d$,

$$P(x[1], \ldots x[N]) = (x[1], \ldots, x[d]).$$

Naimark Theorem

Theorem (Naimark)

Let H be a d-dimensional Hilbert space, $\{e_n\}_{n=1}^N$ be an A-tight frame for H. Then there exists an N-dimensional Hilbert space $\widetilde{H}$, and orthogonal A-normed set $\{\widetilde{e}_n\}_{n=1}^N \subseteq \widetilde{H}$ such that

$$P_H \widetilde{e}_n = e_n$$

where P_H is the orthogonal projection onto H.

The geometry of finite tight frames

- We saw the vertices of platonic solids are FUNTFs.
- However, points that constitute FUNTFs do not have to be equidistributed, e.g., ONBs and Grassmanian frames.
- FUNTFs can be characterized as minimizers of a frame potential function (with Fickus) analogous to Coulomb's Law.

Frame force and potential energy

$$F : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R}^d$$

$$P : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R},$$

where $P(a,b) = p(\|a-b\|)$, $\qquad p'(x) = -xf(x)$

- Coulomb force

$$CF(a,b) = (a-b)/\|a-b\|^3, \quad f(x) = 1/x^3$$

- Frame force

$$FF(a,b) = \langle a,b\rangle(a-b), \quad f(x) = 1 - x^2/2$$

- Total potential energy for the frame force

$$TFP(\{x_n\}) = \sum_{m=1}^{N}\sum_{n=1}^{N} |\langle x_m, x_n\rangle|^2$$

Characterization of FUNTFs

Theorem

Let $N \leq d$. The minimum value of *TFP*, for the frame force and N variables, is N; and the *minimizers* are precisely the orthonormal sets of N elements for $\mathbb{R}^d$.

Let $N \geq d$. The minimum value of *TFP*, for the frame force and N variables, is N^2/d; and the *minimizers* are precisely the FUNTFs of N elements for $\mathbb{R}^d$.

Problem

Find FUNTFs analytically, effectively, computationally.

Construction of FUNTFs

Suppose we want to construct a FUNTF for $\mathbb{F}^d$.

- If $\mathbb{F} = \mathbb{R}$, Let $(x_1, x_2, \ldots, x_N)$ denote a point in $\mathbb{R}^{Nd}$, where each $x_k \in \mathbb{R}^d$. The solutions of the following constrained minimization problem are FUNTFs.

$$\begin{aligned} &\textit{minimize} \qquad TFP(x_1, x_2, \ldots, x_N) = \sum_{m=1}^{N} \sum_{n=1}^{N} |\langle x_m, x_n \rangle|^2 \qquad (1) \\ &\textit{subject to} \qquad \|x_n\|^2 = 1, \quad \forall n = 1, \ldots, N. \end{aligned}$$

  If we view TFP as a function from $\mathbb{R}^{Nd}$ into $\mathbb{R}$, then it is twice differentiable in each argument, so are the constraints. We can solve this problem numerically, e.g., by using Conjugate Gradient minimization algorithm.

- If $\mathbb{F} = \mathbb{C}$, we let $(Re(x_1), Im(x_1), \ldots, Re(x_N), Im(x_N))$ denote a point in $\mathbb{R}^{2Nd}$, view TFP as a function from $\mathbb{R}^{2Nd}$ into $\mathbb{R}$, and solve (1) as in the real case.

Björck CAZAC codes and ambiguity function comparisons

Legendre symbol

Let N be a prime and $(k, N) = 1$.

- k is a quadratic residue mod N if $x^2 = k$ (mod N) has a solution.
- k is a quadratic non–residue mod N if $x^2 = k$ (mod N) has no solution.
- The Legendre symbol:

$$\left(\frac{k}{N}\right) = \begin{cases} 1, & \text{if } k \text{ is a quadratic residue mod } N \text{ ,} \\ -1, & \text{if } k \text{ is a quadratic non–residue mod } N. \end{cases}$$

The diagonal of the product table of $\mathbb{Z}_N$ gives values $k \in \mathbb{Z}$ which are squares. As such we can program Legendre symbol computation.

Example: $N = 7$. $(\frac{k}{N}) = 1$ if $k = 1, 2, 4$.

Definition

Let N be a prime number. A *Björck CAZAC sequence* of length N is

$$u[k] = e^{i\theta_N(k)}, \quad k = 0, 1, \ldots, N-1,$$

where, for $N = 1 \pmod 4$,

$$\theta_N(k) = \arccos\left(\frac{1}{1+\sqrt{N}}\right)\left(\frac{k}{N}\right),$$

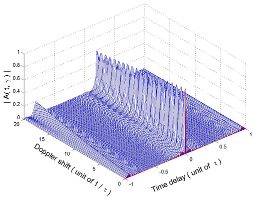
and, for $N = 3 \pmod 4$,

$$\theta_N(k) = \frac{1}{2}\arccos\left(\frac{1-N}{1+N}\right)\left[(1-\delta_k)\left(\frac{k}{N}\right) + \delta_k\right].$$

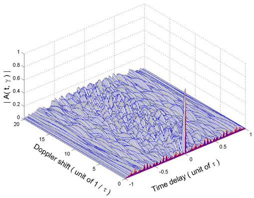
δ_k is Kronecker delta and $\left(\frac{k}{N}\right)$ is Legendre symbol.

Quadratic and Björck ambiguity comparison

- Waveforms associated to Chu-Zadoff and P4 CAZACs are known for their low sidelobes at zero Doppler shift, but their ambiguity functions exhibit strong coupling in the time-frequency plane.
- Waveforms associated to Björck CAZACs can more effectively decouple the effect of time and frequency shifts. However, at zero Doppler shift, their sidelobe behavior is less desirable than quadratic phase CAZACs.
- These differences led to our concatenation idea.



Chu-Zadoff 101



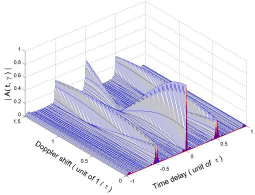
Björck 101

Definition

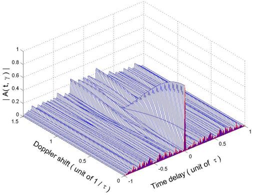
- A *concatenation of partial* CAZACs u and v is $w = Mix(r\%, u, v)$ defined as
$$w[m] = u[m], \text{ if } m = 0, \dots, M$$
and
$$w[m] = v[m], \text{ if } m = M + 1, \dots, N - 1,$$
where M is the nearest integer to $r \times N/100$.
- We show how the ambiguity function can be improved by concatenation of partial CAZACs belonging to two different families. The best choice is obtained with $r = 50$.

Example

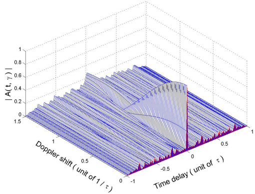
Ambiguity function of a partial concatenation.



Wiener 67



Björck 67

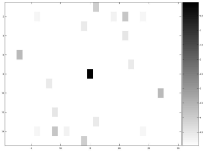


Mix(50%,Wiener 67,Björck 67)

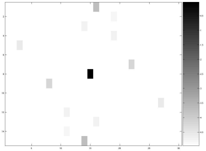
Diversity by averaging technique

- Shifting Wiener CAZACs leads to the same type of discrete aperiodic ambiguity function, i.e., $|A_a(u(\cdot - k_0))| = |A_a(u)|$.
- Discrete aperiodic ambiguity functions of shifted Björck CAZACs exhibit diversity in both the size and location of their sidelobes.
- New families of CAZAC sequences are developed by an averaging technique based on shifting Björck CAZAC sequences.
- This technique is exploited using non-coherent processing (averaging absolute values) in order to achieve lower sidelobe levels.

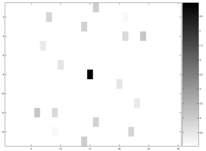
The discrete aperiodic ambiguity function



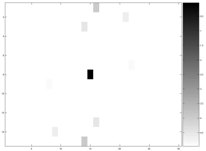
Zero shift



Shift by 7



Shift by 12



Average

Shifted sequences of a Björck CAZAC of length 29; threshold at -10 dB; darker color denotes higher value.

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Shapiro-Rudin polynomials

Shapiro-Rudin polynomials

- The *Shapiro-Rudin polynomials*, $P_n(t)$, $Q_n(t)$, $n = 0, 1, 2, ...$, are defined recursively in the following manner. For $t \in \mathbb{R}/\mathbb{Z}$,

$$\begin{aligned} P_0(t) &= Q_0(t) = 1, \\ P_{n+1}(t) &= P_n(t) + e^{2\pi i 2^n t} Q_n(t), \\ Q_{n+1}(t) &= P_n(t) - e^{2\pi i 2^n t} Q_n(t). \end{aligned}$$

- The number of terms in the n^{th} polynomial, $P_n(t)$ or $Q_n(t)$, is 2^n.
- Thus, the coefficients of each polynomial can be represented as a finite sequence of length 2^n of (± 1)s.

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Golay complementary pairs

- For any sequence $z = \{z_k\}_{k=0}^{n-1} \subseteq \mathbb{C}$ and for any $m \in \{0, 1, ..., n-1\}$, the m^{th} *aperiodic autocorrelation coefficient*, $A_z(m)$, is defined as

$$A_z(m) = \sum_{j=0}^{n-1-m} z_j \overline{z_{m+j}}.$$

- Two sequences, $p = \{p_k\}_{k=0}^{n-1}, q = \{q_k\}_{k=0}^{n-1} \subseteq \mathbb{C}$, are a *Golay complementary pair* if $A_p(0) + A_q(0) \neq 0$, and,

$$\forall m = 1, 2, ..., n-1, \qquad A_p(m) + A_q(m) = 0.$$

- For each n, the coefficients of P_n and Q_n, resp., are a Golay complementary pair.

Cusps

- A parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, defined by $\gamma(t) = (u(t), v(t))$, has a *non-regular point* at $t = t_0$ if $\frac{du}{dt}|_{t=t_0} = \frac{dv}{dt}|_{t=t_0} = 0$. Otherwise, t_0 is a *regular point*.
- A non-regular point t_0 gives rise to a *quadratic cusp* for γ if $\left(\frac{d^2u}{dt^2}|_{t=t_0}, \frac{d^2v}{dt^2}|_{t=t_0}\right) \neq (0,0)$.
- A non-regular point t_0 gives rise to an *ordinary cusp* if it gives rise to a quadratic cusp, and $\left(\frac{d^2u}{dt^2}|_{t=t_0}, \frac{d^2v}{dt^2}|_{t=t_0}\right)$ and $\left(\frac{d^3u}{dt^3}|_{t=t_0}, \frac{d^3v}{dt^3}|_{t=t_0}\right)$ are linearly independent vectors of the real vector space $\mathbb{R}^2$.
- Let $P(z) = z^2 - 2z$ on $\mathbb{C}$, and define $\gamma(t) = P(e^{2\pi it})$. Then, γ has a non-regular point at $t = t_0$ and gives rise to a quadratic cusp there.

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Theorem

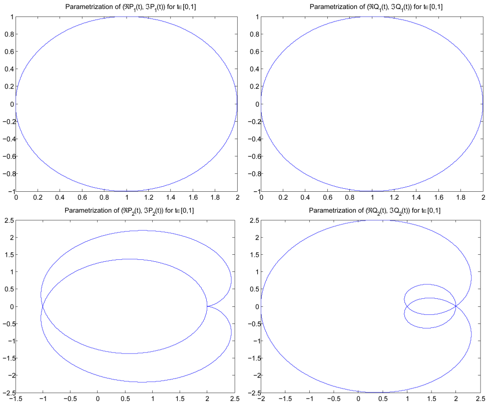
a. For each $n \in \mathbb{N}$, the parametrization $(Re\,(P_{2n}(t))\,, Im\,(P_{2n}(t)))$ gives rise to a quadratic cusp at $(2^n, 0)$, i.e., when $t = 0$.
b. Further, neither $(Re\,(P_{2n+1}(t))\,, Im\,(P_{2n+1}(t)))$ nor $(Re\,(Q_n(t))\,, Im\,(Q_n(t)))$ gives rise to a cusp when $t = 0$.

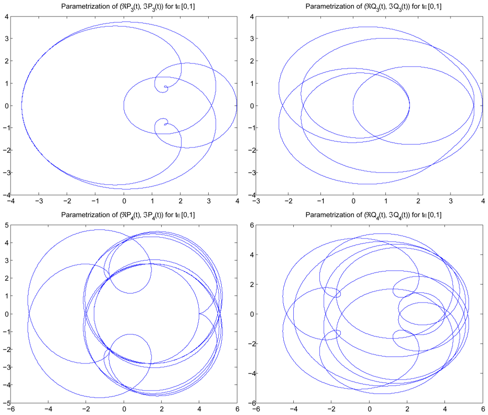
Remark

The Theorem does not contradict the fact that $P_{2n} : \mathbb{R} \longrightarrow \mathbb{C}$ is infinitely differentiable as a 1-periodic polynomial on $\mathbb{R}$.

Graphs of $P_n(t)$ and $Q_n(t)$ for n=1,2,3,4

Graphical parametrizations of $P_n(t)$ and $Q_n(t)$ by means of $(Re\,(P_n(t))\,,Im\,(P_n(t)))$ and $(Re\,(Q_n(t))\,,Im\,(Q_n(t)))$ for $n = 1, 2, 3, 4$.





Parametrization of ($\Re P_3(t)$, $\Im P_3(t)$) for $t\in[0,1]$
Parametrization of ($\Re Q_3(t)$, $\Im Q_3(t)$) for $t\in[0,1]$
Parametrization of ($\Re P_4(t)$, $\Im P_4(t)$) for $t\in[0,1]$
Parametrization of ($\Re Q_4(t)$, $\Im Q_4(t)$) for $t\in[0,1]$

A vector-valued ambiguity function

Outline

1. Problem and goal
2. Frames
3. Multiplication problem and A_p^1
4. $A_p^d : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}^d, u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$
5. $A_p^d(u)$ for DFT frames
6. Figure
7. Epilogue

Background

- Originally, our problem was to construct libraries of phase-coded waveforms v parameterized by design variables, for communications and radar.
- A goal was to achieve diverse ambiguity function behavior of v by defining new classes of quadratic phase and number theoretic perfect autocorrelation codes u with which to define v.
- A realistic more general problem was to construct vector-valued waveforms v in terms of vector-valued perfect autocorrelation codes u. Such codes are relevant in light of vector sensor and MIMO capabilities and modeling.
- Example: Discrete time data vector $u(k)$ for a d-element array,

$$k \longmapsto u(k) = (u_0(k), \ldots, u_{d-1}(k)) \in \mathbb{C}^d.$$

We can have $\mathbb{R}^N \to GL(d, \mathbb{C})$, or even more general.

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General problem and STFT theme

- Establish the theory of vector-valued ambiguity functions to estimate v in terms of ambiguity data.
- First, establish this estimation theory by defining the discrete periodic vector-valued ambiguity function in a natural way.
- Mathematically, this natural way is to formulate the discrete periodic vector-valued ambiguity function in terms of the Short Time Fourier Transform (STFT).

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STFT and ambiguity function

Short time Fourier transform – STFT

- The narrow band cross-correlation ambiguity function of v, w defined on $\mathbb{R}$ is

$$A(v,w)(t,\gamma) = \int_{\mathbb{R}} v(s+t)\overline{w(s)}e^{-2\pi i s\gamma}\,ds.$$

- $A(v,w)$ is the STFT of v with window w.
- The *narrow band radar ambiguity function* $A(v)$ of v on $\mathbb{R}$ is

$$A(v)(t,\gamma) = \int_{\mathbb{R}} v(s+t)\overline{v(s)}e^{-2\pi i s\gamma}\,ds$$

$$= e^{\pi i t\gamma}\int_{\mathbb{R}} v\left(s+\frac{t}{2}\right)\overline{v\left(s-\frac{t}{2}\right)}e^{-2\pi i s\gamma}\,ds, \text{ for } (t,\gamma)\in\mathbb{R}^2.$$

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Goal

- Let v be a phase coded waveform with N lags defined by the code u.
- Let u be N-periodic, and so $u : \mathbb{Z}_N \longrightarrow \mathbb{C}$, where $\mathbb{Z}_N$ is the additive group of integers modulo N.
- The *discrete periodic ambiguity function* $A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}$ is

$$A_p(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m+k)\overline{u(k)} e^{-2\pi i k n/N}.$$

Goal

Given a vector valued N-periodic code $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, construct the following in a meaningful, computable way:

- Generalized $\mathbb{C}$-valued periodic ambiguity function $A_p^1(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}$
- $\mathbb{C}^d$-valued periodic ambiguity function $A_p^d(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}^d$

The STFT is the *guide* and the *theory of frames* is the technology to obtain the goal.

Outline

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Multiplication problem

- Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$.
- If $d = 1$ and $e_n = e^{2\pi i n/N}$, then

$$A_p(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k)e_{nk}\rangle.$$

Multiplication problem

To characterize sequences $\{E_k\} \subseteq \mathbb{C}^d$ and multiplications $*$ so that

$$A_p^1(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * E_{nk}\rangle \in \mathbb{C}$$

is a meaningful and well-defined *ambiguity function.* This formula is clearly motivated by the STFT.

Ambiguity function assumptions

There is a natural way to address the multiplication problem motivated by the fact that $e_m e_n = e_{m+n}$. To this end, we shall make the *ambiguity function assumptions*:

- $\exists \{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ and a multiplication $*$ such that $E_m * E_n = E_{m+n}$ for $m, n \in \mathbb{Z}_N$;
- $\{E_k\}_{k=0}^{N-1} \subseteq \mathbb{C}^d$ is a tight frame for $\mathbb{C}^d$;
- $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$ is bilinear, in particular,

$$\left(\sum_{j=0}^{N-1} c_j E_j\right) * \left(\sum_{k=0}^{N-1} d_k E_k\right) = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} c_j d_k E_j * E_k.$$

Calculation

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Given $u, v : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ and $m, n \in \mathbb{Z}_N$.
- Then, one calculates

$$u(m) * v(n) \quad = \quad \frac{d^2}{N^2} \sum_{j=0}^{N-1} \sum_{s=0}^{N-1} \langle u(m), E_j \rangle \langle v(n), E_s \rangle E_{j+s}.$$

Outline

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$A^1_p(u)$ for DFT frames

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Further, assume that $\{E_j\}_{j=0}^{N-1}$ is a DFT frame, and let r designate a fixed column.
- Without loss of generality, choose the first d columns of the $N \times N$ DFT matrix.
- Then, one calculates

$$\begin{aligned} E_m * E_n(r) &= \frac{d^2}{N^2} \sum_{j=0}^{N-1} \sum_{s=0}^{N-1} \langle E_m, E_j \rangle \langle E_n, E_s \rangle E_{j+s}(r). \\ &= \frac{e_{(m+n)r}}{\sqrt{d}} = E_{m+n}(r). \end{aligned}$$

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$A_p^1(u)$ for DFT frames

- Thus, for DFT frames, $*$ is componentwise multiplication in $\mathbb{C}^d$ with a factor of $\sqrt{d}$.
- In this case $A_p^1(u)$ is well-defined for $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ by

$$\begin{aligned} A_p^1(u)(m,n) &= \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * E_{nk}\rangle \\ &= \frac{d}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\langle E_j, u(k)\rangle\langle u(m+k), E_{j+nk}\rangle. \end{aligned}$$

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Remark

- In the previous DFT example, $*$ is intrinsically related to the "addition" defined on the indices of the frame elements, viz., $E_m * E_n = E_{m+n}$.
- Alternatively, we could have $E_m * E_n = E_{m \bullet n}$ for some function $\bullet : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{Z}_N$, and, thereby, we could use frames which are not FUNTFs.
- Given a bilinear multiplication $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$, we can find a frame $\{E_j\}_j$ and an index operation $\bullet$ with the $E_m * E_n = E_{m \bullet n}$ property.
- If $\bullet$ is the multiplication for a group, possibly non-abelian and/or infinite, we may reverse the process and find a FUNTF and bilinear multiplication $*$ with the $E_m * E_n = E_{m \bullet n}$ property.

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$A^1_p(u)$ for cross product frames

- Take $* : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$ and let $\{i, j, k\}$ be the standard basis.
- $i * j = k$, $j * i = -k$, $k * i = j$, $i * k = -j$, $j * k = i$, $k * j = -i$, $i * i = j * j = k * k = 0$. $\{0, i, j, k, -i, -j, -k, \}$ is a tight frame for $\mathbb{C}^3$ with frame constant 2. Let $E_0 = 0, E_1 = i, E_2 = j, E_3 = k, E_4 = -i, E_5 = -j, E_6 = -k$.
- The index operation corresponding to the frame multiplication is the non-abelian operation $\bullet : \mathbb{Z}_7 \times \mathbb{Z}_7 \longrightarrow \mathbb{Z}_7$, where $1 \bullet 2 = 3, 2 \bullet 1 = 6, 3 \bullet 1 = 2, 1 \bullet 3 = 5, 2 \bullet 3 = 1, 3 \bullet 2 = 4, 1 \bullet 1 = 2 \bullet 2 = 3 \bullet 3 = 0, n \bullet 0 = 0 \bullet n = 0, 1 \bullet 4 = 0, 1 \bullet 5 = 6, 1 \bullet 6 = 2, 4 \bullet 1 = 0, 5 \bullet 1 = 3, 6 \bullet 1 = 5, 2 \bullet 4 = 3, 2 \bullet 5 = 0$, etc.
- The three ambiguity function assumptions are valid and so we can write the cross product as

$$u \times v = u * v = \frac{1}{2^2} \sum_{s=1}^{6} \sum_{t=1}^{6} \langle u, E_s \rangle \langle v, E_t \rangle E_{s \bullet t}.$$

- Consequently, $A^1_p(u)$ can be well-defined.

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Vector-valued ambiguity function $A_p^d(u)$

- Let $\{E_j\}_j^{N-1} \subseteq \mathbb{C}^d$ satisfy the three ambiguity function assumptions.
- Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$.
- The following definition is clearly *motivated* by the STFT.

Definition

$A_p^d(u) : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ is defined by

$$A_p^d(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} u(m+k) * \overline{u(k)} * \overline{E_{nk}}.$$

Outline

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STFT formulation of $A_p(u)$

- The discrete periodic ambiguity function of $u : \mathbb{Z}_N \longrightarrow \mathbb{C}$ can be written as

$$A_p(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} \langle \tau_m u(k), F^{-1}(\tau_n \hat{u})(k) \rangle,$$

  where $\tau_(m)u(k) = u(m+k)$ is translation by m and $F^{-1}(u)(k)) = \check{u}(k)$ is Fourier inversion.
- As such we see that $A_p(u)$ has the form of a STFT.
- We shall develop a vector-valued DFT theory to *verify* (not just *motivate*) that $A_p^d(u)$ is an STFT in the case $\{E_k\}_{k=0}^{N-1}$ is a DFT frame for $\mathbb{C}^d$.

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DFT frames and the vector-valued DFT

Definition

Given $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, and let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$. The *vector-valued discrete Fourier transform* of u is

$$\forall\, n \in \mathbb{Z}_N, \quad F(u)(n) = \hat{u}(n) = \sum_{m=0}^{N-1} u(m) * E_{mn},$$

where $*$ is pointwise (coordinatewise) multiplication.

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Vector-valued Fourier inversion theorem

- Inversion process for the vector-valued case is analogous to the 1-dimensional case.
- We must define a new multiplication in the frequency domain to avoid divisibility problems.
- Define the weighted multiplication $(*) : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$ by $u(*)v = u * v * \omega$ where $\omega = (\omega_1, \ldots, \omega_d)$ has the property that each $\omega_n = \frac{1}{\#\{m \in \mathbb{Z}_N : mn = 0\}}$.
- For the following theorem assume $d << N$ or N prime.

Theorem - Vector-valued Fourier inversion

The vector valued Fourier transform F is an isomorphism from $\ell^2(\mathbb{Z}_N)$ to $\ell^2(\mathbb{Z}_N, \omega)$ with inverse

$$\forall\ m \in \mathbb{Z}_N, \quad F^{-1}(m) = u(m) = \frac{d}{N} \sum_{n=0}^{N-1} \hat{u}(n) * E_{-mn} * \omega.$$

N prime implies F is unitary.

$A_p^d(u)$ as an STFT

- Given $u, v : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$, and let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$.
- $u * \overline{v}$ denotes pointwise (coordinatewise) multiplication with a factor of $\sqrt{d}$.
- We compute

$$A_p^d(u)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} (\tau_m u(k)) * \overline{F^{-1}(\tau_n \hat{u})(k)}.$$

- Thus, $A_p^d(u)$ is compatible with point of view of defining a vector-valued ambiguity function in the context of the STFT.

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Outline

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Epilogue

- If $(G, \bullet)$ is a finite group with representation $\rho : G \longrightarrow GL(\mathbb{C}^d)$, then there is a frame $\{E_n\}_{n \in G}$ and bilinear multiplication, $* : \mathbb{C}^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$, such that $E_m * E_n = E_{m \bullet n}$. Thus, we can develop $A_p^d(u)$ theory in this setting.
- Analyze ambiguity function behavior for (phase-coded) vector-valued waveforms $v : \mathbb{R} \longrightarrow \mathbb{C}^d$, defined by $u : \mathbb{Z}_N \longrightarrow \mathbb{C}^d$ as

$$v = \sum_{k=0}^{N-1} u(k) \mathbb{1}_{[kT,(k+1)T)},$$

in terms of $A_p^d(u)$. (See Figure)

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Computation of $u : \mathbb{Z}_N \to \mathbb{C}^d$ from ambiguity

- CAZAC and waveform computation of $u : \mathbb{Z}_N \to \mathbb{C}^d$ from A(u): Let A_u be the $N \times N$ matix, $(A(u)(m,n))$. Define the $N \times N$ matrix $U = (U_{i,j})$, where $U_{i,j} = \langle u(i+j), u(j) \rangle$. Then

$$U = A_u D_N, \quad \text{where} \quad D_N = DFT \text{ matrix.}$$

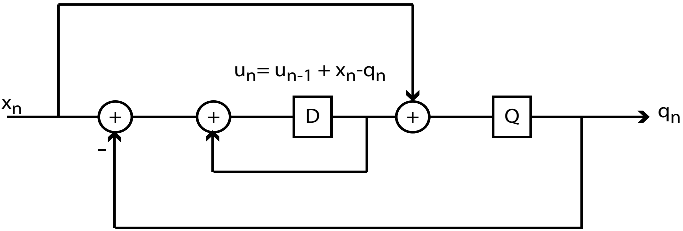
- Let $d = 1$. Note that $U_{k,0} = u(k)\overline{u(0)}$. Hence, if we know the values of the ambiguity function, and, thus, the ambiguity function matrix A_u, then the sequence u, which generates it, can be computed as long as $u(0) \neq 0$. In fact, if $u(0) = 1$ then $u(k) = (A_u D_N)(k, 0)$.
- Similar result for $A_V(u)$ using our vector-valued Fourier analysis.
- Now we can address the classical *radar ambiguity problem*: Find the structure of all $z : \mathbb{Z}_N \to \mathbb{C}^d$ for which $|A(u)| = |A(z)|$ on $X \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$.

Quantization Methods

SIGMA-DELTA QUANTIZATION

Given u_0 and $\{x_n\}_{n=1}$

$$u_n = u_{n-1} + x_n - q_n$$
$$q_n = Q(u_{n-1} + x_n)$$



First Order $\Sigma\Delta$

A quantization problem

Qualitative Problem Obtain *digital* representations for class X, suitable for storage, transmission, recovery.

Quantitative Problem Find dictionary $\{e_n\} \subseteq X$:

1. Sampling [continuous range $\mathbb{K}$ is not digital]

$$\forall x \in X, \quad x = \sum x_n e_n, \quad x_n \in \mathbb{K}.$$

2. Quantization. Construct finite alphabet $\mathcal{A}$ and

$$Q : \ X \to \{\sum q_n e_n : \ q_n \in \mathcal{A} \subseteq \mathbb{K}\}$$

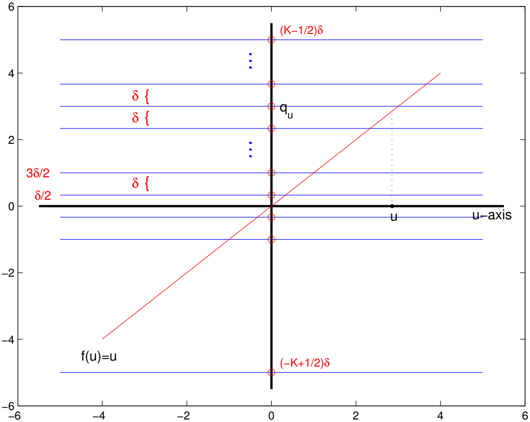
such that $|x_n - q_n|$ and/or $\|x - Qx\|$ small.

Methods

Fine quantization, e.g., PCM. Take $q_n \in \mathcal{A}$ close to given x_n.
Reasonable in 16-bit (65,536 levels)digital audio.
Coarse quantization, e.g., $\Sigma\Delta$. Use fewer bits to exploit redundancy.
SRQP

Quantization

$$\mathcal{A}_K^\delta = \{(-K+1/2)\delta, (-K+3/2)\delta, \ldots, (-1/2)\delta, (1/2)\delta, \ldots, (K-1/2)\delta\}$$



$$Q(u) = \arg\min\{|u - q| : q \in \mathcal{A}_K^\delta\} = q_u$$

PCM

Replace $x_n \leftrightarrow q_n = \arg\{\min |x_n - q| : q \in \mathcal{A}_K^\delta\}$. Then

$$(PCM) \qquad \tilde{x} = \frac{d}{N} \sum_{n=1}^{N} q_n e_n$$

satisfies

$$\|x - \tilde{x}\| \leq \frac{d}{N} \| \sum_{n=1}^{N} (x_n - q_n) e_n \| \leq \frac{d}{N} \frac{\delta}{2} \sum_{n=1}^{N} \|e_n\| = \frac{d}{2}\delta.$$

Not good!

Bennett's white noise assumption

Assume that $(\eta_n) = (x_n - q_n)$ is a sequence of independent, identically distributed random variables with mean 0 and variance $\frac{\delta^2}{12}$. Then the mean square error (MSE) satisfies

$$\text{MSE} = E\|x - \tilde{x}\|^2 \leq \frac{d}{12A}\,\delta^2 = \frac{(d\delta)^2}{12N}$$

$\mathcal{A}_1^2 = \{-1, 1\}$ and E_7

Let $x = (\frac{1}{3}, \frac{1}{2})$, $E_7 = \{(\cos(\frac{2n\pi}{7}), \sin(\frac{2n\pi}{7}))\}_{n=1}^{7}$. Consider quantizers with $\mathcal{A} = \{-1, 1\}$.

$\mathcal{A}_1^2 = \{-1, 1\}$ and E_7

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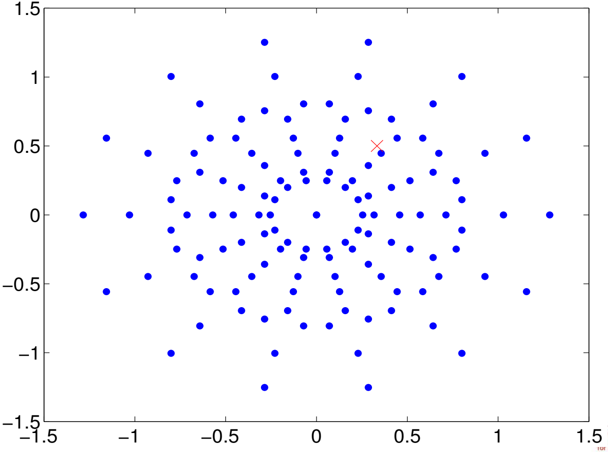

$\mathcal{A}_1^2 = \{-1, 1\}$ and E_7

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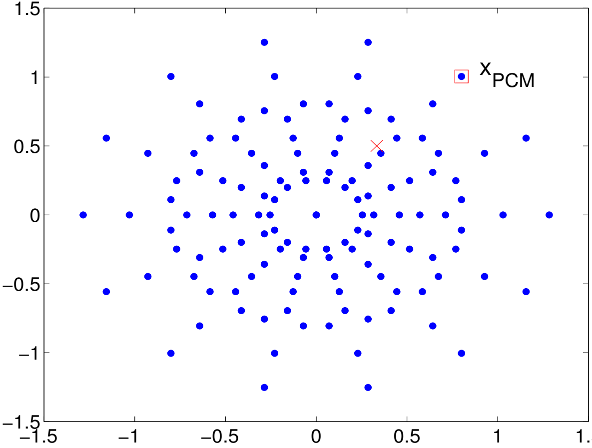

$\mathcal{A}_1^2 = \{-1, 1\}$ and E_7

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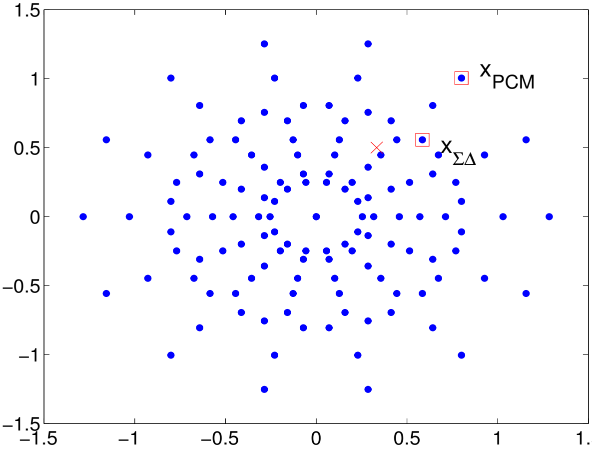

$\Sigma\Delta$ quantizers for finite frames

Let $F = \{e_n\}_{n=1}^{N}$ be a frame for $\mathbb{R}^d$, $x \in \mathbb{R}^d$.

Define $x_n = \langle x, e_n \rangle$.

Fix the ordering p, a permutation of $\{1, 2, \ldots, N\}$.

Quantizer alphabet $\mathcal{A}_K^{\delta}$

Quantizer function $Q(u) = \arg\{\min |u - q| : q \in \mathcal{A}_K^{\delta}\}$

Define the *first-order* $\Sigma\Delta$ *quantizer* with ordering p and with the quantizer alphabet $\mathcal{A}_K^{\delta}$ by means of the following recursion.

$$
\begin{aligned}
u_n - u_{n-1} &= x_{p(n)} - q_n \\
q_n &= Q(u_{n-1} + x_{p(n)})
\end{aligned}
$$

where $u_0 = 0$ and $n = 1, 2, \ldots, N$.

Sigma-Delta quantization – background

- History from 1950s.
- Treatises of Candy, Temes (1992) and Norsworthy, Schreier, Temes (1997).
- PCM for finite frames and $\Sigma\Delta$ for PW_{Ω}:
  Bölcskei, Daubechies, DeVore, Goyal, Güntürk, Kovačevič, Thao, Vetterli.
- Combination of $\Sigma\Delta$ and finite frames:
  Powell, Yılmaz, and B.
- Subsequent work based on this $\Sigma\Delta$ finite frame theory:
  Bodman and Paulsen; Boufounos and Oppenheim; Jimenez and Yang Wang; Lammers, Powell, and Yılmaz.
- Genuinely apply it.

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Stability

The following stability result is used to prove error estimates.

Proposition

If the frame coefficients $\{x_n\}_{n=1}^N$ satisfy

$$|x_n| \leq (K - 1/2)\delta, \quad n = 1, \cdots, N,$$

then the state sequence $\{u_n\}_{n=0}^N$ generated by the first-order $\Sigma\Delta$ quantizer with alphabet $\mathcal{A}_K^\delta$ satisfies $|u_n| \leq \delta/2, n = 1, \cdots, N.$

- The first-order $\Sigma\Delta$ scheme is equivalent to

$$u_n = \sum_{j=1}^{n} x_{p(j)} - \sum_{j=1}^{n} q_j, \quad n = 1, \cdots, N.$$

- Stability results lead to tiling problems for higher order schemes.

Error estimate

Definition

Let $F = \{e_n\}_{n=1}^N$ be a frame for $\mathbb{R}^d$, and let p be a permutation of $\{1, 2, \ldots, N\}$. The *variation* $\sigma(F, p)$ is

$$\sigma(F, p) = \sum_{n=1}^{N-1} \|e_{p(n)} - e_{p(n+1)}\|.$$

Error estimate

Theorem

Let $F = \{e_n\}_{n=1}^{N}$ be an A-FUNTF for $\mathbb{R}^d$. The approximation

$$\tilde{x} = \frac{d}{N} \sum_{n=1}^{N} q_n e_{p(n)}$$

generated by the first-order $\Sigma\Delta$ quantizer with ordering p and with the quantizer alphabet $\mathcal{A}_K^{\delta}$ satisfies

$$\|x - \tilde{x}\| \leq \frac{(\sigma(F,p)+1)d}{N} \frac{\delta}{2}.$$

Harmonic frames

Zimmermann and Goyal, Kelner, Kovačević, Thao, Vetterli.

Definition

$H = \mathbb{C}^d$. An *harmonic frame* $\{e_n\}_{n=1}^N$ for H is defined by the rows of the Bessel map L which is the complex N-DFT $N \times d$ matrix with $N - d$ columns removed.

$H = \mathbb{R}^d$, d even. The harmonic frame $\{e_n\}_{n=1}^N$ is defined by the Bessel map L which is the $N \times d$ matrix whose nth row is

$$e_n^N = \sqrt{\frac{2}{d}}\left(\cos(\frac{2\pi n}{N}), \sin(\frac{2\pi n}{N}), \ldots, \cos(\frac{2\pi(d/2)n}{N}), \sin(\frac{2\pi(d/2)n}{N})\right).$$

- Harmonic frames are FUNTFs.
- Let E_N be the harmonic frame for $\mathbb{R}^d$ and let p_N be the identity permutation. Then

$$\forall N,\ \sigma(E_N, p_N) \leq \pi d(d+1).$$

Error estimate for harmonic frames

Theorem

Let E_N be the harmonic frame for $\mathbb{R}^d$ with frame bound N/d. Consider $x \in \mathbb{R}^d$, $\|x\| \leq 1$, and suppose the approximation $\tilde{x}$ of x is generated by a first-order $\Sigma\Delta$ quantizer as before. Then

$$\|x - \tilde{x}\| \leq \frac{d^2(d+1)+d}{N} \frac{\delta}{2}.$$

- Hence, for harmonic frames (and all those with bounded variation),

$$\mathrm{MSE}_{\Sigma\Delta} \leq \frac{C_d}{N^2} \delta^2.$$

- This bound is clearly superior asymptotically to

$$\mathrm{MSE}_{\mathrm{PCM}} = \frac{(d\delta)^2}{12N}.$$

ΣΔ and ”optimal” PCM

Theorem

The first order ΣΔ scheme achieves the asymptotically optimal $\mathrm{MSE}_{\mathrm{PCM}}$ for harmonic frames.

The digital encoding

$$\mathrm{MSE}_{\mathrm{PCM}} = \frac{(d\delta)^2}{12N}$$

in PCM format leaves open the possibility that decoding (consistent nonlinear reconstruction, with additional numerical complexity this entails) could lead to

$$\text{“}\mathrm{MSE}^{\mathrm{opt}}_{\mathrm{PCM}}\text{”} \ll O(\frac{1}{N}).$$

Goyal, Vetterli, Thao (1998) proved

$$\text{“}\mathrm{MSE}^{\mathrm{opt}}_{\mathrm{PCM}}\text{”} \sim \frac{\tilde{C}_d}{N^2}\delta^2.$$

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A comparison of Σ-Δ and PCM

Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Let $x \in \mathbb{C}^d$, $||x|| \leq 1$.

Definition

- $q_{PCM}(x)$ is the sequence to which x is mapped by PCM.
- $q_{\Sigma\Delta}(x)$ is the sequence to which x is mapped by $\Sigma\Delta$.
-
$$\mathrm{err}_{PCM}(x) = ||x - \frac{d}{N} L^* q_{PCM}(x)||$$
$$\mathrm{err}_{\Sigma\Delta}(x) = ||x - \frac{d}{N} L^* q_{\Sigma\Delta}(x)||$$

Fickus question: We shall analyze to what extent $\mathrm{err}_{\Sigma\Delta}(x) < \mathrm{err}_{PCM}(x)$ beyond our results with Powell and Yilmaz.

Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Definition

A function $e : [a, b] \to \mathbb{C}^d$ is of *bounded variation (BV)* if there is a $K > 0$ such that for every $a \le t_1 < t_2 < \cdots < t_N \le b$,

$$\sum_{n=1}^{N-1} \|e(t_n) - e(t_{n+1})\| \le K.$$

The smallest such K is denoted by $|e|_{BV}$, and defines a seminorm for the space of BV functions.

Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Theorem

Let $e : [0,1] \to \{x \in \mathbb{C}^d : \|x\| = 1\}$ be continuous function of bounded variation such that $F_N = (e(n/N))_{n=1}^N$ is a FUNTF for $\mathbb{C}^d$ for every N. Then,

$$\exists N_0 > 0 \textit{ such that } \forall N \geq N_0 \textit{ and } \forall 0 < \|x\| \leq 1$$

$$\mathtt{err}_{\Sigma\Delta}(x) \leq \mathtt{err}_{PCM}(x).$$

Moreover, a lower bound for N_0 is $d(1 + |e|_{BV})/(\sqrt{d} - 1)$.

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Comparison of 1-bit PCM and 1-bit $\Sigma\Delta$

Example (Roots of unity frames for $\mathbb{R}^2$)

$e_n^N = (\cos(2\pi n/N), \sin(2\pi n/N))$.

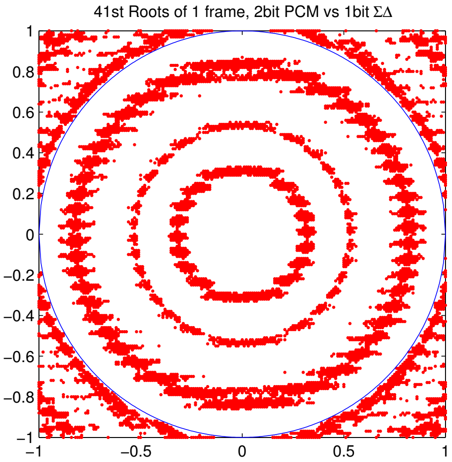
Here, $e(t) = (\cos(2\pi t), \sin(2\pi t))$,
$M = |e|_{BV} = 2\pi$, $lim \alpha_{F_N} = 2/\pi$.

Example (Real Harmonic Frames for $\mathbb{R}^{2k}$)

$e_n^N = \frac{1}{\sqrt{k}}(\cos(2\pi n/N), \sin(2\pi n/N), \ldots, \cos(2\pi kn/N), \sin(2\pi kn/N))$.

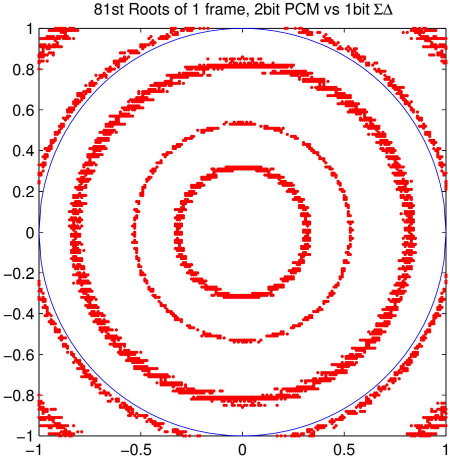
In this case, $e(t) = \frac{1}{\sqrt{k}}(\cos(2\pi t), \sin(2\pi t), \ldots, \cos(2\pi kt), \sin(2\pi kt))$,
$M = |e|_{BV} = 2\pi\sqrt{\frac{1}{d}\sum_{k=1}^{d} k^2}$.

Comparison of 2-bit PCM and 1-bit ΣΔ



Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Comparison of 2-bit PCM and 1-bit ΣΔ



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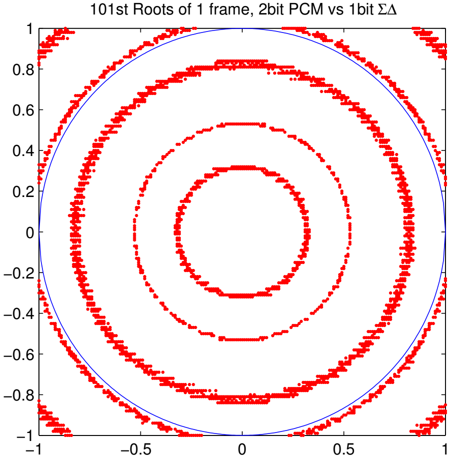
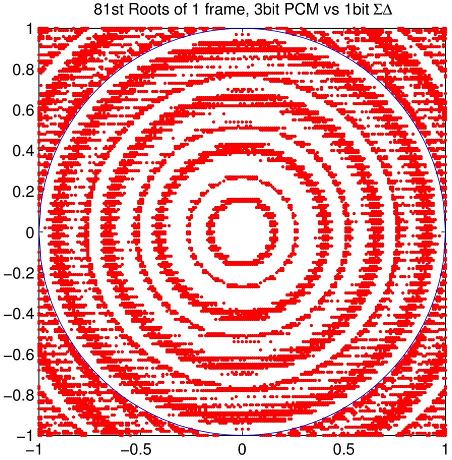
Comparison of 2-bit PCM and 1-bit $\Sigma\Delta$



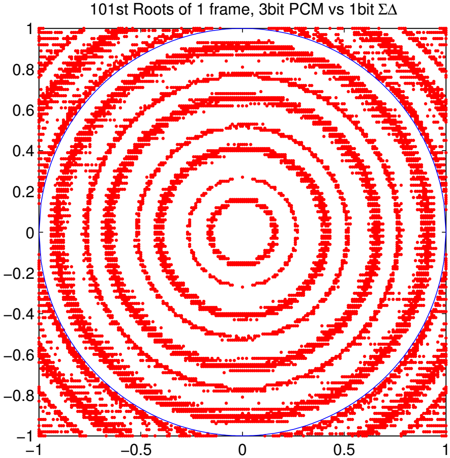
Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Comparison of 3-bit PCM and 1-bit ΣΔ



Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Comparison of 3-bit PCM and 1-bit ΣΔ



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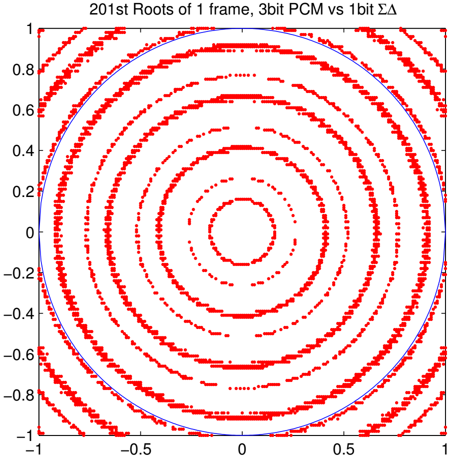
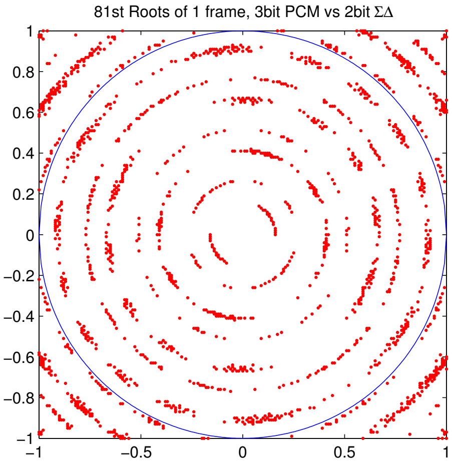


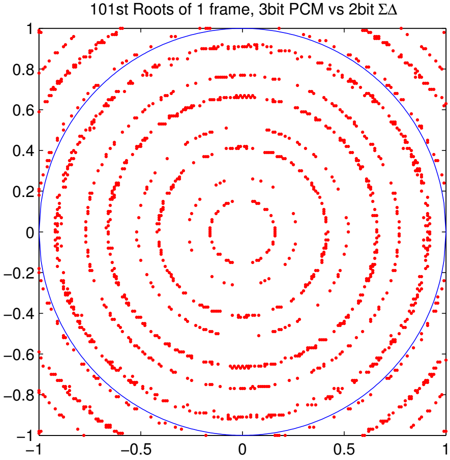
Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Comparison of 3-bit PCM and 2-bit $\Sigma\Delta$



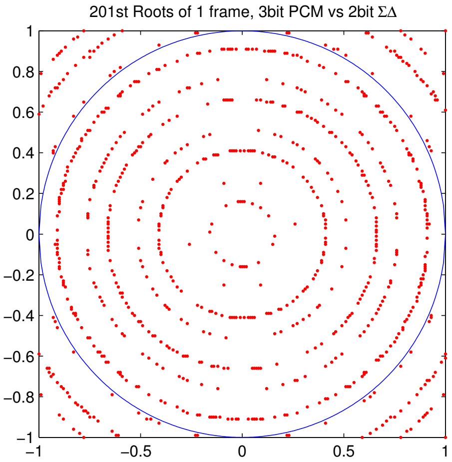
Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Comparison of 3-bit PCM and 2-bit ΣΔ



Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Comparison of 3-bit PCM and 2-bit ΣΔ



Red: $\mathtt{err}_{PCM}(x) < \mathtt{err}_{\Sigma\Delta}(x)$, Green: $\mathtt{err}_{PCM}(x) = \mathtt{err}_{\Sigma\Delta}(x)$.

Complex Σ-Δ and Yang Wang's idea and algorithm

Complex ΣΔ - Alphabet

Let $K \in \mathbb{N}$ and $\delta > 0$. The *midrise* quantization alphabet is

$$\mathcal{A}_K^\delta = \left\{ \left(m + \frac{1}{2} \right) \delta + in\delta : m = -K, \ldots, K-1, \; n = -K, \ldots, K \right\}$$

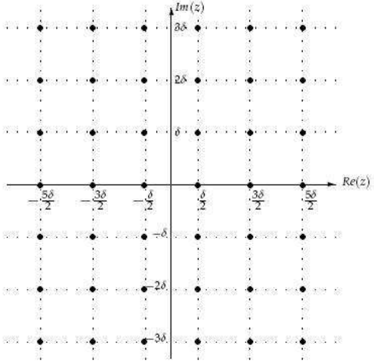


Figure: $\mathcal{A}_K^\delta$ for $K = 3\delta$.

Complex ΣΔ

The *scalar uniform quantizer* associated to $\mathcal{A}_K^\delta$ is

$$Q_\delta(a+ib) = \delta\left(\frac{1}{2} + \left\lfloor \frac{a}{\delta} \right\rfloor + i\left\lfloor \frac{b}{\delta} \right\rfloor\right),$$

where $\lfloor x \rfloor$ is the largest integer smaller than x.
For any $z = a + ib$ with $|a| \leq K$ and $|b| \leq K$, Q satisfies

$$|z - Q_\delta(z)| \leq \min_{\zeta \in \mathcal{A}_K^\delta} |z - \zeta|.$$

Let $\{x_n\}_{n=1}^N \subseteq \mathbb{C}$ and let p be a permutation of $\{1, \ldots, N\}$. Analogous to the real case, the first order ΣΔ quantization is defined by the iteration

$$\begin{aligned} u_n &= u_{n-1} + x_{p(n)} - q_n, \\ q_n &= Q_\delta(u_{n-1} + x_{p(n)}). \end{aligned}$$

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Complex ΣΔ

The following theorem is analogous to BPY

Theorem

Let $F = \{e_n\}_{n=1}^N$ be a finite unit norm frame for $\mathbb{C}^d$, let p be a permutation of $\{1, \ldots, N\}$, let $|u_0| \leq \delta/2$, and let $x \in \mathbb{C}^d$ satisfy $\|x\| \leq (K - 1/2)\delta$. The ΣΔ approximation error $\|x - \widetilde{x}\|$ satisfies

$$\|x - \widetilde{x}\| \leq \sqrt{2}\|S^{-1}\|_{\mathrm{op}} \left(\sigma(F,p)\frac{\delta}{2} + |u_N| + |u_0|\right),$$

where S^{-1} is the inverse frame operator. In particular, if F is a FUNTF, then

$$\|x - \widetilde{x}\| \leq \sqrt{2}\frac{d}{N} \left(\sigma(F,p)\frac{\delta}{2} + |u_N| + |u_0|\right),$$

Complex ΣΔ

Let $\{F_N\}$ be a family of FUNTFs, and p_N be a permutation of $\{1, \ldots, N\}$. Then the frame variation $\sigma(F_N, p_N)$ is a function of N. If $\sigma(F_N, p_N)$ is bounded, then

$$\|x - \widetilde{x}\| = \mathcal{O}(N^{-1}) \text{ as } N \to \infty.$$

Wang gives an upper bound for the frame variation of frames for $\mathbb{R}^d$, using the results from the Travelling Salesman Problem.

Theorem YW

Let $S = \{v_j\}_{j=1}^N \subseteq [-\frac{1}{2}, \frac{1}{2}]^d$ with $d \geq 3$. There exists a permutation p of $\{1, \ldots, N\}$ such that

$$\sum_{j=1}^{N-1} \|v_{p(j)} - v_{p(j+1)}\| \leq 2\sqrt{d+3}N^{1-\frac{1}{d}} - 2\sqrt{d+3}.$$

Complex ΣΔ

Theorem

Let $F = \{e_n\}_{n=1}^N$ be a FUNTF for $\mathbb{R}^d$, $|u_0| \leq \delta/2$, and let $x \in \mathbb{R}^d$ satisfy $\|x\| \leq (K - 1/2)\delta$. Then, there exists a permutation p of $\{1, 2, \ldots, N\}$ such that the approximation error $\|x - \widetilde{x}\|$ satisfies

$$\|x - \widetilde{x}\| \leq \sqrt{2}\delta d \left((1 - \sqrt{d+3})N^{-1} + \sqrt{d+3}N^{-\frac{1}{d}}\right)$$

This theorem guarantees that

$$\|x - \widetilde{x}\| \leq \mathcal{O}(N^{-\frac{1}{d}}) \text{ as } N \to \infty$$

for FUNTFs for $\mathbb{R}^d$.

Σ-Δ and analytic number theory

Even – odd

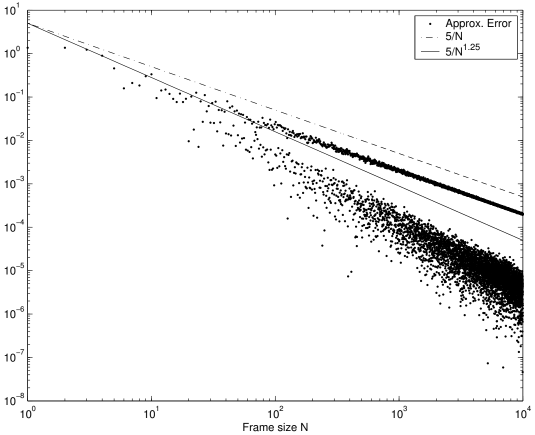

Figure: log-log plot of $\|x - \widetilde{x}_N\|$.

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Even – odd

$E_N = \{e_n^N\}_{n=1}^N$, $e_n^N = (\cos(2\pi n/N), \sin(2\pi n/N))$. Let $x = (\frac{1}{\pi}, \sqrt{\frac{3}{17}})$.

$$x = \frac{d}{N} \sum_{n=1}^{N} x_n^N e_n^N, \quad x_n^N = \langle x, e_n^N \rangle.$$

Let $\widetilde{x}_N$ be the approximation given by the 1st order $\Sigma\Delta$ quantizer with alphabet $\{-1, 1\}$ and natural ordering.

Improved estimates

$E_N = \{e_n^N\}_{n=1}^N$, Nth roots of unity FUNTFs for $\mathbb{R}^2$, $x \in \mathbb{R}^2$, $\|x\| \leq (K - 1/2)\delta$.

$$\text{Quantize} \qquad x = \frac{d}{N} \sum_{n=1}^{N} x_n^N e_n^N, \quad x_n^N = \langle x, e_n^N \rangle$$

using 1st order $\Sigma\Delta$ scheme with alphabet $\mathcal{A}_K^\delta$.

Theorem

If N is even and large then $\|x - \widetilde{x}\| \leq B_x \frac{\delta \log N}{N^{5/4}}$.

If N is odd and large then $A_x \frac{\delta}{N} \leq \|x - \widetilde{x}\| \leq B_x \frac{(2\pi+1)d}{N} \frac{\delta}{2}$.

- The proof uses a theorem of Güntürk (from complex or harmonic analysis); and Koksma and Erdös-Turán inequalities and van der Corput lemma (from analytic number theory).
- The Theorem is true for harmonic frames for $\mathbb{R}^d$.

Proof of Improved Estimates theorem

- If N is even and large then $\|x - \widetilde{x}\| \leq B_x \frac{\delta \log N}{N^{5/4}}$.
  If N is odd and large then $A_x \frac{\delta}{N} \leq \|x - \widetilde{x}\| \leq B_x \frac{(2\pi+1)d}{N} \frac{\delta}{2}$.
- $\forall N, \{e_n^N\}_{n=1}^N$ is a FUNTF.

$$x - \widetilde{x}_N = \frac{d}{N}\left(\sum_{n=1}^{N-2} v_n^N (f_n^N - f_{n+1}^N) + v_{N-1}^N f_{N-1}^N + u_N^N e_N^N\right)$$

$$f_n^N = e_n^N - e_{n+1}^N, \quad v_n^N = \sum_{j=1}^{n} u_j^N, \quad \widetilde{u}_n^N = \frac{u_n^N}{\delta}$$

- To bound v_n^N.

Koksma Inequality

Definition

The *discrepancy* D_N of a finite sequence $x_1, \ldots, x_N$ of real numbers is

$$D_N = D_N(x_1, \ldots, x_N) = \sup_{0 \leq \alpha \langle \beta \leq 1} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[\alpha,\beta)}(\{x_n\}) - (\beta - \alpha) \right|,$$

where $\{x\} = x - \lfloor x \rfloor$.

Theorem(Koksma Inequality)

$g : [-1/2, 1/2) \to \mathbb{R}$ of bounded variation and $\{\omega_j\}_{j=1}^n \subset [-1/2, 1/2) \Longrightarrow$

$$\left| \frac{1}{n} \sum_{j=1}^{n} g(\omega_j) - \int_{-\frac{1}{2}}^{\frac{1}{2}} g(t)dt \right| \leq \mathrm{Var}(g)\mathrm{Disc}\left(\{\omega_j\}_{j=1}^n \right).$$

With $g(t) = t$ and $\omega_j = \widetilde{u}_j^N$,

$$|v_n^N| \leq n\delta \mathrm{Disc}\left(\{\widetilde{u}_j^N\}_{j=1}^n \right).$$

Erdös-Turán Inequality

$$\exists C > 0, \forall K, \mathrm{Disc}\left(\{\widetilde{u}_n^N\}_{n=1}^{j}\right) \leq C\left(\frac{1}{K} + \frac{1}{j}\sum_{k=1}^{K}\frac{1}{k}\left|\sum_{n=1}^{j} e^{2\pi i k \widetilde{u}_n^N}\right|\right).$$

To approximate the exponential sum.

Approximation of Exponential Sum

Güntürk's Proposition (1)

$\forall N, \exists X_N \in \mathcal{B}_{\Omega/N}$ such that , $\forall n = 0, \ldots, N$

$$X_N(n) = u_n^N + c_n \frac{\delta}{2}, \; c_n \in \mathbb{Z}$$

and, $\forall t$,

$$\left| X_N'(t) - h\left(\frac{t}{N}\right) \right| \leq B \frac{1}{N}$$

Bernstein's Inequality (2)

If $x \in \mathcal{B}_\Omega$, then $\|x^{(r)}\|_\infty \leq \Omega^r \|x\|_\infty$

Approximation of Exponential Sum

(1)+(2)

$$\forall t, \left| X_N''(t) - \frac{1}{N} h'\left(\frac{t}{N}\right) \right| \leq B \frac{1}{N^2}$$

- $\widehat{\mathcal{B}}_\Omega = \{ T \in A'(\widehat{\mathbb{R}}) \ : \ \operatorname{supp} T \subseteq [-\Omega, \Omega] \}$
- $\mathcal{M}_\Omega = \{ h \in \mathcal{B}_\Omega \ : \ h' \in$ $L^\infty(\mathbb{R})$ and all zeros of h' on $[0,1]$ are simple$\}$
- We assume $\exists h \in \mathcal{M}_\Omega$ such that $\forall N$ and $\forall\, 1 \leq n \leq N$, $h(n/N) = x_n^N$.

Van der Corput Lemma

- Let a, b be integers with $a < b$, and let f satisfy $f'' \geq \rho > 0$ on $[a, b]$ or $f'' \leq -\rho < 0$ on $[a, b]$. Then

$$\left|\sum_{n=a}^{b} e^{2\pi i f(n)}\right| \leq \left(\left|f'(b) - f'(a)\right| + 2\right)\left(\frac{4}{\sqrt{\rho}} + 3\right).$$

$$\Downarrow$$

- $\forall 0 < \alpha < 1, \exists N_\alpha$ such that $\forall N \geq N_\alpha$,

$$\left|\sum_{n=1}^{j} e^{2\pi i k \widetilde{u}_n^N}\right| \leq B_x N^\alpha + B_x \frac{\sqrt{k} N^{1-\frac{\alpha}{2}}}{\sqrt{\delta}} + B_x \frac{k}{\delta}.$$

Choosing appropriate α and K

Putting $\alpha = 3/4$, $K = N^{1/4}$ yields

$$\exists \widetilde{N} \text{ such that } \forall N \geq \widetilde{N}, \mathrm{Disc}\left(\{\widetilde{u}_n^N\}_{n=1}^{j}\right) \leq B_x \frac{1}{N^{\frac{1}{4}}} + B_x \frac{N^{\frac{3}{4}} \log(N)}{j}$$

$$\Downarrow$$

Conclusion

$$\forall n = 1, \ldots, N, \ |v_n^N| \leq B_x \delta N^{\frac{3}{4}} \log N$$

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Hadamard matrices and infinite CAZAC codes

Accomplishments

- Developed libraries of CAZAC codes parameterized by design variables, proven mathematically and made available by user friendly software (CAZAC Playstation).
- Refined and formulated new, large classes of quadratic phase CAZAC codes and introduced Björck CAZAC codes to achieve diverse discrete periodic ambiguity function behavior.
- Enhanced sidelobe suppression by averaging and mixing techniques for CAZAC codes.
- Constructed vector-valued CAZAC codes with frame properties. This was motivated by the fact that frames lead to robust/stable signal decompositions. Vector-valued CAZAC codes are relevant in light of vector sensor and MIMO capabilities.
- Established the theory of waveforms coded by finite Gabor systems, and made a quantitative comparison with the non-Gabor case (A. Bourouihiya).
- Proved preliminary mathematical results to estimate the number of essentially different CAZAC codes of length N.

Transition and the future

- Our CAZAC software continues to be developed. This is ongoing work in order to develop a useful tool for the community. See

  www.math.umd.edu/~jjb/cazac/

- We shall analyze the wideband radar ambiguity function,

$$WA(u)(x,a) = \sqrt{a}\int u(a(t-x))\overline{u(t)}dt,$$

  in terms of wavelet frames, with the intent of solving the wideband radar ambiguity problem.

- We intend to complete our geometrical analysis of Shapiro-Rudin polynomials, and to extend the study to Golay pairs.

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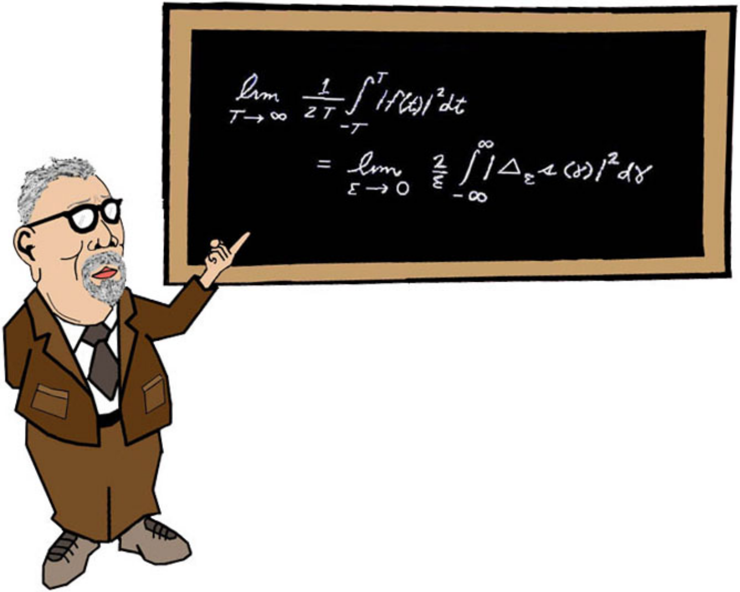
Transition and the future

- We shall further develop and implement our theory of vector-valued ambiguity functions in terms of our notion of frame multiplication and the role of finite groups.
- Our previous MURI results on number theoretic CAZAC codes, such as Björck codes, serving as coefficients for phase-coded waveforms, will be analyzed in the vector-valued setting.
- We shall construct alternatives to the Golay waveform modality by means of our vector-valued theory.
- Gabor frames and pseudodifferential operators will be incorporated in our investigation of the narrow band radar ambiguity function
- We are using our frame potential characterization of FUNTFs in conjunction with L^1-sparse representation criteria in order to construction a new quantization scheme, called SRQP (Sparse Representation Quantization Procedure), which goes beyond $\Sigma - \Delta$.


$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt$
$= \lim_{\varepsilon \to 0} \frac{2}{\varepsilon} \int_{-\infty}^{\infty} |\Delta_\varepsilon s(\gamma)|^2 d\gamma$

Thats all folks!

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