ABSTRACT

Title of dissertation: WIENER'S GENERALIZED HARMONIC ANALYSIS AND WAVEFORM DESIGN

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Bounded codes or waveforms are constructed whose autocorrelation is the inverse Fourier transform of certain positive functions. For the positive function $F \equiv 1$ the corresponding *unimodular* waveform of infinite length, whose autocorrelation is the inverse Fourier transform of F, is constructed using real Hadamard matrices. This waveform has a autocorrelation function that vanishes everywhere on the integers except at zero where it is one. In this case error estimates have been calculated which suggest that for a pre-assigned error the number (finite) of terms from this infinite sequence that are needed so that the autocorrelation at some non-zero k is within this given error range is 'almost' independent of k. In addition, such unimodular codes (both real and complex) whose autocorrelation is the inverse Fourier transform of $F \equiv 1$ has also been constructed by extending Wiener's work on Generalized Harmonic Analysis (GHA) and a certain class of exponential functions. The analogue in higher dimensions is also presented.

Further, for any given positive and even function f defined on the integers that is convex and decreasing to zero on the positive integers, waveforms have been

constructed whose autocorrelation is f. The waveforms constructed are real and bounded with a bound that depends on the value of f at zero.

Wiener's Generalized Harmonic Analysis and Waveform Design

by

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DEDICATION

To my mother, for all the sacrifices she made to bring me up.

To my father, for always showing me the righteous path and strongly motivating me towards mathematics.

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Chapter 1

Introduction

Our ultimate goal is to construct codes, x, which have constant amplitude and whose autocorrelation is the inverse Fourier transform of a given positive bounded Radon measure. We use Wiener's Generalized Harmonic Analysis(GHA) to approach the problem. There are also significant components from measure theory, number theory and functional analysis.

There are two main reasons that the waveforms x should have constant amplitude. First, a transmitter can operate at peak power if x has constant peak amplitude - the system does not have to deal with the surprise of greater than expected amplitudes. Second, amplitude variations during transmission due to additive noise can be (theoretically) eliminated at the receiver end without distorting the message. The problem of waveform design is relevant in several applications in the areas of radar and communications. In the former, the waveforms x can play a role in effective target recognition, e.g., [1], [17], [20], [25]; and in the latter they are used to address synchronization issues in cellular (phone) access technologies, especially code division multiple access (CDMA), e.g., [28], [29]. The radar and communication methods combine in recent advanced multifunction RF systems (AMRFS).

1.1 Notation

We shall use the standard notation from harmonic analysis, e.g., [4], [24]. Let $\mathbb{Z}$ be the set of integers, and its dual group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\mathbb{R}$ the real numbers. $\mathbb{N}$ is the set of natural numbers. In a d-dimensional space, $\mathbb{Z}^d = \mathbb{Z}\times\cdots\times\mathbb{Z}$ (d factors). $C(\mathbb{T}^d)$ is the space of complex valued continuous functions on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, and $A(\mathbb{T}^d)$ is the subspace of absolutely convergent Fourier series. $M(\mathbb{T}^d)$ is the space of Radon measures on $\mathbb{T}^d$, i.e., $M(\mathbb{T}^d)$ is the dual space of the Banach space $C(\mathbb{T}^d)$ taken with the sup norm. We designate the characteristic function of $S \subseteq \mathbb{R}^d$ by $\mathbb{1}_S$. The λ-dilation of a function f is defined by $f_\lambda(t) = \lambda f(\lambda t)$. A sequence $\{p_n\}$ is positive definite if for all $(c_0, \ldots, c_N) \in \mathbb{C}^{N+1}\backslash\{0\}$, $\sum_{0\leqslant j,k\leqslant N} p_{j-k}c_k\overline{c_j} > 0$. A positive definite sequence, $\{p_n\}$, is denoted by $\{p_n\} \gg 0$. Formally, the autocorrelation $A_x : \mathbb{Z} \to \mathbb{C}$ of $x : \mathbb{Z} \to \mathbb{C}$ is defined as

$$\forall k \in \mathbb{Z}, \quad A_x[k] = \lim_{N\to\infty} \frac{1}{(2N+1)} \sum_{-N\leqslant m\leqslant N} x[k+m]\overline{x[m]}.$$

If $F \in A(\mathbb{T}^d)$ we write $\check{F} = f = \{f_k\}$, i.e., $\check{F}[k] = f_k$ and for all $k \in \mathbb{Z}^d$, $\quad f_k = \int_{\mathbb{T}^d} F(\gamma)e^{2\pi i k\cdot\gamma}d\gamma$. There is an analogous definition for $\check{\mu}$ where $\mu \in M(\mathbb{T}^d)$.

1.2 Motivation

Because of recent work in waveform design [1], [5], [26], [2], [13], we are resurrecting certain aspects of Wiener's Generalized Harmonic Analysis (GHA).

Suppose a complicated signal x cannot be analyzed directly but it is possible to quantify its autocorrelation A_x. In GHA, a function x is analyzed for its frequency information by computing its autocorrelation A_x and its power spectrum μ, which is the inverse Fourier transform of the autocorrelation. In signals containing non-square-integrable noise and/or random components the harmonic analysis of a non-square-integrable function is desired. GHA includes the Wiener-Plancherel formula for L^∞, which is an analogue of the L^2-Parseval-Plancherel formula. In one direction, if A_x is the autocorrelation of x, then by the Herglotz-Bochner theorem there exists $\mu \in M_b^+(\mathbb{R})$ such that $\check{\mu} \equiv A_x$. A natural question is the following: for any $\mu \in M_b^+(\widehat{\mathbb{R}})$ does there exist x whose autocorrelation A_x exists, and for which $A_x = \check{\mu}$? The deterministic and constructive affirmative answer to this is the Wiener-Wintner Theorem.

1.3 Background

There are established algebraic approaches [13] for constructing unimodular (amplitude 1) $K-$ periodic sequences u with the property that the autocorrelation A_u vanishes outside of the periodic dc-domain points $nK, n \in \mathbb{Z}$. Such sequences are called CAZAC (constant amplitude zero autocorrelation) codes. The zero autocorrelation ensures minimum interference between signals sharing the same channel.

We would like to construct our codes analytically with the purpose of making the design flexible and the codes stable under modest perturbations.

1.3.1 The Wiener Wintner Theorem in $\mathbb{R}$

In the setting of $\mathbb{R}$, we have the following theorem due to Wiener and Wintner [32], which was later extended to $\mathbb{R}^d$ in [3], [15].

Theorem 1.1 (Wiener-Wintner). *Let μ be a bounded positive measure on $\mathbb{R}$. There is a constructible function $f \in L^\infty_{loc}(\mathbb{R})$ such that its autocorrelation A_f exists for all $t \in \mathbb{R}$, and $A_f = \check{\mu}$ on $\mathbb{R}$, i.e.,*

$$\forall t \in \mathbb{R}, \quad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t+x)\overline{f(x)}dx = \int_{\mathbb{R}} e^{2\pi itx} d\mu(x).$$

1.3.2 Uniform distribution

For a real number x, let $[x]$ denote the integral part of x, that is, the greatest integer $\leqslant x$; let $\{x\} = x - [x]$ be the fractional part of x.

Let $a = (a_1, \cdots, a_d)$ and $b = (b_1, \cdots, b_d)$ be two points in $\mathbb{R}^d$. We say that $a < b$ ($a \leqslant b$) if $a_j < b_j$ ($a_j \leqslant b_j$) for $j = 1, 2, \cdots, d$. The set of points $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ such that $a \leqslant x < b$ will be denoted by $[a, b)$. The other d-dimensional intervals such as $[a, b]$ have similar meanings. The d-dimensional unit cube I^d is the interval $[\mathbf{0}, \mathbf{1})$, where $\mathbf{0} = (0, \cdots, 0)$ and $\mathbf{1} = (1, \cdots, 1)$. The integral part of $x = (x_1, \cdots, x_d)$ is $[x] = ([x_1], \cdots, [x_d])$ and the fractional part of x is $\{x\} = (\{x_1\}, \cdots \{x_d\})$.

Let (x_n), $n = 1, 2, \ldots$, be a sequence of vectors in $\mathbb{R}^d$. For a subset E of I^d, let $A(E; N)$ denote the number of points $\{x_n\}$, $1 \leqslant n \leqslant N$, that lie in E.

Definition 1.2. The sequence (x_n), $n = 1, 2, \ldots$, is *uniformly distributed modulo* 1 *(u.d.mod* 1*)* in $\mathbb{R}^d$ if

$$\lim_{N\to\infty} \frac{A([a, b); N)}{N} = \Pi_{j=1}^{d} (b_j - a_j)$$

for all intervals $[a, b) \subseteq I^d$.

1.4 Outline of the thesis

In Chapter 2 we state and prove the Wiener-Wintner Theorem in $\mathbb{Z}^d$ i.e. we prove that given a positive bounded measure, μ, there exists a locally bounded function x whose autocorrelation is the Fourier transform of μ. Due to our desire of constructing waveforms of constant amplitude we would like the function x constructed in the Wiener-Wintner theorem to be uniformly bounded. This issue is discussed in Chapter 3. The chapter starts by demonstrating how uniformly distributed sequences suggest a way to give uniformly bounded waveforms though we are able to show that the method of using uniformly distributed sequences is not feasible. In Chapter 4 we take the function $F \equiv 1$ on $\mathbb{T}$ and construct several different *unimodular* functions whose autocorrelation is the Fourier series of F. The Fejér function is a positive function whose inverse Fourier transform is an isosceles triangle of height 1 and symmetric about the origin. In Chapter 5 we discuss functions whose autocorrelation is such a triangle (inverse Fourier transform of the Fejér function) and sum of such triangles (inverse Fourier transform of sums of Fejér functions). In this chapter we also show that given a positive and even function f on $\mathbb{Z}$ that is convex and decreases to zero over $\mathbb{Z}^+$, one can construct a function x on $\mathbb{Z}$ whose autocorrelation is f. Chapter 6 gives a summary of the main results of this thesis along with concluding remarks and some avenues for future research.

1.5 Contributive results

The contribution that comes from the work in this thesis can be listed as follows:

(i) In Chapter 2 the Wiener-Wintner Theorem is proved for $\mathbb{Z}^d$.

(ii) As already mentioned, our ultimate goal is to construct waveforms that have constant amplitude. Chapter 3 discusses an approach that could help us construct waveforms that are uniformly bounded. This would be a step towards our goal. Unfortunately, our attempt at using uniformly distributed sequences for this purpose turns out to be futile. Chapter 3 ends with a proof that one cannot use uniformly distributed sequences to get uniformly bounded waveforms.

(iii) Chapter 4 presents numerous cases of *unimodular* functions on $\mathbb{Z}$ whose autocorrelation is one at zero and zero everywhere else. Such functions or waveforms have been constructed using elements of real Hadamard matrices. Error estimates have been calculated which suggest that for a pre-assigned error the number (finite) of terms from this infinite sequence that are needed so that the autocorrelation at some non-zero k is within this given error range depends on the logarithm of k and so is 'almost' independent of k. In addition, it has been shown that such unimodular codes (both real and complex) whose autocorrelation is the Fourier series of $F \equiv 1$ can also be constructed using Wiener's Generalized Harmonic Analysis (GHA) and a certain class of exponential functions. Thus in this chapter we have functions whose autocorrelation is the Fourier series of the positive function $F \equiv 1$ on $\mathbb{T}$. The extension to higher dimensions of the same has also been done.

(iv) Using Wiener's technique, Chapter 5 constructs functions on $\mathbb{Z}$ whose autocorrelation is an isosceles triangle of base length a given integer M and height a given positive number K. Note that such a triangle is the inverse Fourier transform of the Fejér function. Based on this result it has been shown that given a positive and even function f on $\mathbb{Z}$ that is convex and decreasing to zero on $\mathbb{Z}^+$ one can construct a function x on $\mathbb{Z}$ whose autocorrelation is f. Thus even though our ultimate goal was to construct waveforms with constant amplitude whose autocorrelation is the inverse Fourier transform of any given positive function we have reached a point where we can construct bounded waveforms whose autocorrelation is the inverse Fourier transform of a sum of sinusoids.

Chapter 2

Wiener Wintner Theorem in $\mathbb{Z}^d$

The Wiener Wintner Theorem on $\mathbb{R}$ as stated in Section 1.3.1 was extended to $\mathbb{R}^d$ in [3], [15]. In this chapter we state and prove the Wiener Wintner theorem on $\mathbb{Z}^d$.

For any $N \in \mathbb{N}$, we denote the d-dimensional square in $\mathbb{Z}^d$ by $S(N)$ and so by $S(N)$ we shall mean

$$S(N) = \{m = (m_1, m_2, \cdots, m_d) \in \mathbb{Z}^d : -N \leqslant m_i \leqslant N, i = 1, \cdots, d\}.$$

Also, for $k = (k_1, \cdots, k_d) \in \mathbb{Z}^d$,

$$\sum_{S(N)} x[k+m]\overline{x[m]} = \sum_{m \in S(N)} x[k+m]\overline{x[m]} = \sum_{m_1=-N}^{N} \sum_{m_2=-N}^{N} \cdots \sum_{m_d=-N}^{N} x[k+m]\overline{x[m]}.$$

For a function $x : \mathbb{Z}^d \to \mathbb{C}$ the autocorrelation $A_x : \mathbb{Z}^d \to \mathbb{C}$ is defined as

$$A_x[k] = \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{S(N)} x[k+m]\overline{x[m]}.$$

2.1 Approximation by discrete measures

Let $F > 0$ in $A(\mathbb{T}^d)$ have Fourier coefficients $\{p_n\}_{n\in\mathbb{Z}^d}$. $\{p_n\}$ is positive definite, i.e., $\{p_n\} \gg 0$. Let δ_ω be the Dirac measure supported by $\{\omega\}$. Consider the d-dimensional unit square $[0,1) \times \cdots \times [0,1)$ (d factors). Let $\{\omega_{j,n}; j = 1, \ldots, n^d\}$ be points on this square where each edge has n equally spaced points. For each n we choose L_n such that $\forall j = 1, 2, \cdots, n^d$,

$$\left| \sum_{\ell \in S(L_n)} p_\ell e^{-2\pi i \ell . \omega_{j,n}} \right| = \sum_{\ell \in S(L_n)} p_\ell e^{-2\pi i \ell . \omega_{j,n}}. \tag{2.1}$$

This can be done since $F \in A(\mathbb{T}^d)$ and hence $\|F - S_N(F)\|_{L^\infty(\mathbb{T}^d)} \to 0$ as $N \to \infty$.

Define

$$\mu_n = \frac{1}{n^d} \sum_{j=1}^{n^d} \sum_{k \in S(L_n)} p_k e^{-2\pi i k \cdot \omega_{j,n}} \delta_{\omega_{j,n}}. \tag{2.2}$$

Proposition 2.3. *Let $F \in A(\mathbb{T}^d)$ have Fourier coefficients $\{p_k\} \in \ell^1(\mathbb{Z}^d)$ and define μ_n as in (2.2). Then assuming that $L_n \to \infty$ as $n \to \infty$,*

$$\forall f \in C(\mathbb{T}^d), \qquad \lim_{n \to \infty} \int_{\mathbb{T}^d} f(\gamma) d\mu_n(\gamma) = \int_{\mathbb{T}^d} f(\gamma) F(\gamma) d\gamma. \tag{2.4}$$

Proof.

$$\begin{aligned}
&\left| \int_{\mathbb{T}^d} f(\gamma) F(\gamma) d\gamma - \int_{\mathbb{T}^d} f(\gamma) d\mu_n(\gamma) \right| \\
&= \left| \sum_{k \in \mathbb{Z}^d} p_k \left(\int_{\mathbb{T}^d} f(\gamma) e^{-2\pi i k \cdot \gamma} d\gamma \right) - \frac{1}{n^d} \sum_{j=1}^{n^d} \sum_{k \in S(L_n)} p_k e^{-2\pi i k \cdot \omega_{j,n}} f(\omega_{j,n}) \right| \\
&= \left| \sum_{k \in \mathbb{Z}^d} p_k \left(\int_{\mathbb{T}^d} f(\gamma) e^{-2\pi i k \cdot \gamma} d\gamma \right) - \sum_{k \in S(L_n)} \frac{p_k}{n^d} \sum_{j=1}^{n^d} e^{-2\pi i k \cdot \omega_{j,n}} f(\omega_{j,n}) \right| \\
&= \left| \sum_{k \in \mathbb{Z}^d} p_k \left(\int_{\mathbb{T}^d} f(\gamma) e^{-2\pi i k \cdot \gamma} d\gamma \right) - \right. \\
&\left. - \left(\sum_{k \in \mathbb{Z}^d} \frac{p_k}{n^d} \sum_{j=1}^{n^d} e^{-2\pi i k \cdot \omega_{j,n}} f(\omega_{j,n}) - \sum_{k \in \mathbb{Z}^d \setminus S(L_n)} \frac{p_k}{n^d} \sum_{j=1}^{n^d} e^{-2\pi i k \cdot \omega_{j,n}} f(\omega_{j,n}) \right) \right| \\
&\leqslant \left| \sum_{k \in \mathbb{Z}^d} p_k \left(\int_{\mathbb{T}^d} f(\gamma) e^{-2\pi i k \cdot \gamma} d\gamma - \frac{1}{n^d} \sum_{j=1}^{n^d} e^{-2\pi i k \cdot \omega_{j,n}} f(\omega_{j,n}) \right) \right| + \\
&+ \left| \sum_{k \in \mathbb{Z}^d \setminus S(L_n)} \frac{p_k}{n^d} \sum_{j=1}^{n^d} e^{-2\pi i k \cdot \omega_{j,n}} f(\omega_{j,n}) \right|
\end{aligned}$$

$$\leqslant \sum_{k\in\mathbb{Z}^d} |p_k| \left| \int_{\mathbb{T}^d} f(\gamma) e^{-2\pi i k\cdot\gamma} d\gamma - \frac{1}{n^d} \sum_{j=1}^{n^d} e^{-2\pi i k\cdot\omega_{j,n}} f(\omega_{j,n}) \right| +$$

$$+ \sum_{k\in\mathbb{Z}^d\setminus S(L_n)} \frac{|p_k|}{n^d} \sum_{j=1}^{n^d} |f(\omega_{j,n})|$$

$$= \sum_{k\in\mathbb{Z}^d} |p_k| A_{k,n} + B_n \sum_{k\in\mathbb{Z}^d\setminus S(L_n)} |p_k| . \tag{2.5}$$

We now estimate the two sums in the right side of (2.5) and show that both go to zero as n goes to infinity.

$$B_n \sum_{k\in\mathbb{Z}^d\setminus S(L_n)} |p_k| = \sum_{k\in\mathbb{Z}^d\setminus S(L_n)} \frac{|p_k|}{n^d} \sum_{j=1}^{n^d} |f(\omega_{j,n})|$$

$$\leqslant \|f\|_{L^\infty(\mathbb{T}^d)} \sum_{k\in\mathbb{Z}^d\setminus S(L_n)} |p_k| . \tag{2.6}$$

Since $\{p_k\} \in \ell^1(\mathbb{Z}^d)$ and $L_n \to \infty$ as $n \to \infty$, the right side of (2.6) goes to 0 as $n \to \infty$. Thus $B_n \sum_{k\in\mathbb{Z}^d\setminus S(L_n)} |p_k| \to 0$ as $n \to \infty$.

Next we note that $\exists N_1$ such that $\forall n \geqslant N_1$

$$\left| \int_{\mathbb{T}^d} |f(\gamma)|\, d\gamma - \frac{1}{n^d} \sum_{j=1}^{n^d} |f(\omega_{j,n})| \right| < 1. \tag{2.7}$$

So,

$$\left| \int_{\mathbb{T}^d} f(\gamma) e^{-2\pi i k\cdot\gamma} d\gamma - \frac{1}{n^d} \sum_{j=1}^{n^d} e^{-2\pi i k\cdot\omega_{j,n}} f(\omega_{j,n}) \right|$$

$$\leqslant \int_{\mathbb{T}^d} |f(\gamma)|\, d\gamma + \frac{1}{n^d} \sum_{j=1}^{n^d} |f(\omega_{j,n})| \ \text{ (by the triangle inequality)}$$

$$\leqslant \left| \frac{1}{n^d} \sum_{j=1}^{n^d} |f(\omega_{j,n})| - \int_{\mathbb{T}^d} |f(\gamma)|\, d\gamma \right| + 2 \int_{\mathbb{T}^d} |f(\gamma)|\, d\gamma \text{ (again, by the triangle inequality)}$$

$$\leqslant 1 + 2\|f\|_{L^1(\mathbb{T}^d)} \text{ (using (2.7))}. \tag{2.8}$$

Since $f \in C(\mathbb{T})$, it is integrable and so for each k,

$$\lim_{n\to\infty} |p_k| \left| \int_{\mathbb{T}^d} f(\gamma) e^{-2\pi i k\cdot\gamma} d\gamma - \frac{1}{n^d} \sum_{j=1}^{n^d} e^{-2\pi i k\cdot\omega_{j,n}} f(\omega_{j,n}) \right| = 0. \tag{2.9}$$

Due to (2.8) and (2.9) we can apply the Lebesgue Dominated Convergence Theorem for $\mathbb{Z}^d$ to obtain that $\lim_{n\to\infty} \sum_{k\in\mathbb{Z}^d} |p_k| A_{k,n} = 0$.

Since both the terms on the right side of (2.5) go to 0 we have shown that $\lim_{n\to\infty} \left| \int_{\mathbb{T}^d} f(\gamma)F(\gamma)d\gamma - \int_{\mathbb{T}^d} f(\gamma)d\mu_n(\gamma) \right| = 0$ which implies (2.4) and thus proves the proposition. $\square$

We have shown in Proposition 2.3 that a function $F \in A(\mathbb{T}^d)$ can be approximated by discrete measures, μ_n, in the sense of 2.4.

2.2 Constructing x for a given F

Let $F > 0$ in $A(\mathbb{T}^d)$ have Fourier coefficients $\{p_n\}_{n\in\mathbb{Z}^d}$. For $\omega_{j,n}$ as defined in Section 2.1 define

$$x_n[k] = \sum_{j=1}^{n^d} \left(\frac{1}{n^d} \sum_{\ell\in S(L_n)} p_l e^{-2\pi i \ell\cdot\omega_{j,n}} \right)^{\frac{1}{2}} e^{2\pi i k\cdot\omega_{j,n}}. \tag{2.10}$$

Then,

$$\begin{aligned} \|x_n\|_\infty &\leqslant \sum_{j=1}^{n^d} \left| \frac{1}{n^d} \sum_{\ell\in S(L_n)} p_\ell e^{-2\pi i \ell\cdot\omega_{j,n}} \right|^{\frac{1}{2}} \\ &\leqslant \sqrt{n^d} \left(\sum_{\ell\in S(L_n)} |p_\ell| \right)^{\frac{1}{2}} = \sqrt{n^d} \|F\|_{A(\mathbb{T}^d)}^{\frac{1}{2}}. \end{aligned} \tag{2.11}$$

Lemma 2.12. *For each n,*

$$\lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{m\in S(N)} x_n[k+m]\overline{x_n[m]} = \check{\mu}_n[k] \tag{2.13}$$

uniformly on $\mathbb{Z}^d$ where x_n and μ_n are as defined in (2.10) and (2.2) respectively.

Proof. Let $a_{j,n} = \frac{1}{n^d}\sum_{\ell \in S(L_n)} p_\ell e^{-2\pi i \ell\cdot\omega_{j,n}}$. Then

$$\begin{aligned} \mu_n &= \sum_{j=1}^{n^d} a_{j,n}\delta_{\omega_{j,n}}, \qquad \text{where } \omega_{j,n} = (\omega_{j,n}^{(1)}, \cdots, \omega_{j,n}^{(d)}). \\ \check{\mu_n}[k] &= \sum_{j=1}^{n^d} a_{j,n} e^{2\pi i k\cdot\omega_{j,n}}, \qquad \text{and} \\ x_n[m] &= \sum_{j=1}^{n^d} a_{j,n}^{\frac{1}{2}} e^{2\pi i m\cdot\omega_{j,n}}. \end{aligned} \tag{2.14}$$

Now let us try to evaluate the limit in (2.13).

$$\begin{aligned} &\frac{1}{(2N+1)^d}\sum_{m\in S(N)} x_n[k+m]\overline{x_n[m]} \\ &= \frac{1}{(2N+1)^d}\sum_{m\in S(N)}\left[\sum_{j=1}^{n^d} a_{j,n}^{\frac{1}{2}} e^{2\pi i(k+m)\cdot\omega_{j,n}} \times \sum_{r=1}^{n^d}\overline{a_{r,n}}^{\frac{1}{2}} e^{-2\pi i m\cdot\omega_{r,n}}\right] \\ &= \frac{1}{(2N+1)^d}\sum_{m\in S(N)}\left[\sum_{j=r;j=1}^{n^d} a_{j,n}^{\frac{1}{2}}\overline{a_{j,n}}^{\frac{1}{2}} e^{2\pi i k\cdot\omega_{j,n}} + \sum_{j\neq r;j,r=1}^{n^d} a_{j,n}^{\frac{1}{2}}\overline{a_{r,n}}^{\frac{1}{2}} e^{2\pi i(k\cdot\omega_{j,n}+m\cdot(\omega_{j,n}-\omega_{r,n}))}\right]. \end{aligned} \tag{2.15}$$

Since F is positive we can choose L_n (in the definition of μ_n and x_n, (2.2) and (2.10) respectively) such that $a_{j,n}$ is positive. So for the sum involving $j = r$ we have

$$\begin{aligned} &\frac{1}{(2N+1)^d}\sum_{m\in S(N)}\sum_{j=r;j=1}^{n^d} a_{j,n}^{\frac{1}{2}}\overline{a_{j,n}}^{\frac{1}{2}} e^{2\pi i k\cdot\omega_{j,n}} = \frac{1}{(2N+1)^d}\sum_{m\in S(N)}\sum_{j=1}^{n^d} |a_{j,n}| e^{2\pi i k\cdot\omega_{j,n}} \\ &= \frac{1}{(2N+1)^d}\sum_{m\in S(N)}\sum_{j=1}^{n^d} a_{j,n} e^{2\pi i k\cdot\omega_{j,n}} = \frac{(2N+1)^d}{(2N+1)^d}\sum_{j=1}^{n^d} a_{j,n} e^{2\pi i k\cdot\omega_{j,n}} \\ &= \sum_{j=1}^{n^d} a_{j,n} e^{2\pi i k\cdot\omega_{j,n}} = \check{\mu_n}[k]. \end{aligned} \tag{2.16}$$

For $j \neq r$,

$$\begin{aligned}
&\left| \frac{1}{(2N+1)^d} \sum_{m \in S(N)} \sum_{j \neq r; j,r=1}^{n^d} a_{j,n}^{\frac{1}{2}} \overline{a_{r,n}}^{\frac{1}{2}} e^{2\pi i (k \cdot \omega_{j,n} + m \cdot (\omega_{j,n} - \omega_{r,n}))} \right| \\
= \; & \left| \frac{1}{(2N+1)^d} \sum_{j \neq r; j,r=1}^{n^d} a_{j,n}^{\frac{1}{2}} \overline{a_{r,n}}^{\frac{1}{2}} e^{2\pi i k \cdot \omega_{j,n}} \left(\sum_{m \in S(N)} e^{2\pi i m \cdot (\omega_{j,n} - \omega_{r,n})} \right) \right| \\
\leqslant \; & \frac{1}{(2N+1)^d} \sum_{j \neq r; j,r=1}^{n^d} |a_{j,n}|^{\frac{1}{2}} |a_{r,n}|^{\frac{1}{2}} \left| \sum_{m \in S(N)} e^{2\pi i m \cdot (\omega_{j,n} - \omega_{r,n})} \right| \\
= \; & \frac{1}{(2N+1)^d} \sum_{j \neq r; j,r=1}^{n^d} |a_{j,n}|^{\frac{1}{2}} |a_{r,n}|^{\frac{1}{2}} \left| \frac{\prod_{i=1}^d \sin(N + \frac{1}{2}) 2\pi (\omega_{j,n}^{(i)} - \omega_{r,n}^{(i)})}{\prod_{i=1}^d \sin \pi (\omega_{j,n}^{(i)} - \omega_{r,n}^{(i)})} \right| \\
\leqslant \; & \frac{1}{(2N+1)^d} \sum_{j \neq r; j,r=1}^{n^d} |a_{j,n}|^{\frac{1}{2}} |a_{r,n}|^{\frac{1}{2}} \frac{1}{\prod_{i=1}^d \left| \sin \pi (\omega_{j,n}^{(i)} - \omega_{r,n}^{(i)}) \right|} \qquad (2.17)
\end{aligned}$$

The right side of (2.17), being independent of k and the sum there being finite, goes to 0 uniformly as $N \to \infty$. This in combination with (2.16) proves that

$$\lim_{N \to \infty} \frac{1}{(2N+1)^d} \sum_{m \in S(N)} x[m+k] \overline{x[m]} = \check{\mu}_n[k].$$

□

It is easy to prove the following lemma.

Lemma 2.18. *Let* $\{N_n : n = 1, 2, \cdots\} \subseteq \mathbb{N}$ *increase to* ∞, *and let* $y : \mathbb{Z}^d \to \mathbb{C}$. *For* $k \in \mathbb{Z}^d$ *and* $q \in \mathbb{N}$,

$$\lim_{N \to \infty} \frac{1}{(2N+1)^d} \sum_{m \in S(N_q)} y[k+m] \overline{y[m]} = 0. \qquad (2.19)$$

We are now in a position to define a waveform x on $\mathbb{Z}^d$ but first let us make a few important observations.

By the uniform convergence in Lemma 2.12, we have an increasing sequence,

$\{K_m : m \in \mathbb{N}\} \subseteq \mathbb{Z}^+$, such that $\forall k \in \mathbb{Z}^d$ and $\forall N \geqslant K_n$,

$$\left| \frac{1}{(2N+1)^d} \sum_{m \in S(N)} x_n[k+m]\overline{x_n[m]} - \check{\mu}_n[k] \right| < \frac{1}{2^{n+1}}. \tag{2.20}$$

Set

$$N_n = (K_1+1)(K_2+2)\cdots(K_n+n). \tag{2.21}$$

Therefore, $N_n \geqslant n!$ and the sequences

$$\begin{aligned} &\{N_n\}, \left\{\frac{N_{n+1}}{N_n}\right\} = \{K_{n+1}+n+1\} \quad \text{and} \\ &\{N_{n+1} - N_n\} = \{(K_1+1)\cdots(K_n+n)(K_{n+1}+n)\} \quad \text{increase to infinity.} \end{aligned} \tag{2.22}$$

For $\{N_n\}$ defined in (2.21), set

$$\boxed{x[k] = \begin{cases} x_n[k] & \text{for } k \in S(N_{n+1}) \setminus S(N_n) \\ 0 & \text{for } k \in S(N_1). \end{cases} \qquad (2.23)}$$

Lemma 2.24.

$$\lim_{n\to\infty} \frac{1}{n!} \sum_{j=2}^{n} \max(\|x_{j-1}\|_\infty, \|x_j\|_\infty) \max(\|x_j\|_\infty, \|x_{j+1}\|_\infty) = 0. \tag{2.25}$$

Proof. We provide the proof for the one dimensional case but it can be extended to d dimensions.

$$\begin{aligned} &\frac{1}{n!} \sum_{j=2}^{n} \max(\|x_{j-1}\|_\infty, \|x_j\|_\infty) \max(\|x_j\|_\infty, \|x_{j+1}\|_\infty) \\ &\leqslant \frac{1}{n!} \sum_{j=2}^{n} \|F\|_{A(\mathbb{T})} \sqrt{j}\sqrt{j+1} \quad \text{(due to (2.11))} \end{aligned}$$

$$\leqslant \frac{1}{n!}\|F\|_{A(\mathbb{T})}\sum_{j=2}^{n}(j+1) = \frac{1}{n!}\|F\|_{A(\mathbb{T})}\sum_{j=3}^{n+1} j$$
$$\leqslant \frac{1}{n!}\|F\|_{A(\mathbb{T})}\sum_{j=1}^{n+1} j = \frac{(n+1)(n+2)}{2n!}\|F\|_{A(\mathbb{T})} \to 0$$

as n goes to infinity. $\square$

2.3 Statement of the theorem and proof

Theorem 2.26 (The Wiener Wintner Theorem on $\mathbb{Z}^d$). *Given a positive $\mu \equiv F \in A(\mathbb{T}^d)$ with corresponding functions $\{x_n\}$ and x. Then, for each $k \in \mathbb{Z}^d$,*

$$\lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{S(N)} x[k+m]\overline{x[m]} = \check{\mu}[k]. \tag{2.27}$$

Proof of the Wiener Wintner Theorem on $\mathbb{Z}^d$. (a) Given $k \in \mathbb{Z}^d, k = (k_1, k_2, \cdots, k_d)$ and $\epsilon > 0$. Recall the sequence $\{N_n\}$ defined in (2.21). Choose $q_1 = q_1(k)$ such that if $k_0 = \max_{1\leqslant i\leqslant d}\{|k_i|\}$ then

$$\forall m > q_1 = q_1(k), \qquad N_{m+1} - k_0 > N_m + k_0. \tag{2.28}$$

This is possible due to the fact (2.22). Choose $q_2 = q_2(\epsilon, k) > q_1$ such that

$$\forall n \geqslant q_2, \qquad |\check{\mu}_n[k] - \check{\mu}[k]| < \epsilon. \tag{2.29}$$

Note that (2.29) is obtained by using $f(\gamma) = e^{2\pi i\gamma k}$ in (2.4) of Proposition 2.3.

To prove the result of the theorem we shall find $q = q(\epsilon, k) > q_2$ such that for all sufficiently large N,

$$\left|\frac{1}{(2N+1)^d} \sum_{m\in S(N_q)} x[k+m]\overline{x[m]}\right| < \epsilon \tag{2.30}$$

and

$$\left|\frac{1}{(2N+1)^d}\sum_{m\in S(N)\setminus S(N_q)} x[k+m]\overline{x[m]}-\check{\mu}[k]\right|<\epsilon(8+4^{d+1}+2^{2d+1}). \tag{2.31}$$

Then for all large N,

$$\begin{aligned}&\left|\frac{1}{(2N+1)^d}\sum_{m\in S(N)} x[k+m]\overline{x[m]}-\check{\mu}[k]\right|\leqslant\\ &\left|\frac{1}{(2N+1)^d}\sum_{m\in S(N_q)} x[k+m]\overline{x[m]}\right|+\left|\frac{1}{(2N+1)^d}\sum_{m\in S(N)\setminus S(N_q)} x[k+m]\overline{x[m]}-\check{\mu}[k]\right|\\ &<\epsilon(9+4^{d+1}+2^{2d+1})\end{aligned} \tag{2.32}$$

which is what is required to prove the theorem. (2.30) is valid due to Lemma 2.18 and so we just need to prove (2.31).

(b) For each $q>q_2$ and each $N>N_q$, write

$$\begin{aligned}&\frac{1}{(2N+1)^d}\sum_{m\in S(N)\setminus S(N_q)} x[k+m]\overline{x[m]}\\ &=\sum_{j=q}^{n-1}(b(j,N)+c(j,N))+\frac{1}{(2N+1)^d}\sum_{m\in S(N)\setminus S(N_n)} x[k+m]\overline{x[m]}\end{aligned} \tag{2.33}$$

where $b(j,N)=\frac{1}{(2N+1)^d}\sum_{S(N_{j+1}-k_0)\smallsetminus S(N_j)} x[k+m]\overline{x[m]}$,

$c(j,N)=\frac{1}{(2N+1)^d}\sum_{S(N_{j+1})\smallsetminus S(N_{j+1}-k_0)} x[k+m]\overline{x[m]}$ and

$n=n(N)$ is the largest integer n for which $N_n\leqslant N$.

(c) In this part we shall verify that for $q>q_1$ and the $c(j,N)$ defined in part (b),

$$\lim_{N\to\infty}\sum_{j=q}^{n-1}c(j,N)=0. \tag{2.34}$$

$$\left|\sum_{j=q}^{n-1} c(j,N)\right| = \left|\sum_{j=q}^{n-1} \frac{1}{(2N+1)^d} \sum_{S(N_{j+1})\setminus S(N_{j+1}-k_0)} x[k+m]\overline{x[m]}\right|$$
$$\leqslant \frac{1}{(2N+1)^d} \sum_{j=q}^{n-1} \sum_{S(N_{j+1})\setminus S(N_{j+1}-k_0)} \left|x[k+m]\overline{x[m]}\right|$$
$$= \frac{1}{(2N+1)^d} \sum_{j=q}^{n-1} \sum_{S(N_{j+1})\setminus S(N_{j+1}-k_0)} \left|x[k+m]\overline{x_j[m]}\right| \quad (2.35)$$

If $m \in S(N_{j+1}) \setminus S(N_{j+1} - k_0)$ then $k + m \in S(N_{j+2}) \setminus S(N_j)$. Thus, $x[k+m]$ is either $x_j[k+m]$ or $x_{j+1}[k+m]$. So, setting $p_j = \|x_j\|_\infty \max(\|x_j\|_\infty, \|x_{j+1}\|_\infty)$, we have from (2.35)

$$\left|\sum_{j=q}^{n-1} c(j,N)\right| \leqslant \frac{1}{(2N+1)^d} \sum_{j=q}^{n-1} p_j (2k_0)^d$$
$$= \frac{(2k_0)^d}{(2N+1)^d} \sum_{j=q}^{n-1} p_j \leqslant \frac{(2k_0)^d}{(2N)^d} \sum_{j=q}^{n-1} p_j$$
$$= \frac{k_0^d}{N^d} \sum_{j=q}^{n-1} p_j \leqslant \frac{k_0^d}{(n!)^d} \sum_{j=q}^{n-1} p_j \quad \text{(since } N > N_n \geqslant n!\text{)}. \quad (2.36)$$

Due to (2.25) the right side of (2.36) goes to zero as N goes to infinity and thus we have shown (2.34).

Parts (d) - (h) are devoted to showing that there are $q_3 = q_3(\epsilon, k) > q_2$ and $N(\epsilon, k) > q_3$ such that, for $N(\epsilon, k) > q \geqslant q_3$ and $\forall N > N(\epsilon, k)$

$$\sum_{j=q}^{n-1} b(j,N) = \frac{\check{\mu}[k]}{(2N+1)^d} \sum_{j=q}^{n-1} \left((2(N_{j+1}-k_0)+1)^d - (2N_j+1)^d\right) + r(q,N,k) \quad (2.37)$$

where $|r(q,N,k)| < \epsilon(4^{d+1} + 2^{2d+1} + 1) + \frac{1}{2^{q-1}}$. Parts (i) - (k) contain the proof that $\forall N > N(\epsilon, k)$

$$\frac{1}{(2N+1)^d} \sum_{S(N)\setminus S(N_n)} x[k+m]\overline{x[m]} = \check{\mu}[k]\left(1 - \left(\frac{2N_n+1}{2N+1}\right)^d\right) + s(N,k) \quad (2.38)$$

where $s(N,k) < 4\epsilon + \frac{1}{2^n}$. The proof of (2.31) is completed in part (l) by invoking (2.34), (2.37) and (2.38) in (2.33).

(d) In order to estimate $b(j, N)$ we define the sets

$$\begin{aligned} A_j(N) &= \{m \in S(N_{j+1} - k_0) \setminus S(N_j) : m + k \in S(N_{j+1}) \setminus S(N_j)\} \qquad \text{and} \\ A_{j-1}(N) &= \{m \in S(N_{j+1} - k_0) \setminus S(N_j) : m + k \in S(N_j) \setminus S(N_{j-1})\}. \end{aligned}$$

Then for $q > q_1$, $S(N_{j+1} - k_0) \setminus S(N_j) = A_j(N) \cup A_{j-1}(N)$, a disjoint union, and we have

$$\begin{aligned} b(j, N) &= \frac{1}{(2N+1)^d} \sum_{A_j(N)} x_j[k+m]\overline{x_j[m]} + \frac{1}{(2N+1)^d} \sum_{A_{j-1}(N)} x_{j-1}[k+m]\overline{x_j[m]} \\ &= \frac{1}{(2N+1)^d} \sum_{A_j(N) \cup S(N_j)} x_j[k+m]\overline{x_j[m]} - \frac{1}{(2N+1)^d} \sum_{S(N_j)} x_j[k+m]\overline{x_j[m]} + \\ &\quad + \frac{1}{(2N+1)^d} \sum_{A_{j-1}(N)} x_{j-1}[k+m]\overline{x_j[m]}. \end{aligned} \tag{2.39}$$

We shall estimate the three sums on the right side of (2.39). In the process of making these estimates we need the following lemma.

Lemma 2.40. *For an integer d and a sequence $\{N_n\}$ as defined in (2.21)*

$$\frac{1}{(2N_n+1)^d} \sum_{j=1}^{n} (2N_j + 1)^d \leqslant 2^{2d+1}. \tag{2.41}$$

Proof. We first show that

$$\frac{1}{N_n^d} \sum_{j=1}^{n} N_j^d \leqslant 2. \tag{2.42}$$

In fact, the left side of (2.42) is

$$\begin{aligned} &\frac{1}{N_n^d}\left(N_1^d + N_2^d + \cdots + N_{n-1}^d + N_n^d\right) \\ &= 1 + \left(\frac{N_{n-1}}{N_n}\right)^d + \cdots + \left(\frac{N_2}{N_n}\right)^d + \left(\frac{N_1}{N_n}\right)^d \\ &= 1 + \frac{(K_1+1)^d \cdots (K_{n-1}+n-1)^d}{(K_1+1)^d \cdots (K_{n-1}+n-1)^d (K_n+n)^d} + \cdots + \frac{1}{(K_2+2)^d \cdots (K_n+n)^d} \\ &= 1 + \frac{1}{(K_n+n)^d} + \cdots + \frac{1}{(K_2+2)^d \cdots (K_n+n)^d} \end{aligned}$$

$$
\begin{aligned}
&\leqslant 1+\frac{1}{n^d}+\frac{1}{(n(n-1))^d}+\cdots+\frac{1}{(2\cdot 3\cdots n)^d} \\
&\leqslant 1+\frac{1}{n^d}+\frac{1}{(n(n-1))^d}+\cdots+\frac{1}{(2\cdot 3\cdots n)^d}+\frac{1}{(n!)^d} \\
&= \frac{1}{(n!)^d}\sum_{j=0}^{n}(j!)^d = \frac{1}{(n!)^d}\left((n!)^d+((n-1)!)^d+\cdots+(2!)^d+(1!)^d+(0!)^d\right) \\
&\leqslant 1+\frac{n((n-1)!)^d}{(n!)^d} \leqslant 1+\frac{(n!)^d}{(n!)^d} = 1+1 = 2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{1}{(2N_n+1)^d}\sum_{j=1}^{n}(2N_j+1)^d &\leqslant \frac{\sum_{j=1}^{n}(2N_j+2N_j)^d}{(2N_n+1)^d} = \frac{4^d\sum_{j=1}^{n}N_j^d}{(2N_n+1)^d} \\
&\leqslant \frac{4^d}{N_n^d}\sum_{j=1}^{n}N_j^d \leqslant 4^d\cdot 2 = 2^{2d+1}
\end{aligned}
$$

which establishes (2.41). □

(e) Our initial step in estimating the first sum on the right side of (2.39) is to prove that $\exists\ q_3 = q_3(\epsilon, k)$ such that $\forall j \geqslant q_3$,

$$
X = \left|\frac{1}{(2(N_{j+1}-k_0)+1)^d}\left(\sum_{S(N_{j+1}-k_0)} x_j[k+m]\overline{x_j[m]} - \sum_{A_j(N)\cup S(N_j)} x_j[k+m]\overline{x_j[m]}\right)\right| < \epsilon. \tag{2.43}
$$

The difference of the two sums in the left side of (2.43) is the summation over $A_{j-1}(N)$. Thus

$$
\begin{aligned}
X &= \left|\frac{1}{(2(N_{j+1}-k_0)+1)^d}\left(\sum_{A_{j-1}(N)} x_j[k+m]\overline{x_j[m]}\right)\right| \\
&\leqslant \frac{1}{(2(N_{j+1}-k_0)+1)^d}\sum_{A_{j-1}(N)}\left|x_j[k+m]\overline{x_j[m]}\right|.
\end{aligned} \tag{2.44}
$$

Again if $m \in A_{j-1}(N)$ then $m \in S(N_j + k_0) \setminus S(N_j)$. So from (2.44)

$$\begin{aligned} X &\leqslant \frac{1}{(2(N_{j+1}-k_0)+1)^d} \sum_{S(N_j+k_0)\setminus S(N_j)} \left| x_j[k+m]\overline{x_j[m]} \right| \\ &\leqslant \frac{\|x_j\|_\infty^2}{(2(N_{j+1}-k_0)+1)^d}(2k_0)^d. \end{aligned} \tag{2.45}$$

The right side of (2.45) goes to 0 as $j \to \infty$ due to (2.25) and we have (2.43).

Therefore,

$$\begin{aligned} &\left| \frac{1}{(2(N_{j+1}-k_0)+1)^d} \sum_{A_j(N)\cup S(N_j)} x_j[m+k]\overline{x_j[m]} - \check{\mu}[k] \right| \\ = &\left| \frac{1}{(2(N_{j+1}-k_0)+1)^d} \left(\sum_{A_j(N)\cup S(N_j)} x_j[k+m]\overline{x_j[m]} - \sum_{S(N_{j+1}-k_0)} x_j[k+m]\overline{x_j[m]} \right) + \right. \\ &\left. + \frac{1}{(2(N_{j+1}-k_0)+1)^d} \sum_{S(N_{j+1}-k_0)} x_j[k+m]\overline{x_j[m]} - \check{\mu}_j[k] + \check{\mu}_j[k] - \check{\mu}[k] \right| \\ \leqslant &\ \epsilon + \frac{1}{2^{j+1}} + \epsilon = 2\epsilon + \frac{1}{2^{j+1}} \quad \text{(by the triangle inequality, (2.43), (2.20) and (2.29)).} \end{aligned} \tag{2.46}$$

We can write (2.46) in the form

$$\frac{1}{(2(N_{j+1}-k_0)+1)^d} \sum_{A_j(N)\cup S(N_j)} x_j[m+k]\overline{x_j[m]} = \check{\mu}[k] + \beta_j(k)\left(2\epsilon + \frac{1}{2^{j+1}}\right) \tag{2.47}$$

where $|\beta_j(k)| < 1$ and $j \geqslant q_3$.

(f) To estimate the second sum on the right side of (2.39) we use the triangle inequality, (2.20) and (2.29) to get for all $j > q_2$,

$$\begin{aligned} &\left| \frac{1}{(2N_j+1)^d} \sum_{S(N_j)} x_j[k+m]\overline{x_j[m]} - \check{\mu}[k] \right| \\ = &\left| \frac{1}{(2N_j+1)^d} \sum_{S(N_j)} x_j[k+m]\overline{x_j[m]} - \check{\mu}_j[k] + \check{\mu}_j[k] - \check{\mu}[k] \right| \\ \leqslant &\ \frac{1}{2^{j+1}} + \epsilon. \end{aligned} \tag{2.48}$$

This yields

$$\frac{1}{(2N_j+1)^d}\sum_{S(N_j)} x_j[k+m]\overline{x_j[m]} = \check{\mu}[k] + \gamma_j(k)\left(\epsilon + \frac{1}{2^{j+1}}\right) \tag{2.49}$$

where $|\gamma_j(k)| < 1$ and $j > q_2$.

(g) Since $A_{j-1}(k) \subseteq S(N_j + k_0) \setminus S(N_j)$, the third sum on the right side of (2.39) is bounded by

$$\frac{1}{(2N+1)^d}\sum_{S(N_j+k_0)\setminus S(N_j)} |x_{j-1}[k+m]\overline{x_j[m]}| \leqslant \frac{1}{(2N+1)^d}\|x_{j-1}\|_\infty \|x_j\|_\infty (2k_0)^d. \tag{2.50}$$

(h) By parts (e), (f) and (g) we can write

$$\begin{aligned}
\sum_{j=q}^{n-1} b(j,N) &= \frac{1}{(2N+1)^d}\sum_{j=q}^{n-1}(2(N_{j+1}-k_0)+1)^d\left(\check{\mu}[k] + \beta_j(k)\left(2\epsilon + \frac{1}{2^{j+1}}\right)\right) - \\
&\quad - \frac{1}{(2N+1)^d}\sum_{j=q}^{n-1}(2N_j+1)^d\left(\check{\mu}[k] + \gamma_j(k)\left(\epsilon + \frac{1}{2^{j+1}}\right)\right) + \\
&\quad + \frac{1}{(2N+1)^d}\sum_{j=q}^{n-1}\sum_{A_{j-1}(k)} x_{j-1}[k+m]\overline{x_j[m]} \\
&= \frac{\check{\mu}[k]}{(2N+1)^d}\sum_{j=q}^{n-1}\left((2(N_{j+1}-k_0)+1)^d - (2N_j+1)^d\right) + r(q,N,k)
\end{aligned} \tag{2.51}$$

where

$$\begin{aligned}
r(q,N,k) &= \frac{1}{(2N+1)^d}\sum_{j=q}^{n-1}(2(N_{j+1}-k_0)+1)^d\beta_j(k)\left(2\epsilon + \frac{1}{2^{j+1}}\right) - \\
&\quad - \frac{1}{(2N+1)^d}\sum_{j=q}^{n-1}(2N_j+1)^d\gamma_j(k)\left(\epsilon + \frac{1}{2^{j+1}}\right) + \\
&\quad + \frac{1}{(2N+1)^d}\sum_{j=q}^{n-1}\sum_{A_{j-1}(k)} x_{j-1}[k+m]\overline{x_j[m]}.
\end{aligned}$$

Using the triangle inequality and (2.50)

$$\begin{aligned}|r(q,N,k)| &\leqslant 2\epsilon\sum_{j=q}^{n-1}\frac{(2(N_{j+1}-k_0)+1)^d}{(2N+1)^d}+\sum_{j=q}^{n-1}\frac{1}{2^{j+1}}+\epsilon\sum_{j=q}^{n-1}\frac{(2N_j+1)^d}{(2N+1)^d}+\\ &+\sum_{j=q}^{n-1}\frac{1}{2^{j+1}}+\frac{(2k_0)^d}{(2N+1)^d}\sum_{j=q}^{n-1}\|x_{j-1}\|_\infty\|x_j\|_\infty.\end{aligned}$$

Using (2.41) we further get

$$|r(q,N,k)|\leqslant 2\epsilon\cdot 2^{2d+1}+\sum_{j=q}^{n-1}\frac{1}{2^j}+\epsilon\cdot 2^{2d+1}+\frac{(2k_0)^d}{(2N+1)^d}\sum_{j=q}^{n-1}\|x_{j-1}\|_\infty\|x_j\|_\infty. \tag{2.52}$$

Due to (2.25) the last term in the right side of (2.52) is less than ϵ for large enough N. Therefore,

$$|r(q,N,k)|\leqslant \epsilon\cdot 4^{d+1}+\frac{1}{2^{q-1}}+\epsilon\cdot 2^{2d+1}+\epsilon.$$

So, at this point we have estimated $\sum_{j=q}^{n-1}b(j,N)$ as described in (2.37).

(i) Looking back at part (b) we see that having estimated $\sum_{j=q}^{n-1}b(j,N)$ and $\sum_{j=q}^{n-1}c(j,N)$ we now have to estimate the last sum on the right side of (2.33). We define the sets

$$\begin{aligned}A_{n-1}(k) &= \{m : m\in S(N)\setminus S(N_n) \text{ and } m+k\in S(N_n)\setminus S(N_{n-1})\}\\ A_n(k) &= \{m : m\in S(N)\setminus S(N_n) \text{ and } m+k\in S(N_{n+1})\setminus S(N_n)\}\\ A_{n+1}(k) &= \{m : m\in S(N)\setminus S(N_n) \text{ and } m+k\in S(N_{n+2})\setminus S(N_{n+1})\}.\end{aligned}$$

Then,

$$\begin{aligned}\frac{1}{(2N+1)^d}\sum_{S(N)\setminus S(N_n)}x[k+m]\overline{x[m]} &= \frac{1}{(2N+1)^d}\sum_{A_{n-1}(k)}x_{n-1}[k+m]\overline{x_n[m]}+\\ &+\frac{1}{(2N+1)^d}\sum_{A_n(k)}x_n[k+m]\overline{x_n[m]}+\frac{1}{(2N+1)^d}\sum_{A_{n+1}(k)}x_{n+1}[k+m]\overline{x_n[m]}\end{aligned}$$

$$= \frac{1}{(2N+1)^d} \sum_{A_n(k)\cup S(N_n)} x_n[k+m]\overline{x_n[m]} - \frac{1}{(2N+1)^d} \sum_{S(N_n)} x_n[k+m]\overline{x_n[m]} + \\ + \frac{1}{(2N+1)^d} \left(\sum_{A_{n-1}(k)} x_{n-1}[k+m]\overline{x_n[m]} + \sum_{A_{n+1}(k)} x_{n+1}[k+m]\overline{x_n[m]} \right). \tag{2.53}$$

(j) The inclusions $A_{n-1}(k) \subseteq S(N_n + k_0) \setminus S(N_n)$ and $A_{n+1}(k) \subseteq S(N) \setminus S(N_{n+1} - k_0)$ are valid and it should be observed that $A_{n+1}(k)$ can be empty if $N < N_{n+1} - k_0$. In any case the last term in (2.53) is dominated by

$$\begin{aligned} &\frac{1}{(2N+1)^d}\Big(\sum_{S(N_n+k_0)\setminus S(N_n)} |x_{n-1}[k+m]\overline{x_n[m]}| + \\ &\qquad \sum_{S(N)\setminus S(N_{n+1}-k_0)} |x_{n+1}[k+m]\overline{x_n[m]}|\Big) \\ &\leqslant \frac{1}{(2N+1)^d} \left\{ \|x_{n-1}\|_\infty \|x_n\|_\infty (2k_0)^d + \|x_{n+1}\|_\infty \|x_n\|_\infty (2k_0)^d \right\} \\ &= \frac{(2k_0)^d}{(2N+1)^d} \|x_n\|_\infty \left(\|x_{n-1}\|_\infty + \|x_{n+1}\|_\infty \right) \\ &< \epsilon \qquad \text{for large enough } N. \end{aligned} \tag{2.54}$$

(k) To estimate the first sum of (2.53) we note that

$$\begin{aligned} &\left| \frac{1}{(2N+1)^d} \left(\sum_{S(N)} x_n[k+m]\overline{x_n[m]} - \sum_{A_n(k)\cup S(N_n)} x_n[k+m]\overline{x_n[m]} \right) \right| \\ = &\left| \frac{1}{(2N+1)^d} \sum_{A_{n-1}(k)\cup A_{n+1}(k)} x_n[k+m]\overline{x_n[m]} \right| \\ \leqslant & \frac{\|x_n\|_\infty^2}{(2N+1)^d} \left(|A_{n-1}(k)| + |A_{n+1}(k)| \right) \\ \leqslant & \frac{\|x_n\|_\infty^2}{(2N+1)^d} \left((2k_0)^d + (2k_0)^d \right) \\ < & \ \epsilon \qquad \text{for large enough } N. \end{aligned} \tag{2.55}$$

So,

$$
\begin{aligned}
&\left|\frac{1}{(2N+1)^d}\sum_{A_n(k)\cup S(N_n)} x_n[k+m]\overline{x_n[m]} - \check{\mu}[k]\right| \\
= &\left|\frac{1}{(2N+1)^d}\left(\sum_{A_n(k)\cup S(N_n)} x_n[k+m]\overline{x_n[m]} - \sum_{S(N)} x_n[k+m]\overline{x_n[m]}\right) + \right. \\
&\left. + \frac{1}{(2N+1)^d}\sum_{S(N)} x_n[k+m]\overline{x_n[m]} - \check{\mu_n}[k] + \check{\mu_n}[k] - \check{\mu}[k]\right| \\
\leqslant\ &\epsilon + \frac{1}{2^{n+1}} + \epsilon = 2\epsilon + \frac{1}{2^{n+1}} \quad \text{(by the triangle inequality, (2.55), (2.20) and (2.29)).}
\end{aligned}
$$

So,

$$
\frac{1}{(2N+1)^d}\sum_{A_n(k)\cup S(N_n)} x_n[k+m]\overline{x_n[m]} = \check{\mu[k]} + \beta_{N,n}(k)\left(2\epsilon + \frac{1}{2^{n+1}}\right) \tag{2.56}
$$

where $|\beta_{N,n}(k)| < 1$.

For the second term of (2.53) we have

$$
\begin{aligned}
&\left|\frac{1}{(2N_n+1)^d}\sum_{S(N_n)} x_n[k+m]\overline{x_n[m]} - \check{\mu}[k]\right| \\
= &\left|\frac{1}{(2N_n+1)^d}\sum_{S(N_n)} x_n[k+m]\overline{x_n[m]} - \check{\mu}_n[k] + \check{\mu}_n[k] - \check{\mu}[k]\right| \\
\leqslant\ &\frac{1}{2^{n+1}} + \epsilon \quad \text{(by the triangle inequality, (2.20) and (2.29)).}
\end{aligned}
\tag{2.57}
$$

So,

$$
\frac{1}{(2N_n+1)^d}\sum_{S(N_n)} x_n[k+m]\overline{x_n[m]} = \check{\mu}[k] + \gamma_{N,n}(k)\left(\epsilon + \frac{1}{2^{n+1}}\right) \tag{2.58}
$$

where $|\gamma_{N,n}(k)| < 1$.

Therefore, the last term in the right side of (2.33) which we are in the process of estimating is

$$\begin{aligned}
&\frac{1}{(2N+1)^d}\sum_{S(N)\setminus S(N_n)} x[k+m]\overline{x[m]} = \check{\mu}[k] + \beta_n(k)\left(2\epsilon + \frac{1}{2^{n+1}}\right) - \\
&- \frac{(2N_n+1)^d}{(2N+1)^d}\left(\check{\mu}[k] + \gamma_n(k)\left(\epsilon + \frac{1}{2^{n+1}}\right)\right) + \\
&+ \frac{1}{(2N+1)^d}\left(\sum_{A_{n-1}(k)} x_{n-1}[k+m]\overline{x_n[m]} + \sum_{A_{n+1}(k)} x_{n+1}[k+m]\overline{x_n[m]}\right) \\
= \ & \check{\mu}[k]\left(1 - \left(\frac{2N_n+1}{2N+1}\right)^d\right) + s(N,k)
\end{aligned}$$

where

$$\begin{aligned}
s(N,k) \ = \ & \beta_n(k)\left(2\epsilon + \frac{1}{2^{n+1}}\right) - \gamma_n(k)\left(\epsilon + \frac{1}{2^{n+1}}\right)\left(\frac{2N_n+1}{2N+1}\right)^d + \\
& + \frac{1}{(2N+1)^d}\left(\sum_{A_{n-1}(k)} x_{n-1}[k+m]\overline{x_n[m]} + \sum_{A_{n+1}(k)} x_{n+1}[k+m]\overline{x_n[m]}\right).
\end{aligned}$$

By the triangle inequality, (2.56), (2.58) and (2.54),

$$|s(N,k)| \leqslant 2\epsilon + \frac{1}{2^{n+1}} + \epsilon + \frac{1}{2^{n+1}} + \epsilon = 4\epsilon + \frac{1}{2^n}.$$

By this we have verified (2.38).

(l) This part completes the proof of the theorem by combining what we obtained in the previous parts. For $q \geqslant q_3$ and for all large N, $n(N) > q$, the right side of (2.33) is

$$\begin{aligned}
&\frac{\check{\mu}[k]}{(2N+1)^d}\left(\sum_{j=q}^{n-1}\left((2(N_{j+1}-k_0)+1)^d - (2N_j+1)^d\right)\right) + r(q,N,k) + \\
&+ \check{\mu}[k]\left(1 - \left(\frac{2N_n+1}{2N+1}\right)^d\right) + s(N,k) + \sum_{j=q}^{n-1} c(j,N) \\
= \ & \check{\mu}[k] + \frac{\check{\mu}[k]}{(2N+1)^d}\left[\sum_{j=q}^{n-1}\left((2(N_{j+1}-k_0)+1)^d - (2N_j+1)^d\right) - (2N_n+1)^d\right] + \\
&+ s(N,k) + r(q,N,k) + \sum_{j=q}^{n-1} c(j,N). \qquad (2.59)
\end{aligned}$$

On the right side of (2.59) we see that $\exists\ N_c$ such that $\forall N > N_c$,

$$\left|s(N,k) + r(q,N,k) + \sum_{j=q}^{n-1} c(j,N)\right| \leqslant 4\epsilon + \frac{1}{2^n} + \epsilon \cdot 4^{d+1} + \frac{1}{2^{q-1}} + \epsilon \cdot 2^{2d+1} + \epsilon + \epsilon. \quad (2.60)$$

The second term on the right side of (2.59) is,

$$\begin{aligned}
& \frac{\check{\mu}[k]}{(2N+1)^d}\Big[\, (2(N_n - k_0) + 1)^d - (2N_{n-1} + 1)^d + (2(N_{n-1} - k_0) + 1)^d - \\
& - (2N_{n-2} + 1)^d + (2(N_{n-2} - k_0) + 1)^d - (2N_{n-3} + 1)^d + \cdots + \\
& + (2(N_{q+1} - k_0) + 1)^d - (2N_q + 1)^d - (2N_n + 1)^d \Big] \\
\leqslant \quad & \frac{\check{\mu}[k]}{(2N+1)^d}\Big[(2N_n + 1)^d - (2N_{n-1} + 1)^d + (2N_{n-1} + 1)^d - (2N_{n-2} + 1)^d + \\
& + (2N_{n-2} + 1)^d - (2N_{n-3} + 1)^d + \cdots + (2N_{q+1} + 1)^d - (2N_q + 1)^d - (2N_n + 1)^d \Big]
\end{aligned}$$

which after the appropriate cancellations is in absolute value

$$= \quad \left|\frac{\check{\mu}[k]}{(2N+1)^d}(2N_q + 1)^d\right| < \epsilon \qquad \text{for all } N \text{ large enough.} \quad (2.61)$$

Using (2.60) and (2.61) we get

$$\left|\frac{1}{(2N+1)^d} \sum_{S(N)\backslash S(N_q)} x[k+m]\overline{x[m]} - \check{\mu}[k]\right| < \epsilon(8 + 4^{d+1} + 2^{2d+1}) \quad (2.62)$$

and this proves (2.31). One can refer to part (a) to recall that this is what we needed in order to finish the proof of the theorem. $\square$

Chapter 3

Construction of Uniformly Bounded Waveforms

We have shown in Chapter 2 that the function $x : \mathbb{Z}^d \to \mathbb{C}$ as constructed in the Wiener-Wintner Theorem on $\mathbb{Z}^d$ is locally bounded. Having uniform boundedness in x would be a step towards our ultimate goal of constructing waveforms with constant amplitude. In this chapter we discuss how one can try to get the waveform x to be uniformly bounded. We try to use uniformly distributed sequences to achieve this. Though this might seem a natural approach it can be shown that it is impossible to use uniformly distributed sequences to make x uniformly bounded.

For any positive function $F \in A(\mathbb{T}^d)$, the space of absolutely convergent Fourier series on $\mathbb{T}^d$, our *problem* is to construct a complex-valued *bounded* function $x : \mathbb{Z}^d \to \mathbb{C}$ whose autocorrelation A_x is the Fourier transform of F. Formally, the autocorrelation A_x is defined as

$$\forall k \in \mathbb{Z}^d, \quad A_x[k] = \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{m \in S(N)} x[k+m]\overline{x[m]} \ ,$$

where

$$S(N) = \{m = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d : -N \leqslant m_i \leqslant N, i = 1, ..., d\}$$

and the Fourier transform of $F \in A(\mathbb{T}^d)$ is

$$\forall k \in \mathbb{Z}^d, \quad p_k = \int_{\mathbb{T}^d} F(\gamma) e^{2\pi i k.\gamma} d\gamma.$$

3.1 Theory of uniform distribution

Definition 3.1. Let $A = (a_{nk})$ be a matrix which satisfies the following two conditions:

(i) For all n and k, $a_{nk} \geqslant 0$.

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

Definition 3.2. Let $A = (a_{n,k})$ be a matrix as in Definition 3.1, and let (x_n), $n = 1, 2, \cdots$, be a sequence of points in $\mathbb{R}^d$. For $\mathbf{0} \leqslant x < \mathbf{1}$, let $\mathbb{1}_x$ be the characteristic function of the interval $[\mathbf{0}, x)$. The function $g(x)$, $x \in \mathbb{T}^d$, is the *A-asymptotic distribution function modulo* 1 *(A-a.d.f(mod 1))* of (x_n) if the sequence $(\mathbb{1}_x(\{x_n\}))$, $n = 1, 2, \cdots$, is summable by A to the value $g(x)$ for $x \in \mathbb{T}^d$; that is, if

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} \mathbb{1}_x(\{x_k\}) = g(x) \tag{3.3}$$

In the case $g(x) = x$ for $x \in \mathbb{T}^d$, the sequence (x_n) is called *A-u.d. mod 1.* The following result is found in [16].

Theorem 3.4. *Let $A = (a_{nk})$ be a matrix as in Definition 3.1. A sequence (x_n) has the continuous A-a.d.f.(mod 1) $g(x)$ if and only if for every real-valued continuous function f on $\mathbb{T}^d$,*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} f(\{x_k\}) = \int_{\mathbb{T}^d} f(x) dg(x). \tag{3.5}$$

Suppose $g(x) = x$ and $A = (a_{nk})$ is a matrix as defined in Defintion 3.1 which is *lower triangular.* Then (3.5) becomes

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} f(\{x_k\}) = \int_{\mathbb{T}^d} f(x) dx \tag{3.6}$$

for every real valued continuous function f on $\mathbb{T}^d$.

3.2 Solving the problem using uniformly distributed sequences

As already said we would like to establish the following assertion.

Let $F \in A(\mathbb{T}^d)$ be non-negative. There is $x \in \ell^\infty(\mathbb{Z}^d)$ such that, for each $k \in \mathbb{Z}^d$,

$$\lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{m\in S(N)} x[k+m]\overline{x[m]} = \check{F}[k] \tag{3.7}$$

where $S(N) = \{m = (m_1, m_2, \cdots, m_d) \in \mathbb{Z}^d : -N \leqslant m_i \leqslant N, i = 1, \cdots, d\}$.

Due to the Wiener Wintner Theorem on $\mathbb{Z}^d$ which we proved in Chapter 2 it is natural to try to prove this in the following way. Suppose the given positive function $F \in A(\mathbb{T}^d)$ has Fourier coefficients $\{p_k\}_{k\in\mathbb{Z}^d}$ i.e., $\check{F} = \{p_k\}$. In order to construct the desired uniformly bounded waveform x, F is first approximated by a sequence of discrete measures, i.e., a sequence $\{\mu_n\}_{n\in\mathbb{Z}^d} \subseteq M(\mathbb{T}^d)$. For each n, the discrete measure μ_n is supported in the set $\{\gamma_j : j = 1, \ldots, n\} \subseteq I^d$ where $\{\gamma_j\}$ is A-u.d. mod 1. We define

$$\mu_n = \sum_{j=1}^{n} a_{nj} \left(\sum_{k\in S(L_n)} p_k e^{-2\pi i k\cdot\gamma_j} \right) \delta_{\gamma_j}. \tag{3.8}$$

The sequence $\{\mu_n\}$ converges to F in the weak-$*$ $\sigma(M(\mathbb{T}^d), C(\mathbb{T}^d))$ topology. This can be shown as a result of 3.6 and following the proof of Proposition 2.3.

Next a sequence $\{x_n\} \subseteq \ell^\infty(\mathbb{Z}^d)$ is constructed based on the μ_ns such that each $A_{x_n} = \check{\mu_n}$, where $\check{\mu_n} : \mathbb{Z}^d \to \mathbb{C}$ is the inverse Fourier transform of μ_n. Define

$$x_n[k] = \sum_{j=1}^{n} a_{nj}^{\frac{1}{2}} \left(\sum_{\ell\in S(L_n)} p_\ell e^{-2\pi i \ell\cdot\gamma_j} \right)^{\frac{1}{2}} e^{-2\pi i k\cdot\gamma_j}. \tag{3.9}$$

In fact, it can be shown that there is an increasing positive sequence $\{K_n\}$ such that

$\forall k \in \mathbb{Z}^d$ and $\forall N \geqslant K_n$,

$$\left| \frac{1}{(2N+1)^d} \sum_{m \in S(N)} x_n[k+m]\overline{x_n}[m] - \check{\mu_n}[k] \right| < \frac{1}{2^{n+1}}.$$

Then, setting

$$N_n = (K_1+1)(K_2+2)\cdots(K_n+n)$$

the x_ns already constructed are used to define $x : \mathbb{Z}^d \to \mathbb{C}$ as follows:

$$x[k] = \begin{cases} 0 & \text{for } k \in S(N_1) \\ x_n[k] & \text{for } k \in S(N_{n+1}) \setminus S(N_n). \end{cases}$$

Finally, it is checked that $A_x[k] = p_k$.

$\square$

One observes that

$$\begin{aligned} |x_n[k]| &\leqslant \sum_{j=1}^{n} a_{nj}^{\frac{1}{2}} \left(\sum_{\ell \in S(L_n)} |p_\ell| \right)^{\frac{1}{2}} \leqslant \sum_{j=1}^{\infty} a_{nj}^{\frac{1}{2}} \left(\sum_{\ell \in S(L_n)} |p_\ell| \right)^{\frac{1}{2}} \\ &< \left(\sum_{j=1}^{\infty} a_{nj}^{\frac{1}{2}} \right) \|F\|_{A(\mathbb{T}^d)}^{\frac{1}{2}} < \infty. \end{aligned} \tag{3.10}$$

Thus we can get $\{x_n\}$ and so also x to be bounded if we require the matrix $A = (a_{nk})$ to satisfy

$$\forall n, \quad \sum_{k=1}^{\infty} \sqrt{a_{nk}} \leqslant C, \quad C \text{ independent of } n. \tag{3.11}$$

As a possible approach to construct an appropriate uniformly distributed sequence $\{\gamma_j\}$ and a corresponding matrix A, [9] and [16] show that the sequence $\{j\theta\}$ with θ irrational is $A-$u.d. mod 1 if the matrix A satisfies $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k+1} - a_{nk}| = 0$ and for all $k = 1, 2, \cdots$, $\lim_{n\to\infty} a_{nk} = 0$ along with conditions 1 and 2 of Definition 3.2. Thus the matrix $A = (a_{n,k})$ we need so that $\{n\theta\}$ is A-u.d. mod 1 and the waveform $x : \mathbb{Z} \to \mathbb{C}$ is bounded should satisfy the following properties:

(i) $a_{n,k} \geqslant 0$.

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1$.

(iii) For all n, $\sum_{k=1}^{\infty} \sqrt{a_{n,k}} \leqslant C$, where C is a constant independent of n.

(iv) For all $k = 1, 2, \ldots,\quad \lim_{n\to\infty} a_{n,k} = 0$.

(v) $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k+1} - a_{n,k}| = 0$.

(vi) There are only finitely many terms on each row.

Condition (vi) is needed to prove (3.7), the outline of the proof of which has been given.

We have the following examples each of which miss at least one of the above six conditions.

3.2.1 Illustrative examples

Example 1.

$$a_{n,k} = \begin{cases} \frac{1}{2^k} & \textit{if } 1 \leqslant k \leqslant n \\ 0 & \textit{if } k > n. \end{cases}$$

So,

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \ldots \\ \frac{1}{2} & \frac{1}{2^2} & 0 & \ldots \\ \frac{1}{2} & \frac{1}{2^2} & \frac{1}{2^3} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Properties:

(i) $a_{n,k} \geqslant 0$.

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$.

(iii) $\sum_{k=1}^{\infty} \sqrt{a_{n,k}} = \sum_{k=1}^{n} \frac{1}{\sqrt{2^k}} = \sum_{k=1}^{n} \left(\frac{1}{\sqrt{2}}\right)^k \leqslant \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^k = \frac{1}{\sqrt{2}} \frac{1}{\left(1-\frac{1}{\sqrt{2}}\right)}$.

(iv) For any k, $\lim_{n\to\infty} a_{n,k} = \frac{1}{2^k} \neq 0$.

(v) $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k+1} - a_{n,k}| = \left(\frac{1}{2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{2^3}\right) + \cdots \left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right) + \left(\frac{1}{2^n} - 0\right) = \frac{1}{2} \neq 0$.

(vi) There are n terms in row n.

Example 2 (Cesàro summation matrix of order 1).

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Properties:

(i) $a_{n,k} \geqslant 0$.

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = \sum_{k=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = 1$.

(iii) $\sum_{k=1}^{\infty} \sqrt{a_{n,k}} = \sum_{k=1}^{n} \frac{1}{\sqrt{n}} = n \cdot \frac{1}{\sqrt{n}} = \sqrt{n}$ which cannot be bounded by some constant independent of n.

(iv) For $k = 1, 2, \dots,\ \lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty} \frac{1}{n} = 0$.

(v) $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k+1} - a_{n,k}| = \lim_{n\to\infty} \frac{1}{n} = 0$.

(vi) There are n terms in row n.

Example 3.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2^2} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{3^2} & \frac{1}{3^2} & \frac{7}{9} & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{4^2} & \frac{1}{4^2} & \frac{1}{4^2} & \frac{13}{16} & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n^2} & \frac{1}{n^2} & \frac{1}{n^2} & \dots & \frac{1}{n^2} & \frac{n^2-n+1}{n^2} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Properties:

(i) $a_{n,k} \geqslant 0$.

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1$.

(iii) For any n, $\sum_{k=1}^{\infty} \sqrt{a_{n,k}} = \sum_{k=1}^{n-1} \left(\frac{1}{n}\right) + \sqrt{\frac{n^2-n+1}{n^2}} = \frac{n-1}{n} + \sqrt{1 + \left(\frac{-1}{n}\right) + \frac{1}{n^2}}$
$\leqslant 1 + \sqrt{1 + \frac{1}{n^2}} \leqslant 1 + \sqrt{2}$.

(iv) For $k = 1, 2, \dots,\ \lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty} \frac{1}{n^2} = 0$.

(v) $\lim_{n\to\infty} |a_{n,k+1} - a_{n,k}| = \left|\frac{1}{n^2} - \frac{n^2-n+1}{n^2}\right| + \frac{n^2-n+1}{n^2} = \frac{n^2-n}{n^2} + 1 - \frac{1}{n} + \frac{1}{n^2} \nrightarrow 0$.

(vi) There are n terms in row n.

Example 4.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2\cdot\frac{1}{2^2} & 2\cdot\frac{2}{2^2} & 0 & 0 & 0 & 0 & 0 & \dots \\ 2\cdot\frac{1}{3^2} & 2\cdot\frac{2}{3^2} & 2\cdot\frac{3}{3^2} & 0 & 0 & 0 & 0 & \dots \\ 2\cdot\frac{1}{4^2} & 2\cdot\frac{2}{4^2} & 2\cdot\frac{3}{4^2} & 2\cdot\frac{4}{4^2} & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2\cdot\frac{1}{n^2} & 2\cdot\frac{2}{n^2} & 2\cdot\frac{3}{n^2} & 2\cdot\frac{4}{n^2} & \dots & 2\cdot\frac{n}{n^2} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Properties:

(i) $a_{n,k} \geqslant 0$.

(ii) $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k} = \lim_{n\to\infty}\frac{2(1+2+\cdots n)}{n^2} = \lim_{n\to\infty}\frac{2n(n+1)}{2n^2} = \lim_{n\to\infty}\frac{n^2+n}{n^2} = 1.$

(iii) $\sum_{k=1}^{\infty}\sqrt{a_{n,k}} = \frac{\sqrt{2}\left(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}\right)}{n} = \frac{\sqrt{2}}{n}\sum_{k=1}^{n}\sqrt{k} \sim \frac{\sqrt{2}}{n}\int_1^n \sqrt{x}dx = \frac{2\sqrt{2}}{3n}(n^{\frac{3}{2}}-1)$

which cannot be bounded by any constant independent of n.

(iv) For $k = 1, 2\cdots,\ \lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty} 2\cdot\frac{k}{n^2} = 0.$

(v) $\lim_{n\to\infty}\sum_{k=1}^{\infty}|a_{n,k+1} - a_{n,k}| = \lim_{n\to\infty}\left|-\frac{2}{n^2} + \frac{2n}{n^2} + \frac{2}{n}\right|$

$= \lim_{n\to\infty}\left\{(n-1)\cdot 2\cdot\frac{1}{n^2} + \frac{2}{n}\right\} = 0.$

(vi) There are n elements in row n.

Example 5. *$A = a_{n,k}$ is a matrix such that the n^{th} row is*

$$\underbrace{\frac{3}{n^4}\quad 3\cdot\frac{2^2}{n^4}\ \cdots\ 3\cdot\frac{n^2}{n^4}}_{\textit{block } 1}\quad \underbrace{3\cdot\frac{n^2}{n^4}\ \cdots\ 3\cdot\frac{2^2}{n^4}\quad 3\cdot\frac{1}{n^4}}_{\textit{block } 2}\quad \underbrace{\frac{3}{n^4}\quad 3\cdot\frac{2^2}{n^4}\ \cdots\ 3\cdot\frac{n^2}{n^4}}_{\textit{block } 3}\quad \cdots$$

and there are n such blocks in the n^{th} row.

Properties:

(i) $a_{n,k} \geqslant 0$.

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = \lim_{n\to\infty} \underbrace{\dfrac{3(1^2+2^2+\cdots+n^2)}{n^4}}_{\text{sum of each block}} \cdot n = \lim_{n\to\infty} \frac{3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$

$= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{2n^3} = 1.$

(iii) $\sum_{k=1}^{\infty} \sqrt{a_{n,k}} = \frac{\sqrt{3}(1+2+\cdots+n)}{n^2} \cdot n = \frac{\sqrt{3}}{n} \frac{n(n+1)}{2} = \frac{\sqrt{3}}{2}(n+1)$ which is not bounded by any constant independent of n.

(iv) For $k = 1, 2, \cdots$ $\lim_{n\to\infty} a_{n,k} = 0$.

(v) To compute $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k+1} - a_{n,k}|$ first observe that from each block the sum of $|a_{n,k+1} - a_{n,k}|$ is either $3\left|\left(\frac{2^2}{n^4} - \frac{1^2}{n^4}\right) + \left(\frac{3^2}{n^4} - \frac{2^2}{n^4}\right) + \cdots + \left(\frac{n^2}{n^4} - \frac{(n-1)^2}{n^4}\right)\right|$ or $3\left|\left(\frac{n^2}{n^4} - \frac{(n-1)^2}{n^4}\right) + \cdots \left(\frac{3^2}{n^4} - \frac{2^2}{n^4}\right) + \left(\frac{2^2}{n^4} - \frac{1^2}{n^4}\right)\right|$ each of which is equal to $3\left(\frac{n^2-1}{n^4}\right)$. Also, the last term of each block is the same as the first term of the next block and so the difference is 0. Thus,

$$\sum_{k=1}^{\infty} |a_{n,k+1} - a_{n,k}| =$$

$$\begin{cases} \left(n \cdot \frac{3(n^2-1)}{n^4} + 3 \cdot \frac{n^2}{n^4}\right) = \frac{3(n^2-1)}{n^3} + \frac{3}{n^2} & \text{if the } n^{\text{th}} \text{ block ends in } 3\frac{n^2}{n^4} \\ \left(n \cdot \frac{3(n^2-1)}{n^4} + 3 \cdot \frac{1}{n^4}\right) = \frac{3(n^2-1)}{n^3} + \frac{3}{n^4} & \text{if the } n^{\text{th}} \text{ block ends in } 3\frac{1}{n^4}. \end{cases}$$

So, $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k+1} - a_{n,k}| = 0$.

(vi) There are n^2 elements in row n.

Example 6. *Let the matrix A be defined by*

$$a_{n,k} = \begin{cases} 0 & \text{if } k < K_n \\ \frac{c_n}{k^{J_n}} & \text{if } k \geqslant K_n. \text{ (Actually } K_n \leqslant k \leqslant K_n + K_{(n)}, \text{ where } K_{(n)} \text{ is large.)} \end{cases}$$

Properties:

(i) Due the way the matrix elements are defined we have $a_{n,k} \geqslant 0$.

(ii) $\sum_{k=1}^{\infty} a_{n,k} \sim c_n \int_{K_n}^{\infty} \frac{dx}{x^{J_n}} = \frac{c_n}{(-J_n+1)} \left[x^{-J_n+1}\right]_{K_n}^{\infty} = \frac{c_n}{J_n-1} K_n^{-J_n+1}$.

Set $c_n = (J_n - 1)K_n^{J_n-1}$.

Then $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \sim 1$.

(iii) Based on the calculation in part (ii)

$$a_{n,k} = \begin{cases} 0 & \text{if } k < K_n \\ \frac{(J_n-1)K_n^{J_n-1}}{k^{J_n}} & \text{if } k \geqslant K_n. \end{cases} \tag{3.12}$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{\infty} \sqrt{a_{n,k}} &= \sum_{k=K_n}^{\infty} \frac{\sqrt{(J_n-1)}K_n^{\frac{J_n}{2}}}{\sqrt{K_n}k^{\frac{J_n}{2}}} \\ &\sim \frac{\sqrt{(J_n-1)}K_n^{\frac{J_n}{2}}}{\sqrt{K_n}} \int_{K_n}^{\infty} \frac{dx}{x^{\frac{J_n}{2}}} \\ &= \frac{\sqrt{(J_n-1)}K_n^{\frac{J_n}{2}}}{\sqrt{K_n}} \left[\frac{x^{-\frac{J_n}{2}+1}}{\left(-\frac{J_n}{2}+1\right)}\right]_{K_n}^{\infty} \\ &= \frac{\sqrt{(J_n-1)}K_n^{\frac{J_n}{2}}}{\sqrt{K_n}} \frac{K_n^{-\frac{J_n}{2}+1}}{\left(\frac{J_n}{2}-1\right)} \\ &= \sqrt{K_n}\frac{\sqrt{J_n-1}}{\left(\frac{J_n}{2}-1\right)} \leqslant \frac{\sqrt{K_n J_n}}{\left(\frac{J_n}{2}-1\right)}. \end{aligned} \tag{3.13}$$

We want the right side of (3.13) to be bounded by some constant independent of n.

(iv) For $k \geqslant K_n$,

$$a_{n,k} = \frac{(J_n-1)K_n^{J_n-1}}{k^{J_n}} = \frac{(J_n-1)}{K_n}\left(\frac{K_n}{k}\right)^{J_n}. \tag{3.14}$$

We would like the right hand side of (3.14) to go to zero as n goes to infinity.

(v) Row n of our matrix looks like

$$0\ 0\ \cdots\ 0\ 0\ \tfrac{c_n}{K_n^{J_n}}\ \tfrac{c_n}{(K_n+1)^{J_n}}\ \tfrac{c_n}{(K_n+2)^{J_n}}\ \cdots\ \tfrac{c_n}{(K_n+K_{(n)})^{J_n}}\ 0\ \cdots\ 0.$$

$$\begin{aligned}\sum_{k=1}^{\infty}|a_{n,k+1}-a_{n,k}| &= a_{n,K_n}+(a_{n,K_n}-a_{n,K_n+1})+(a_{n,K_n+1}-a_{n,K_n+2})+\cdots\\ &+ (a_{n,K_n+K_{(n)}-1}-a_{n,K_n+K_{(n)}})+a_{n,K_n+K_{(n)}}\\ &= 2a_{n,K_n}=\frac{2(J_n-1)}{K_n}.\end{aligned} \tag{3.15}$$

We would like the right side of (3.15) to go to zero as n goes to infinity.

If we want the right side of (3.15) to go to zero as n goes to infinity we ought to have $K_n \to \infty$ and $\frac{J_n}{K_n} \to 0$, i.e., $K_n \to \infty$ faster than J_n. Then automatically the right side of (3.14) goes to 0 as n goes to ∞ but the right side of (3.13) is not bounded some constant independent of n. To bound the right side of (3.13) by a constant independent of n we may take K_n to be bounded or even $K_n, J_n \to \infty$ at the same rate, for example, $J_n = n, K_n = n$ but then the right side of (3.14) does not go to 0 as n goes to ∞. If $J_n = n^x$ and $K_n = n^y$, for the right side of (3.14) to go to 0 we need $y > x$ and for the right side of (3.13) to go to 0, we need, $\frac{y}{2} + \frac{x}{2} \leqslant x \Rightarrow y \leqslant x$. A contradiction.

The next theorem shows that there cannot be a matrix which satisfies all of the six conditions at the same time which means that the sequence $(n\theta)$ with θ irrational cannot be used for our purpose.

Theorem 3.16. *It is impossible to create a matrix satisfying*

(i) $a_{nk} \geqslant 0$.

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

(iii) $\sum_{k=1}^{\infty} \sqrt{a_{nk}} \leqslant C$, *where* C *is independent of* n.

(iv) *For* $k = 1, 2, 3, \ldots$ $\lim_{n\to\infty} a_{nk} = 0$.

(v) $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k+1} - a_{nk}| = 0$.

(vi) *There are finitely many elements in each row.*

Proof. For a fixed n we can write $\sum_{k=1}^{\infty} a_{nk}$ as

$$\sum_{k=1}^{\infty} a_{nk} = \sum_{k=1}^{\infty} \sqrt{a_{nk}}\sqrt{a_{nk}}$$

which implies,

$$\sum_{k=1}^{\infty} a_{nk} \leqslant ||\sqrt{a_{nk}}||_{\ell^\infty(k)} \sum_{k=1}^{\infty} \sqrt{a_{nk}}.$$

Since there are finitely many terms in each row

$$||\sqrt{a_{nk}}||_{\ell^\infty(k)} = \max_k \sqrt{a_{nk}} = \sqrt{\max_k a_{nk}} = \sqrt{||a_{nk}||_{\ell^\infty(k)}}.$$

So,

$$\sum_{k=1}^{\infty} a_{nk} \leqslant \sqrt{||a_{nk}||_{\ell^\infty(k)}} \sum_{k=1}^{\infty} \sqrt{a_{nk}} \tag{3.17}$$

Claim: $||a_{nk}||_{\ell^\infty(k)} \to 0$ as $n \to \infty$.

The left side of 3.17 goes to one as n goes to infinity and given our claim the right side goes to zero as n goes to infinity because we require $\sum_{k=1}^{\infty} \sqrt{a_{nk}} \leqslant C$, where C is independent of n. This gives us the contradiction that $1 \leqslant 0$ and so such a matrix cannot exist.

Proof of claim. Since $\{||a_{nk}||_{\ell^\infty(k)}\}$ is a bounded sequence, it has a convergent subsequence, $\{||a_{n_m k}||_{\ell^\infty(k)}\}$. We prove the claim by contradiction. We assume that $||a_{n_m k}||_{\ell^\infty(k)} \to a > 0$. For notational convenience let us write a_{nk} instead of $a_{n_m k}$. So we assume that $||a_{nk}||_{\ell^\infty(k)} \to a > 0$.

Since $\lim_{n\to\infty} \sum_{k=1}^\infty a_{nk} = 1$, given $\epsilon > 0$, there exists N_1 such that

$$\forall n > N_1, \quad \left| \sum_{k=1}^{\infty} a_{nk} - 1 \right| < \epsilon.$$

Define $M \in \mathbb{N} \setminus \{1\}$ by $Ma \geqslant 1 + 3\epsilon$.

Define γ by $\gamma = \frac{\epsilon}{\frac{M(M-1)}{2}}$.

Define δ by $\delta = \frac{\epsilon}{M}$.

Take N_2 such that for all $n \geqslant N_2$

$$\big| ||a_{nk}||_{\ell^\infty(k)} - a \big| < \delta$$

i.e.,

$$a - \delta < ||a_{nk}||_{\ell^\infty(k)} < a + \delta.$$

Take N_3 such that for all $n \geqslant N_3$

$$\sum_{k=1}^{\infty} |a_{n,k+1} - a_{nk}| < \gamma.$$

This implies that

$$|a_{n,k+1} - a_{nk}| < \gamma$$

i.e.,

$$a_{nk} - \gamma < a_{n,k+1} < a_{nk} + \gamma.$$

Let $N = \max\{M, N_1, N_2, N_3\}$.

For a given n suppose $\sup_k a_{nk}$ is at a_{nk_1}, $k_1 = k_1(n)$. Then for all $n \geqslant N$

$$
\begin{aligned}
\sum_{k=1}^{\infty} a_{nk} &\geqslant a_{nk_1} + a_{n,k_1+1} + a_{n,k_1+2} + \cdots \\
&> (a-\delta) + (a-\delta-\gamma) + (a-\delta-2\gamma) + \cdots + ((a-\delta) - (M-1)\gamma) \\
&= Ma - M\delta - \gamma(1 + 2 + \cdots + (M-1)) \\
&= Ma - M\delta - \frac{\gamma M(M-1)}{2} \\
&> 1 + 3\epsilon - \epsilon - \epsilon = 1 + \epsilon, \qquad (3.18)
\end{aligned}
$$

a contradiction. □

3.2.2 Generalization of the notion of uniform distribution mod 1 due to M. Tsuji ([27])

In this section we investigate the prospect of using summation methods other than the matrix summation method discussed in the last section.

Let $\lambda_n > 0$ be a sequence which satisfies the following condition

$$
(A): \begin{cases} \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n > 0, \\ \sum_{n=1}^{\infty} \lambda_n = \infty. \end{cases}
$$

If for any open interval I in $[0,1]$,

$$
\lim_{n\to\infty} \frac{\lambda_1 \mathbb{1}(\{x_1\}) + \cdots + \lambda_n \mathbb{1}(\{x_n\})}{\lambda_1 + \lambda_2 + \cdots + \lambda_n} = |I|
$$

then we say that (x_n) is (λ_n) - uniformly distributed mod 1. The uniform distribution mod 1 is a special case, where $\lambda_n = 1 (n = 1, 2, \ldots)$.

The following theorem has been proved in [27].

Theorem 3.19. *The necessary and sufficient condition that (x_n) is (λ_n) - uniformly distributed mod 1 is that, for any Riemann integrable function $f(x)$ in $[0,1]$,*

$$\lim_{n\to\infty} \frac{\lambda_1 f(\{x_1\}) + \cdots + \lambda_n f(\{x_n\})}{\lambda_1 + \lambda_2 + \cdots + \lambda_n} = \int_0^1 f(x)dx. \tag{3.20}$$

Weyl's Criterion: The necessary and sufficient condition that (x_n) is (λ_n) - uniformly distributed mod 1 is that, for $m = \pm 1, \pm 2, \ldots$

$$\sum_{k=1}^{n} \lambda_k e^{2\pi i m x_k} = o\left(\sum_{k=1}^{n} \lambda_k\right)$$

as n goes to infinity.

Let the symbol $\log_k n$ mean $\log(\log_{k-1} n)$. Let $g(t) = at^{\sigma}(\log t)^{\sigma_1} \cdots (\log_k t)^{\sigma_k}$ where $a > 0, 0 \leqslant \sigma < 1$ and $\sigma, \sigma_1, \ldots, \sigma_k$ are such that the first one of $\sigma, \sigma_1, \ldots, \sigma_k$, which is not zero, is positive and the other σ_i may be greater than, equal to or less than 0. Then $g(t)$ goes to infinity as t goes to infinity and it was shown in [27] that $(g(n))$ is $\left(\frac{1}{n \log n \cdots \log_{k-1} n}\right)$ - uniformly distributed mod 1. So here $\lambda_n = \left(\frac{1}{n \log n \cdots \log_{k-1} n}\right)$.

Comparing this situation to what we discussed earlier we see that the corresponding weights of the matrix summation method are $a_m = \frac{\lambda_m}{\sum_{m=1}^{n} \lambda_m}$. For our purpose we want

$$\begin{aligned}\sum_{m=1}^{n} a_m = \sum_{m=1}^{n} \sqrt{\frac{\lambda_m}{\sum_{m=1}^{n} \lambda_m}} &= \frac{\sum_{m=1}^{n} \sqrt{\lambda_m}}{\sqrt{\sum_{m=1}^{n} \lambda_m}} \\ &= \frac{\sum_{m=1}^{n} \frac{1}{\sqrt{n \log n \cdots \log_{k-1} n}}}{\sqrt{\sum_{m=1}^{n} \frac{1}{n \log n \cdots \log_{k-1} n}}} \sim \frac{\int_1^n \frac{dx}{\sqrt{x \log x \cdots \log_{k-1} x}}}{\sqrt{\int_1^n \frac{dx}{x \log x \cdots \log_{k-1} x}}}\end{aligned} \tag{3.21}$$

to be bounded by some constant independent of n. We will now show that this cannot be done. First let us try to evaluate the denominator on the right hand side of (3.21). Note that

$$\begin{aligned}
\frac{d(\log_k x)}{dx} &= \frac{d(\log(\log_{k-1} x))}{dx} = \frac{1}{\log_{k-1} x}\frac{d(\log(\log_{k-2} x))}{dx} \\
&= \frac{1}{\log_{k-1} x \log_{k-2} x}\frac{d(\log_{k-2} x)}{dx} = \cdots = \frac{1}{\log_{k-1} x \log_{k-2} x \cdots \log x}\frac{d(\log x)}{dx} \\
&= \frac{1}{\log_{k-1} x \log_{k-2} x \cdots \log x \cdot x}. \qquad (3.22)
\end{aligned}$$

So

$$\int_1^n \frac{dx}{x \log x \cdots \log_{k-1} x} = \log_k x\Big|_1^n = \log_k n. \qquad (3.23)$$

Therefore, the denominator on the right hand side of (3.21) is just $\sqrt{\log_k n}$. To estimate the numerator on the right side of (3.21) suppose that t_k is a value for which $\log t_k \geqslant 0, \log_2 t_k \geqslant 0, \ldots \log_{k-1} t_k \geqslant 0$. Note that t_k has to be greater than 1. Then

$$\begin{aligned}
&\int_{t_k}^n \frac{dx_1}{\sqrt{x_1 \log x_1 \cdots \log_{k-1} x_1}} \geqslant \int_{t_k}^n \frac{dx_1}{x_1\sqrt{\log x_1 \cdots \log_{k-1} x_1}} \\
&\overbrace{=}^{x_2=\log(x_1)} \int_{\log t_k}^{\log n} \frac{dx_2}{\sqrt{x_2 \log x_2 \cdots \log_{k-2} x_2}} \geqslant \int_{\log t_k}^{\log n} \frac{dx_2}{x_2\sqrt{\log x_2 \cdots \log_{k-2} x_1}} \\
&\overbrace{=}^{x_3=\log(x_2)} \int_{\log(\log t_k)}^{\log(\log n)} \frac{dx_3}{\sqrt{x_3 \log x_3 \cdots \log_{k-3} x_3}} \\
&\geqslant \int_{\log_2 t_k}^{\log_2 n} \frac{dx_3}{x_3\sqrt{\log x_3 \cdots \log_{k-3} x_3}} \cdots \geqslant \int_{\log_{k-2} t_k}^{\log_{k-2} n} \frac{dx_{k-1}}{x_{k-1}\sqrt{\log x_{k-1}}} \\
&\overbrace{=}^{x_k=\log(x_{k-1})} \int_{\log_{k-1} t_k}^{\log_{k-1} n} \frac{dx_k}{\sqrt{x_k}} = 2\sqrt{x_k}\Big|_{\log_{k-1} t_k}^{\log_{k-1} n} = 2\sqrt{\log_{k-1} n} - 2\sqrt{\log_{k-1} t_k}.
\end{aligned}$$

So the right side of (3.21) is greater than or equal to

$$\frac{2\sqrt{\log_{k-1} n} - 2\sqrt{\log_{k-1} t_k}}{\sqrt{\log_k n}}$$

which goes to infinity as n goes to infinity and so cannot be bounded.

3.3 Why this method fails

At this point one is inclined to ask the following question. Can there exist a sequence (x_k) and a matrix $A = (a_{n_k})$ satisfying

(i) $a_{n_k} \geqslant 0$

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n_k} = 1$ and

(iii) $\forall n, \sum_{k=1}^{\infty} \sqrt{a_{n_k}} \leqslant C$ (C does not depend on n)

so that (x_k) is $A-$u.d. mod 1 i.e.,

$$\forall f \in C(\mathbb{T}),\ \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n_k} f(x_k) = \int_0^1 f(x)dx \quad ? \tag{3.24}$$

The answer is no and that suggests that we cannot apply the technique of using sequences which are uniformly distributed with respect to some matrix A for obtaining the desired boundedness in the function x that we construct.

Let us now prove why we cannot have such a sequence and a corresponding matrix.

Claim. $\sup_k \sqrt{a_{nk}} \to 0$ as $n \to \infty$.

Once we prove that the claim is true we can argue as follows:

$$\begin{aligned} \sum_{k=1}^{\infty} a_{nk} &= \sum_{k=1}^{\infty} \sqrt{a_{nk}}\sqrt{a_{nk}} \\ &\leqslant \sup_k \sqrt{a_{nk}} \sum_{k=1}^{\infty} \sqrt{a_{nk}} \\ &\leqslant \sup_k \sqrt{a_{nk}} C. \end{aligned} \tag{3.25}$$

Taking limits on both sides of (3.25) we get $1 \leqslant 0$ which is impossible.

Proof of claim. Suppose $\sup_k a_{nk} \to \delta$ as $n \to \infty$. We will show that $\delta = 0$.

Given ϵ ($\epsilon << \delta$), there exists a $N_1 \in \mathbb{N}$ such that for all $n \geqslant N_1$,

$$\left|\sup_k a_{nk} - \delta\right| < \epsilon_1$$

or,

$$\delta - \epsilon_1 < \sup_k a_{nk} < \epsilon_1 + \delta.$$

Given ϵ_2, there exists $N_2 \in \mathbb{N}$ such that for all $n \geqslant N_2$ and for all $f \in C(\mathbb{T})$,

$$\left|\sum_{k=1}^{\infty} a_{nk} f(x_k) - \int_0^1 f(x)dx\right| < \epsilon_2$$

or,

$$\sum_{k=1}^{\infty} a_{nk} f(x_k) - \epsilon_2 < \int_0^1 f(x)dx < \sum_{k=1}^{\infty} a_{nk} f(x_k) + \epsilon_2. \tag{3.26}$$

Let $N = \max(N_1, N_2)$. By the definition of supremum there exists a K such that given ϵ_3,

$$\left|a_{NK} - \sup_k a_{Nk}\right| < \epsilon_3 \tag{3.27}$$

or,

$$\sup_k a_{Nk} - \epsilon_3 < a_{NK} < \sup_k a_{Nk} + \epsilon_3.$$

So,

$$\begin{aligned} |a_{NK} - \delta| &= |a_{NK} - \sup_k a_{Nk} + \sup_k a_{Nk} - \delta| \\ &\leqslant |a_{NK} - \sup_k a_{Nk}| + |\sup_k a_{Nk} - \delta| < \epsilon_3 + \epsilon_1. \end{aligned}$$

Let $\{\tilde{\epsilon}_m\}$ be a sequence which goes to zero aas m goes to infinity. For the K in (3.27) let $E_m = [x_K - \tilde{\epsilon}_m, x_K + \tilde{\epsilon}_m]$.

Let $\mathbb{1}_{E_m}$ be the characteristic function of E_m. For each m there is a sequence of functions, $\{f_{j(m)}\} \in C(\mathbb{T})$ such that $f_{j(m)} \to \mathbb{1}_{E_m}$ in $\mathcal{L}^1$, i.e.,

$$\lim_{j\to\infty} \int |f_j(x) - \mathbb{1}_{E_m}(x)|\, dx = 0$$

or, given ϵ_4 there exists a $J \in \mathbb{N}$ so that for all $j \geqslant J$

$$\left| \int_0^1 (f_j(x) - \mathbb{1}_{E_m}(x))\, dx \right| \leqslant \int_0^1 |f_j(x) - \mathbb{1}_{E_m}(x)|\, dx < \epsilon_4$$

or,

$$\int_0^1 \mathbb{1}_{E_m}(x)dx - \epsilon_4 < \int_0^1 f_j(x)dx < \int_0^1 \mathbb{1}_{E_m}(x)dx + \epsilon_4. \tag{3.28}$$

From (3.26)

$$\sum_{k=1}^{\infty} a_{Nk} f_J(x_k) - \epsilon_2 < \int_0^1 f_J(x)dx.$$

Also, since for all k, $a_{Nk} \geqslant 0$ and $f_J \geqslant 0$ for the same K in (3.27)

$$a_{Nk} f_J(x_K) - \epsilon_2 < \sum_{k=1}^{\infty} a_{Nk} f_J(x_k) - \epsilon_2 < \int_0^1 f_J(x)dx. \tag{3.29}$$

Since $f_J(x_K) = 1$ (3.29) becomes

$$a_{Nk} - \epsilon_2 < \sum_{k=1}^{\infty} a_{Nk} f_J(x_k) - \epsilon_2 < \int_0^1 f_J(x)dx \underbrace{<}_{\text{by (3.28)}} \int_0^1 \mathbb{1}_{E_m}(x)dx + \epsilon_4.$$

Using (3.27) we get

$$\sup_k a_{Nk} - \epsilon_3 - \epsilon_2 < a_{Nk} - \epsilon_2 < \int_0^1 \mathbb{1}_{E_m}(x)dx + \epsilon_4.$$

Therefore,

$$\sup_k a_{Nk} < \int_0^1 \mathbb{1}_{E_m}(x)dx + \epsilon_2 + \epsilon_3 + \epsilon_4. \tag{3.30}$$

When $m \to \infty$, $\int_0^1 \mathbb{1}_{E_m}(x)dx \to 0$. Thus in (3.30) $\sup_k a_{Nk} < \epsilon_2 + \epsilon_3 + \epsilon_4 = \epsilon$. Since any $n > N$ would yield $\sup_k a_{nk} < \epsilon$ therefore $\sup_k a_{nk} \to 0$ as $n \to \infty$. This completes the proof of the claim. $\square$

Chapter 4

Unimodular Sequences whose Autocorrelation is δ

Let us define the function δ as

$$\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}$$

If we consider the positive function $F \equiv 1$ on $\mathbb{T}$ then the Fourier coefficients of F are $p_0 = 1$ and for all other $n \neq 0$, $p_n = 0$. In other words, $p_k = \delta(k)$ or $\{p_k\} = \delta$. In this chapter we construct several examples of *unimodular* sequences whose autocorrelation is δ.

4.1 A sequence of the form $e^{2\pi i n^\alpha \theta}$, $\alpha \in \mathbb{N} \setminus \{1\}$ and θ irrational

To begin with let us recall two important theorems of the theory of uniform distribution.

Theorem 4.1 (Weyl Criterion [16]). *The sequence $(x_n), n = 1, 2, \cdots,$ is u.d. mod 1 if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0, \qquad \textit{for all integers } h \neq 0. \tag{4.2}$$

Theorem 4.3. *[16] Let $p(x) = \alpha_m x^m + \alpha_{m-1} x^{m-1} + \cdots + \alpha_0$, $m > 1$, be a polynomial with real coefficients and let at least one of the coefficients α_j with $j > 0$ be irrational. Then the sequence $(p(n)), n = 1, 2, \cdots,$ is u.d. mod 1.*

4.1.1 Autocorrelation of the function $e^{2\pi i n^\alpha \theta}$

Let $x : \mathbb{Z} \to \mathbb{C}$ be defined for all $n \in \mathbb{N}$ by

$$x[n] = e^{2\pi i n^\alpha \theta}, \qquad \theta \notin \mathbb{Q}, \alpha \in \mathbb{N} \setminus \{1\}. \tag{4.4}$$

The autocorrelation of x at k, denoted by $A_x[k]$, is, from the definition,

$$\begin{aligned} A_x[k] &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} e^{2\pi i (m+k)^\alpha \theta} e^{-2\pi i m^\alpha \theta} \\ &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} e^{2\pi i \theta (m^\alpha + \binom{\alpha}{1} m^{\alpha-1} k + \cdots + \binom{\alpha}{\alpha-1} m k^{\alpha-1} + k^\alpha)} e^{-2\pi i m^\alpha \theta} \\ &= \lim_{N\to\infty} \frac{e^{2\pi i k^\alpha \theta}}{2N+1} \sum_{m=-N}^{N} e^{2\pi i \theta (\binom{\alpha}{1} m^{\alpha-1} k + \binom{\alpha}{2} m^{\alpha-2} k^2 + \cdots + \binom{\alpha}{\alpha-1} m k^{\alpha-1})}. \end{aligned} \tag{4.5}$$

Let $p(x) = \theta\binom{\alpha}{1} k x^{\alpha-1} + \theta\binom{\alpha}{2} k^2 x^{\alpha-2} + \cdots + \theta\binom{\alpha}{\alpha-1} k^{\alpha-1} x$. Since $\theta \notin \mathbb{Q}$ we can apply Theorem 4.3 when $k \neq 0$ to say that the sequence $(p(n))$ is u.d. mod 1. Therefore, according to the Weyl criterion, taking $h = 1$ and $x_n = p(n)$ in (4.2), the right side of (4.5) is zero if $k \neq 0$. If $k = 0$ then

$$\begin{aligned} A_x[0] &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} e^{2\pi i m^\alpha \theta} e^{-2\pi i m^\alpha \theta} \\ &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} 1 = 1. \end{aligned} \tag{4.6}$$

Thus,

$$A_x[k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}$$

So we have a unimodular code whose autocorrelation is the same as the Fourier coefficients of the function $F \equiv 1$ on $\mathbb{T}$.

4.1.2 Cross-correlation of $e^{2\pi i n^\alpha \theta}$

Let $x : \mathbb{Z} \to \mathbb{C}$ be defined for all $n \in \mathbb{Z}$ as

$$x[n] = e^{2\pi i n^\alpha \theta}, \qquad \theta \notin \mathbb{Q}, \alpha \in \mathbb{N} \setminus \{1\}$$

and let $y : \mathbb{Z} \to \mathbb{C}$ be defined for all $n \in \mathbb{Z}$ by

$$y[n] = e^{2\pi i n^\beta \phi}, \qquad \phi \notin \mathbb{Q}, \beta \in \mathbb{N} \setminus \{1\}.$$

The cross-correlation $C_{xy}[k]$ between x and y is defined as

$$C_{xy}[k] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} e^{2\pi i (m+k)^\beta \phi} e^{-2\pi i m^\alpha \theta} \tag{4.7}$$

where $\theta, \phi \notin \mathbb{Q}$ and $\alpha, \beta \in \mathbb{N} \setminus \{1\}$.

Case: $\alpha < \beta$.

$$\begin{aligned} C_{xy}[k] &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} e^{2\pi i \phi (m^\beta + \binom{\beta}{1} m^{\beta-1} k + \cdots + \binom{\beta}{\beta-1} m k^{\beta-1} + k^\beta)} e^{-2\pi i m^\alpha \theta} \\ &= \lim_{N\to\infty} \frac{e^{2\pi i k^\beta \phi}}{2N+1} \sum_{m=-N}^{N} e^{2\pi i \phi (m^\beta + \binom{\beta}{1} m^{\beta-1} k + \cdots + \binom{\beta}{\beta-1} m k^{\beta-1})} e^{-2\pi i m^\alpha \theta}. \end{aligned}$$

Since $\alpha < \beta$, there exists n, $1 \leqslant n \leqslant \beta - 1$, such that $\alpha = \beta - n$. Therefore,

$$\begin{aligned} C_{xy}[k] &= \lim_{n\to\infty} \frac{e^{2\pi i k^\beta \phi}}{2N+1} \sum_{m=-N}^{N} e^{2\pi i \phi (m^\beta + \binom{\beta}{1} m^{\beta-1} k + \cdots + \binom{\beta}{n} \phi k^n m^{\beta-n} + \cdots + \binom{\beta}{\beta-1} m k^{\beta-1})} e^{-2\pi i m^{\beta-n} \theta} \\ &= \lim_{N\to\infty} \frac{e^{2\pi i k^\beta \phi}}{2N+1} \sum_{m=-N}^{N} e^{2\pi i (m^\beta \phi + \binom{\beta}{1} m^{\beta-1} k \phi + \cdots + (\binom{\beta}{n} \phi k^n - \theta) m^{\beta-n} + \cdots + \binom{\beta}{\beta-1} m k^{\beta-1} \phi)}. \end{aligned} \tag{4.8}$$

Let $p(x) = \phi x^\beta + \binom{\beta}{1} k \phi x^{\beta-1} + \cdots + (\binom{\beta}{n} \phi k^n - \theta) x^{\beta-n} + \cdots + \binom{\beta}{\beta-1} k^{\beta-1} \phi x$. When $k = 0$, $p(x) = x^\beta \phi - \theta x^{\beta-n}$. So for all k, p is a polynomial satisfying the hypotheses of Theorem 4.3 and so the sequence $(p(n))$ is u.d. mod 1. Therefore, by the Weyl criterion, taking $h = 1$ and $x_n = p(n)$ in (4.2), we get the right side of (4.8) to be zero. Thus in this case, for all values of k, $C_{xy}[k] = 0$.

Case: $\alpha > \beta$.

$$\begin{aligned} C_{xy}[k] &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} e^{2\pi i\phi(m^\beta + \binom{\beta}{1} m^{\beta-1}k + \cdots + \binom{\beta}{\beta-1} mk^{\beta-1} + k^\beta)} e^{-2\pi i m^\alpha \theta} \\ &= \lim_{N\to\infty} \frac{e^{2\pi i k^\beta \phi}}{2N+1} \sum_{m=-N}^{N} e^{2\pi i(-m^\alpha\theta + m^\beta\phi + \binom{\beta}{1} m^{\beta-1}k\phi + \cdots + \binom{\beta}{\beta-1} mk^{\beta-1}\phi)}. \qquad (4.9) \end{aligned}$$

Let $p(x) = -\theta x^\alpha + \phi x^\beta + \binom{\beta}{1} k\phi x^{\beta-1} + \cdots + \binom{\beta}{\beta-1} k^{\beta-1}\phi x$. Note that when $k = 0$, $p(x) = -\theta x^\alpha + \phi x^\beta$. Once again the hypotheses of Theorem 4.3 are satisfied and so $(p(n))$ is u.d. mod 1. By the Weyl Criterion, taking $h = 1$ and $x_n = p(n)$ in (4.2), we get from (4.9) that for all values of k, $C_{xy}[k] = 0$.

Case: $\alpha = \beta$, $\theta \neq \phi$.

$$\begin{aligned} C_{xy}[k] &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} e^{2\pi i\phi(m^\beta + \binom{\beta}{1} m^{\beta-1}k + \cdots + \binom{\beta}{\beta-1} mk^{\beta-1} + k^\beta)} e^{-2\pi i m^\alpha \theta} \\ &= \lim_{N\to\infty} \frac{e^{2\pi i k^\beta \phi}}{2N+1} \sum_{m=-N}^{N} e^{2\pi i(m^\beta(\phi-\theta) + \binom{\beta}{1} m^{\beta-1}k\phi + \cdots + \binom{\beta}{\beta-1} mk^{\beta-1}\phi)}. \qquad (4.10) \end{aligned}$$

Let $p(x) = (\phi - \theta)x^\beta + \binom{\beta}{1} k\phi x^{\beta-1} + \cdots + \binom{\beta}{\beta-1} k^{\beta-1}\phi x$. If $k \neq 0$ the hypotheses of Theorem 4.3 are satisfied and so $(p(n))$ is u.d. mod 1. Therefore, in this case by the Weyl Criterion, letting $h = 1$ and $x_n = p(n)$ in (4.2), we get from (4.10) that for $k \neq 0$, $C_{xy}[k] = 0$.

If $k = 0$, $p(x) = (\phi - \theta)x^\beta$ and we have to consider the following two cases.

(a) Let $\phi - \theta$ be irrational. Then the hypotheses of Theorem 4.3 are satisfied and

so by the Weyl criterion $C_{xy}[0] = 0$.

(b) Let $\phi - \theta$ be rational. Suppose $\phi - \theta = \frac{p}{q}$ where p and q have been reduced to their lowest terms and $p, q \in \mathbb{Z}$. Then

$$C_{xy}[0] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} e^{2\pi i \frac{p}{q} m^{\alpha}}. \tag{4.11}$$

If $\phi - \theta = \frac{p}{q}$ where $q = 1$, that is, when $\phi - \theta$ is an integer,

$$C_{xy}[0] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} e^{2\pi i p m^{\alpha}} = 1. \tag{4.12}$$

Results are inconclusive when $\phi - \theta = \frac{p}{q}$ and $q \neq 1$.

Thus for this case we can summarize as follows:

$$C_{xy}[k] = \begin{cases} 0 & \text{if } k \neq 0. \\ 0 & \text{if } k = 0 \text{ and } \phi - \theta \text{ is irrational.} \\ 1 & \text{if } k = 0 \text{ and } \phi - \theta \text{ is an integer.} \\ \text{Inconclusive} & \text{if } k = 0 \text{ and } \phi - \theta \text{ is a rational fraction.} \end{cases}$$

Case: $\alpha \neq \beta, \theta = \phi$.

$$\begin{aligned} C_{xy}[k] &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} e^{2\pi i \theta (m^{\beta} + \binom{\beta}{1} m^{\beta-1} k + \cdots + \binom{\beta}{\beta-1} m k^{\beta-1} + k^{\beta})} e^{-2\pi i m^{\alpha} \theta} \\ &= \lim_{N\to\infty} \frac{e^{2\pi i \theta k^{\beta}}}{2N+1} \sum_{m=-N}^{N} e^{2\pi i \theta (-m^{\alpha} + m^{\beta} + \binom{\beta}{1} m^{\beta-1} k + \cdots + \binom{\beta}{\beta-1} m k^{\beta-1})}. \end{aligned} \tag{4.13}$$

Let $p(x) = -x^{\alpha} + x^{\beta} + \binom{\beta}{1} x^{\beta-1} k + \cdots + \binom{\beta}{\beta-1} x k^{\beta-1}$. If $\beta < \alpha$ then $p(x)$ is a polynomial of degree α otherwise it is a polynomial of degree β. If $k = 0$ then $p(x) = -\theta x^{\alpha} + \theta x^{\beta}$. In any case, the sequence $(p(n))$ is u.d. mod 1 by Theorem 4.3 and so according to the Weyl Criterion if we take $h = 1$ and $x_n = p(n)$ in (4.2) we get from (4.13) that $\forall k$, $C_{xy}[k] = 0$.

Case: $\alpha = \beta$, $\theta = \phi$.

$$\begin{aligned}
C_{xy}[k] &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} e^{2\pi i (m+k)^{\beta}\phi} e^{-2\pi i m^{\alpha}\theta} \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} e^{2\pi i (m+k)^{\alpha}\theta} e^{-2\pi i m^{\alpha}\theta} \\
&= A_x[k].
\end{aligned}$$

So the computations are exactly as in Section 4.1.1 and we have

$$C_{xy}[k] = A_x[k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}$$

4.1.3 Autocorrelation of the function $e^{2\pi i n^{\alpha}\theta}$ when α is not an integer.

Numerical results support the claim that in this case the sum

$$\frac{1}{2N+1} \sum_{m=-N}^{N} x[m+k]\overline{x[m]}$$

grows unbounded for non-zero integers k and so the autocorrelation will not be δ.

4.1.4 Higher dimensions

Suppose we would like to construct unimodular codes on $\mathbb{Z}^d$ whose autocorrelation can be computed in a manner discussed so far on $\mathbb{Z}$. Let $n = (n_1, n_2, \cdots n_d) \in \mathbb{Z}^d$ and $|n| = \sqrt{n_1^2 + n_2^2 + \cdots + n_d^2}$. A natural way of extending the unimodular codes to $\mathbb{Z}^d$ is to define codes $x : \mathbb{Z}^d \to \mathbb{C}$ as

$$\forall n \in \mathbb{Z}^d, \qquad x[n] = e^{2\pi i |n|^{\alpha}\theta}, \qquad \theta \notin \mathbb{Q}, \alpha \in \mathbb{N}. \tag{4.14}$$

Let

$$S(N) = \left\{ m = (m_1, m_2, \cdots, m_d) \in \mathbb{Z}^d : -N \leq m_i \leq N, i = 1, \cdots, d \right\}.$$

Let us define the autocorrelation $A_x : \mathbb{Z}^d \to \mathbb{C}$ of x as

$$A_x[k] = \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{S(N)} x[k+m]\overline{x[m]}. \tag{4.15}$$

So for the x defined in (4.14),

$$A_x[k] = \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{m_1=-N}^{N} \cdots \sum_{m_d=-N}^{N} e^{2\pi i |m+k|^\alpha \theta} e^{-2\pi i |m|^\alpha \theta}. \tag{4.16}$$

Let $\alpha = 2$. Then

$$\begin{aligned}
A_x[k] &= \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{m_1=-N}^{N} \cdots \sum_{m_d=-N}^{N} e^{2\pi i (|m+k|^2 - |m|^2)\theta} \\
&= \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{m_1=-N}^{N} \cdots \sum_{m_d=-N}^{N} e^{2\pi i (|k|^2 + 2(m_1k_1+\cdots+m_dk_d))\theta} \\
&= \lim_{N\to\infty} \frac{e^{2\pi i|k|^2\theta}}{(2N+1)^d} \sum_{m_1=-N}^{N} \cdots \sum_{m_d=-N}^{N} e^{4\pi i (m_1k_1+\cdots+m_dk_d)\theta} \\
&= \lim_{N\to\infty} e^{2\pi i|k|^2\theta} \left(\frac{1}{2N+1} \sum_{m_1=-N}^{N} e^{4\pi i m_1 k_1 \theta} \right) \cdots \left(\frac{1}{2N+1} \sum_{m_d=-N}^{N} e^{4\pi i m_d k_d \theta} \right) \\
&= \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0. \end{cases}
\end{aligned}$$

Another way to define unimodular codes $x : \mathbb{Z}^d \to \mathbb{C}$ is as follows. Let $n = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d$ and $\theta = (\theta_1, \cdots, \theta_d)$ has the property that $\theta_1, \cdots, \theta_d$ are not in $\mathbb{Q}$. Define $n^\alpha := (n_1^\alpha, n_2^\alpha, \cdots, n_d^\alpha)$. Let

$$x[n] = e^{2\pi i n^\alpha \cdot \theta}. \tag{4.17}$$

Let $\alpha = 2$. Then

$$A_x[k] = \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{m_1=-N}^{N} \cdots \sum_{m_d=-N}^{N} e^{2\pi i (m+k)^2\cdot\theta} e^{-2\pi i m^2\cdot\theta}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{m_1=-N}^{N} \cdots \sum_{m_d=-N}^{N} e^{2\pi i(k_1^2\theta_1+\cdots+k_d^2\theta_d+2(m_1k_1\theta_1+\cdots+m_dk_d\theta_d))} \\
&= \lim_{N\to\infty} \frac{e^{2\pi i k^2\cdot\theta}}{(2N+1)^d} \sum_{m_1=-N}^{N} \cdots \sum_{m_d=-N}^{N} e^{4\pi i(m_1k_1\theta_1+\cdots+m_dk_d\theta_d)} \\
&= \lim_{N\to\infty} e^{2\pi i k^2\cdot\theta} \left(\frac{1}{2N+1} \sum_{m_1=-N}^{N} e^{4\pi i m_1k_1\theta_1}\right) \cdots \left(\frac{1}{2N+1} \sum_{m_d=-N}^{N} e^{4\pi i m_dk_d\theta_d}\right) \\
&= \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0. \end{cases}
\end{aligned}
$$

Yet another way to define a code $x : \mathbb{Z}^d \to \mathbb{C}$ is as follows. For $\alpha \in \mathbb{N} \setminus \{1\}$ and some irrational number θ suppose $y : \mathbb{Z} \to \mathbb{C}$ is defined by $y[n] = e^{2\pi i n^\alpha \theta}$ and for $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$ let $x[n] = x(n_1, \ldots, n_d) = y[n_1]$. Then for any given $k = (k_1, k_2, \ldots, k_d)$,

$$
\begin{aligned}
A_x[k] &= \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{m_1=-N}^{N} \cdots \sum_{m_d=-N}^{N} x[m+k]\overline{x[m]} \\
&= \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{m_1=-N}^{N} \cdots \sum_{m_d=-N}^{N} y[m_1+k_1]\overline{y[m_1]} \\
&= \lim_{N\to\infty} \frac{1}{(2N+1)} \sum_{m_1=-N}^{N} e^{2\pi i(m_1+k_1)^\alpha\theta} e^{-2\pi i m_1^\alpha\theta} \\
&= A_y[k_1] \\
&= \begin{cases} 0 & \text{if } k_1 \neq 0 \\ 1 & \text{if } k_1 = 0. \end{cases}
\end{aligned}
$$

Here the autocorrelation is 1 not just at the origin but at all points on the hyperplane $k_1 = 0$. If one defines $x[n] = y[n_i],\ i = 1, \cdots, d$ then the autocorrelation is 1 along the hyperplane $k_i = 0$.

4.2 Sequence obtained from Wiener's Generalized Harmonic Analysis

It will be discussed in detail in Chapter 5 that if $\lambda \in (0,1)$ has binary expansion $0.\alpha_1\alpha_2\ldots$ then the function $x : \mathbb{Z} \to \mathbb{C}$ given as

$$x[k] = \begin{cases} 2\alpha_{2n+1} - 1 & \text{if } k = n+1, n \in \mathbb{N} \cup \{0\} \\ 2\alpha_{2n} - 1 & \text{if } k = -n+1, n \in \mathbb{N} \end{cases}$$

has autocorrelation

$$A_x[k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases} \tag{4.18}$$

for almost every $\lambda \in (0,1)$. Note that x takes values ± 1. Here the aim is to show that such a x can be constructed deterministically [34].

Proposition 4.19. *The result in (4.18) would not be true for any rational λ in $(0,1)$.*

Proof. We will show that there exists some non-zero k for which $A_x[k]$ cannot be zero. Every rational number has a non-periodic part followed by a periodic part. Let $\lambda = 0.\underbrace{q_1q_2\ldots q_k}_{\text{non-periodic part}}\ \underbrace{q_{k+1}q_{k+2}\ldots q_{k+r}}_{\text{periodic with period } r}q_{k+1}q_{k+2}\ldots q_{k+r}q_{k+1}\ldots$

Case 1. When $k = \frac{r}{2}$ and r is even.

$$A_x[k] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} x[m+k]x[m]. \tag{4.20}$$

For $p > 0$, let $x[p]$ come from q_{k+1}. We denote this by $x[p] : q_{k+1}$. Then $x[p+1] : q_{k+3}$ and so on.

$x[p] : q_{k+1}$

$x[p+1] : q_{k+3}$

$x[p+2] : q_{k+5}$

$\vdots$

$x[p+\frac{r}{2}] : q_{k+r+1} = q_{k+1}$

$x[p+\frac{r}{2}+1] : q_{k+3}$

$\vdots$

Note that $x[p]$ and $x[p+\frac{r}{2}]$ both come from q_{k+1} and so have the same value. This is true for any two xs separated by $\frac{r}{2}$. Thus, $x[p]x[p+\frac{r}{2}] = 1$, $x[p+1]x[p+1+\frac{r}{2}] = 1$, so on. Then from (4.20)

$$\begin{aligned}
A_x[k] = A_x[\frac{r}{2}] &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{p-1} x[m+\frac{r}{2}]x[m] + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=p}^{N} x[m+\frac{r}{2}]x[m] \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{p-1} x[m+\frac{r}{2}]x[m] + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=p}^{N} 1 \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{p-1} x[m+\frac{r}{2}]x[m] + \lim_{N\to\infty} \frac{N-p+1}{2N+1} \neq 0.
\end{aligned}$$

Case 2. When $k = r$ is odd.

As before, for some $p > 0$ suppose that $x[p]$ comes from q_{k+1} and so on.

$x[p] : q_{k+1}$

$x[p+1] : q_{k+3}$

$x[p+2] : q_{k+5}$

$\vdots$

$x[p+\frac{r-1}{2}] : q_{k+r}$

$x[p+\frac{r+1}{2}] : q_{k+r+2} = q_{k+2}$

$\vdots$

$x[p+r-1] : q_{k+2r-1} = q_{k+r-1}$

$x[p+r] : q_{k+2r+1} = q_{k+1}$

$\vdots$

Note that in this case $x[p]$ and $x[p+r]$ come from the same bit, q_{k+1}, and so will have the same value of either $+1$ or -1. Two xs at points differing by r will have the same value. Thus $x[p]x[p+r] = x[p+1]x[p+r+1] = \cdots = 1$.

$$\begin{aligned}
A_x[r] &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} x[m+r]x[m] \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{p+1} x[m+r]x[m] + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=p}^{N} x[m+r]x[m] \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{p+1} x[m+r]x[m] + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=p}^{N} 1 \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{p+1} x[m+r]x[m] + \lim_{N\to\infty} \frac{N-p+1}{2N+1} \neq 0.
\end{aligned}$$

$\square$

Now we discuss Wiener's technique of deterministically constructing a sequence out of ± 1s whose autocorrelation is given by (4.18). On the positive integers let x take values in the following order

$[1, -1]$ (this row is repeated $2^0 = 1$ time and has 1.2^1 elements).

$[1, 1; 1, -1; -1, 1; -1, -1]$ (is repeated $2^1 = 2$ times and has 2.2^2 elements).

$[1, 1, 1; 1, 1, -1; 1, -1, 1; 1, -1, -1; -1, 1, 1; -1, 1, -1;$

$-1, -1, 1; -1, -1, -1]$ (is repeated $2^2 = 4$ times and has 3.2^3 elements).

$\vdots$

i.e., $x[1] = 1, x[2] = -1, x[3] = 1, x[4] = 1, \ldots$. In addition let $x[0] = 1$ and for $k \in \mathbb{N}$ let $x[-k] = x[k]$. One can observe that the number of elements in each row is a number of the form $n2^n$, $n = 1, 2, \ldots$.

Consider any row of length $n2^n$. There are 2^{n-1} such rows. Take $p < n$. Consider any p tuple of ± 1s.

Question: How many such equivalent p tuples are there? (i.e., how many repetitions?) Call this k_p. Take any non-equivalent p tuple (i.e. non-equivalent to the given one). Prove there are the same number as the original one.

Answer: From the way the construction is being done, each p tuple is equally likely to occur in a row as any other p tuple. In other words, if one randomly picks a p tuple in any row then the probability of getting a given p tuple is $\frac{1}{2^p}$. This should be the same as the relative frequency of the occurrence of the given p tuple in the row. i.e.,

$$\frac{k_p}{n2^n} = \frac{1}{2^p}.$$

Thus for each $p < n$, $k_p = n2^{n-p}$.

Suppose we wish to calculate the relative frequency of the occurrence of a particular sequence of p consecutive terms among the first N terms. Then we might have to stop in the middle of a row. Recall that a row with $j2^j$ elements is repeated 2^{j-1} times. So N would be

$$N = 1.2^1.2^0 + 2.2^2.2^1 + 3.2^3.2^2 + \ldots + M2^M.2^{M-1} + P(M+1)2^{M+1} + Q$$

where $0 \leqslant P < 2^M$ and $0 \leqslant Q < (M+1)2^{M+1}$. Here $M \to \infty$ as $N \to \infty$. Thus,

$$\begin{aligned} N &= \sum_{j=1}^{M} j2^j 2^{j-1} + P(M+1)2^{M+1} + Q \\ &= \sum_{j=1}^{M} j2^{2j-1} + P(M+1)2^{M+1} + Q \\ &= S + Q. \end{aligned}$$

For a fixed p, let us denote the number of occurrences of a particular p tuple in a row of length 1.2^1 by n_1, the number of occurrences in a row of length 2.2^2 by n_2, so on. Therefore,

$$\begin{aligned} &\lim_{N\to\infty} \frac{\text{No. of repetitions of a fixed } p \text{ tuple in the first } N}{N} \\ &= \lim_{N\to\infty} \left[\frac{2^0.n_1}{N} + \frac{2^1.n_2}{N} + \cdots + \frac{2^{M-1}n_M}{N} + \frac{Pn_{M+1}}{N} + \frac{\text{no. of repetitions in the last } Q}{N} \right] \\ &= \lim_{N\to\infty} \left[\frac{2^0.n_1}{S+Q} + \frac{2^1.n_2}{S+Q} + \cdots + \frac{2^{M-1}n_M}{S+Q} + \frac{Pn_{M+1}}{S+Q} + \frac{\text{no. of repetitions in the last } Q}{N} \right] \\ &= \lim_{N\to\infty} \left[\frac{2^0.\frac{n_1}{S}}{1+\frac{Q}{S}} + \frac{2^1.\frac{n_2}{S}}{1+\frac{Q}{S}} + \cdots + \frac{2^{M-1}\frac{n_M}{S}}{1+\frac{Q}{S}} + \frac{P\frac{n_{M+1}}{S}}{1+\frac{Q}{S}} + \frac{\text{no. of repetitions in the last } Q}{N} \right]. \end{aligned} \tag{4.21}$$

We shall soon show that $\frac{Q}{S} \to 0$ as $N \to \infty$ and so $\frac{Q}{S+Q} = \frac{Q}{N} \to 0$ as $N \to \infty$ which means that the last term in (4.21) is zero. Thus the right side of (4.21) is

$$\begin{aligned} &2^0\frac{n_1}{S} + 2^1\frac{n_2}{S} + \cdots + 2^{M-1}\frac{n_M}{S} + P\frac{n_{M+1}}{S} \\ =\ & \frac{2^0.n_1 + 2^1.n_2 + \cdots + 2^{M-1}n_M + Pn_{M+1}}{S} \\ =\ & \frac{2^0.n_1 + 2^1.n_2 + \cdots + 2^{M-1}n_M + Pn_{M+1}}{\sum_{j=1}^{M} j2^{2j-1} + P(M+1)2^{M+1}} \\ =\ & \frac{1}{2^p}. \end{aligned}$$

To summarize,

$$\lim_{n\to\infty}\frac{\text{No. of repetitions of a fixed sequence of } p \text{ consecutive terms in the first } N}{N}=\frac{1}{2^p}. \tag{4.22}$$

Now let us show that $\frac{Q}{S}\to 0$ as $N\to\infty$.

$$\begin{aligned}
0<\frac{Q}{S} &< \frac{(M+1)2^{M+1}}{\sum_{j=1}^{M} j2^{2j-1}+P(M+1)2^{M+1}} \leqslant \frac{(M+1)2^{M+1}}{\sum_{j=1}^{M} 2^{2j-1}+P(M+1)2^{M+1}}\\
&= \frac{(M+1)2^{M+1}}{\sum_{j=0}^{M-1} 2^{2(j-1)-1}+P(M+1)2^{M+1}}\\
&= \frac{(M+1)2^{M+1}}{\frac{1}{2^3}\sum_{j=0}^{M-1} 2^{2j}+P(M+1)2^{M+1}} = \frac{(M+1)2^{M+1}}{\frac{1}{2^3}\frac{2^{2M}-1}{3}+P(M+1)2^{M+1}}.
\end{aligned}\tag{4.23}$$

The right side of (4.23) goes to 0 as $M\to\infty$. Therefore, $\frac{Q}{S}\to 0$, as $N\to\infty$.

Note: The above calculation also suggests that the ratio of the number of terms in a row to that in all previous rows approaches zero. It is for this reason that it is necessary to have to repeat each row a certain number of times, otherwise, this wouldn't be true.

Theorem 4.24. *Given an integer k the autocorrelation of x is*

$$A_x[k]=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{m=-N}^{N}x[m+k]x[m]=\begin{cases}1 & \textit{if } k=0\\ 0 & \textit{if } k\neq 0.\end{cases}$$

Proof. For $k=0$,

$$A_x[0]=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{m=-N}^{N}x[m]x[m]=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{m=-N}^{N}1=1.$$

For the case when $k\neq 0$ it is enough to prove the result just for positive k as the autocorrelation function is even. Also, it is enough to prove that

$$\lim_{N\to\infty}\sum_{m=1}^{N}x[m+k]x[m]=0.$$

For clarity, first consider $k = 1$. Then $x[m]x[m+1]$ comes from sequences of length 2. These sequences look like

$\underbrace{11}_{x[m]x[m+1]}$, $1\ -1, -1\ 1, -1\ -1$. The first and the last combinations give $x[m]x[m+1]$ a value of 1 while the middle two combinations give $x[m]x[m+1]$ a value of -1. Note that out of 4 possible combinations 2 give the value 1 and the remaining 2 give the value -1. Also as we have discussed (see (4.22)), each of these sequences occur equally often, their relative frequency approaching $\frac{1}{2^2} = \frac{1}{4}$. Therefore,

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{m=1}^{N} x[m+1]x[m] \\
&= \lim_{N\to\infty} \Big[\frac{\text{no. of times } (11) \text{ is repeated in } N \text{ terms}}{N} \cdot (+1) \\
&\quad + \frac{\text{no. of times } (1\ \ -1) \text{ is repeated in } N \text{ terms}}{N} \cdot (-1) \\
&\quad + \frac{\text{no. of times } (-1\ 1) \text{ is repeated in } N \text{ terms}}{N} \cdot (-1) \\
&\quad + \frac{\text{no. of times } (-1\ \ -1) \text{ is repeated in } N \text{ terms}}{N} \cdot (+1) \Big] \\
&= \frac{1}{2^2} \cdot (+1) + \frac{1}{2^2} \cdot (-1) + \frac{1}{2^2} \cdot (-1) + \frac{1}{2^2} \cdot (+1) = 0.
\end{aligned}$$

Now let $k = 2$. The values of $x[m]x[m+2]$ now come from the product of the first and last elements of sequences of length 3 i.e., from the following 2^3 possibilities.

$$\underbrace{1}_{x[m]}\ 1\ \underbrace{1}_{x[m+2]} \longrightarrow x[m]x[m+2] = 1,$$

$$11-1 \longrightarrow x[m]x[m+2] = -1,$$

$$1-11 \longrightarrow x[m]x[m+2] = 1,$$

$$1-1-1 \longrightarrow x[m]x[m+2] = -1,$$

$$-111 \longrightarrow x[m]x[m+2] = -1,$$

$$-11-1 \longrightarrow x[m]x[m+2]=1,$$

$$-1-11 \longrightarrow x[m]x[m+2]=-1,$$

$$-1-1-1 \longrightarrow x[m]x[m+2]=1.$$

Again exactly half the number of sequences (four in this case) give $x[m]x[m+2]$ a value of $+1$ and the remaining four sequences give $x[m]x[m+2]$ a value of -1. So,

$$\begin{aligned}
\lim_{N\to\infty}\sum_{m=1}^{N} x[m+2]x[m] &= \lim_{N\to\infty}\Big[\frac{\text{No. of repetitions of 111 in } N \text{ terms}}{N}(+1)+ \\
&\quad + \frac{\text{No. of repetitions of } 11-1 \text{ in } N \text{ terms}}{N}(-1)+\cdots\Big] \\
&= \frac{1}{2^3}(+1)+\frac{1}{2^3}(-1)+\cdots \\
&= 4\cdot\frac{1}{2^3}(+1)+4\cdot\frac{1}{2^3}(-1)=0.
\end{aligned}$$

So, for any general $k=p$, the values of $x[m]x[m+p]$ would come from the product of the first and last element of sequences of length $p+1$ and there are 2^{p+1} possible sequences each of which would be repeated equally often, the relative frequency approaching $\frac{1}{2^{p+1}}$. Out of the 2^{p+1} sequences, 2^p of the sequences cause $x[m]x[m+p]$ to be $+1$ and 2^p of the sequences cause $x[m]x[m+p]$ to be -1. Thus,

$$\lim_{N\to\infty}\sum_{m=1}^{N} x[m+p]x[m] = 2^p\cdot\frac{1}{2^{p+1}}(+1)+2^p\cdot\frac{1}{2^{p+1}}(-1)=0.$$

□

4.3 Sequence obtained from n roots of unity

Instead of the function x taking values $+1$ or -1 let us construct x so that it takes values which are one of the n roots of unity where $n>2$. For example, if we choose $n = 3$ then x takes values from $\{a_1 = 1, a_2 = e^{i\frac{2\pi}{3}}, a_3 = e^{i\frac{4\pi}{3}}\}$. The following sequence represents the values of x over the positive integers.

$[a_1, a_2, a_3]$ repeated $3^0 = 1$ times.

$[a_1, a_1; a_1, a_2; a_1, a_3; a_2, a_1; a_2, a_2; a_2, a_3; a_3, a_1; a_3, a_2; a_3, a_3]$ repeated $3^1 = 3$ times.

$[a_1, a_1, a_1; a_1, a_1, a_2; a_1, a_1, a_3; a_1, a_2, a_1; a_1, a_2, a_2; a_1, a_2, a_3;$

$a_1, a_3, a_1; a_1, a_3, a_2; a_1, a_3, a_3; a_2, a_1, a_1; a_2, a_1, a_2; a_2, a_1, a_3; \ldots]$ repeated $3^2 = 9$ times.

$\vdots$

Let $x[0] = 1$ and for all $k \in \mathbb{N}$, $x[-k] = x[k]$. The number of elements in each row is a number of the form $j3^j$. Let us fix p. In a particular row having $j3^j$ elements where $p \leqslant j$, any finite sequence of length p occurs as often as any other finite sequence of the same length. At this point using an identical argument to that used in Section 4.2 we can say that if one is dealing with n roots of unity then

$$\lim_{N\to\infty} \frac{\text{No. of repetitions of a fixed sequence of } p \text{ consecutive terms in the first } N}{N} = \frac{1}{n^p}. \tag{4.25}$$

Theorem 4.26. *Given an integer k the autocorrelation of x is*

$$A_x[k] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} x[m+k]\overline{x[m]} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}$$

Proof. When $k = 0$,

$$\begin{aligned} A_x[0] &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} x[m]\overline{x[m]} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} |x[m]|^2 \\ &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} 1 = 1. \end{aligned}$$

For non-zero values of k it is enough to prove the result only for positive k since the autocorrelation is an even function. Besides, it is enough to show that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N} x[m+k]\overline{x[m]} = 0.$$

For simplicity let us start with discussing the case when $k = 1$ and we are dealing with just $n = 3$ roots of unity. One should note that if $k = p$ the value of $x[m+p]\overline{x[m]}$ is the product of the first (actually, its conjugate) and last element of some sequence $a_{i_1} \dots a_{i_{p+1}},\ a_{i_n} \in \{a_1, a_2, a_3\}$. There would be 3^{p+1} such sequences. When $k = 1$, $x[m+1]\overline{x[m]}$ can come from any one of the following $3^2 = 9$ tuples.

$\underbrace{\overline{a_1}a_1}_{\overline{x[m]}x[m+1]} = 1$,

$\overline{a_1}a_2 = e^{\frac{2\pi i}{3}}$,

$\overline{a_1}a_3 = e^{\frac{4\pi i}{3}}$,

$\overline{a_2}a_1 = e^{\frac{-2\pi i}{3}} = e^{\frac{4\pi i}{3}}$,

$\overline{a_2}a_2 = e^{\frac{-2\pi i}{3}}e^{\frac{2\pi i}{3}} = 1$,

$\overline{a_2}a_3 = e^{\frac{-2\pi i}{3}}e^{\frac{4\pi i}{3}} = e^{\frac{2\pi i}{3}}$,

$\overline{a_3}a_1 = e^{\frac{-4\pi i}{3}} = e^{\frac{2\pi i}{3}}$,

$\overline{a_3}a_2 = e^{\frac{-4\pi i}{3}}e^{\frac{2\pi i}{3}} = e^{\frac{4\pi i}{3}}$,

$\overline{a_3}a_3 = e^{\frac{-4\pi i}{3}}e^{\frac{4\pi i}{3}} = 1$. Therefore,

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N} x[m+1]\overline{x[m]} &= \lim_{N\to\infty}\frac{\text{No. of repetitions of } a_1a_1}{N}\cdot 1\\ &+ \lim_{N\to\infty}\frac{\text{No. of repetitions of } a_1a_2}{N}\cdot e^{\frac{2\pi i}{3}} + \lim_{N\to\infty}\frac{\text{No. of repetitions of } a_1a_3}{N}\cdot e^{\frac{4\pi i}{3}}\\ &+ \lim_{N\to\infty}\frac{\text{No. of repetitions of } a_2a_1}{N}\cdot e^{\frac{4\pi i}{3}} + \lim_{N\to\infty}\frac{\text{No. of repetitions of } a_2a_2}{N}\cdot 1\end{aligned}$$

$$+\lim_{N\to\infty}\frac{\text{No. of repetitions of } a_2a_3}{N}\cdot e^{\frac{2\pi i}{3}}+\lim_{N\to\infty}\frac{\text{No. of repetitions of } a_3a_1}{N}\cdot e^{\frac{2\pi i}{3}}$$
$$+\lim_{N\to\infty}\frac{\text{No. of repetitions of } a_3a_2}{N}\cdot e^{\frac{4\pi i}{3}}+\lim_{N\to\infty}\frac{\text{No. of repetitions of } a_3a_3}{N}\cdot 1. \tag{4.27}$$

Using (4.25) with $p = 2$ and $n = 3$, (4.27) reduces to

$$\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N}x[m+1]\overline{x[m]} = \frac{1}{3^2}\cdot 1+\frac{1}{3^2}\cdot e^{\frac{2\pi i}{3}}+\frac{1}{3^2}\cdot e^{\frac{4\pi i}{3}}+\frac{1}{3^2}\cdot e^{\frac{4\pi i}{3}}+\frac{1}{3^2}\cdot 1+$$
$$+\frac{1}{3^2}\cdot e^{\frac{2\pi i}{3}}+\frac{1}{3^2}\cdot e^{\frac{2\pi i}{3}}+\frac{1}{3^2}\cdot e^{\frac{4\pi i}{3}}+\frac{1}{3^2}\cdot 1$$
$$=\frac{1}{3^2}\cdot\left(1+e^{\frac{2\pi i}{3}}+e^{\frac{4\pi i}{3}}\right)\cdot 3$$
$$=0 \quad (\text{since } 1+e^{\frac{2\pi i}{3}}+e^{\frac{4\pi i}{3}}=0).$$

In general, for $k = p$ and $n = 3$ roots of unity

$$\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N}x[m+p]\overline{x[m]} = \frac{1}{3^{p+1}}\left(1+e^{\frac{2\pi i}{3}}+e^{\frac{4\pi i}{3}}\right)\cdot 3^p = 0.$$

Generalizing the argument further to $n(n > 3)$ roots of unity, let $w_k = e^{\frac{2\pi ik}{n}}$, $k = 0, 1, 2, \ldots, n-1$ be the n roots on unity. Let x be a function which can take values from $\{w_0, w_1, \ldots, w_{n-1}\}$. On the positive integers let x be defined to take values in the following order:

$[w_0, w_1, \ldots, w_{n-1}]$ repeated $n^0 = 1$ times.

$[w_0, w_0; w_0, w_1; \ldots; w_0, w_{n-1}; w_1, w_0; w_1, w_1; \ldots; w_1, w_{n-1}; \ldots;$

$w_{n-1}, w_0, w_{n-1}, w_1, \ldots, w_{n-1}, w_{n-1}]$ repeated $n^1 = n$ times.

$[w_0, w_0, w_0; w_0, w_0, w_1; \ldots w_0, w_0, w_{n-1}; \ldots; w_{n-1}, w_{n-1}, w_0; \ldots;$

$w_{n-1}, w_{n-1}, w_{n-1}]$ repeated n^2 times.

$\vdots$

For $k \in \mathbb{N}$, let $x[-k] = x[k]$ and $x[0] = 1$.

Following a working identical to what we did for $n = 3$ we can show that

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N}\sum_{m=1}^{N} x[m+p]\overline{x[m]} &= \frac{1}{n^{p+1}}\left(1+e^{\frac{2\pi i}{n}}+\cdots+e^{\frac{2\pi i(n-1)}{n}}\right)\cdot n^p = 0 \\ &= 0 \quad (\text{since } 1+e^{\frac{2\pi i}{n}}+\cdots+e^{\frac{2\pi i(n-1)}{n}} = 0).\end{aligned}$$

□

4.4 Sequence obtained from Hadamard matrices

Definition 4.28. A real *Hadamard matrix* is a square matrix whose entries are either $+1$ or -1 and whose rows are mutually orthogonal.

Examples of Hadamard matrices were actually first constructed by James Joseph Sylvester. Let H be a Hadamard matrix of order n. Then the matrix

$$\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

is a Hadamard matrix of order $2n$. This observation can applied repeatedly to obtain the following sequence of Hadamard matrices.

$$H_1 = \begin{bmatrix} 1 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} H_1 & H_1 \\ H_1 & -H_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

and $H_{2^k} = \begin{bmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{bmatrix}$.

We say that the Hadamard matrix H_{2^k} is of size $2^k \times 2^k$ or has size 2^k. To construct our desired unimodular function x let H_1 be repeated once $(2^0 = 1)$, H_2 be repeated twice (2^1), H_4 be repeated four (2^2) times, H_8 be repeated eight (2^3) times and so on. For the positive integers let x take values from the elements of the sequence of matrices

$$H_1, H_2, H_2, H_4, H_4, H_4, H_4, H_8, \cdots$$

Let $x[0] = 1$ and for any $k \in \mathbb{N}$, $x[-k] = x[k]$. Having defined x on the set of integers we proceed to show that the autocorrelation of x is one at zero and zero everywhere else.

Lemma 4.29. *For a fixed $k = 2^j$ (for some $j \in \mathbb{N}$), let H_k be the $k \times k$ Hadamard matrix. For every Hadamard matrix of size $m \times m$ where $m > k$ (i.e., $m = 2^{j+1}, 2^{j+2}, \cdots$) let*

$p =$ number of occurrences of $H_k H_k$ or $-H_k - H_k$ in all the rows of the matrix H_m

and

$n =$ number of occurrences of $H_k - H_k$ or $-H_k H_k$ in all the rows of the matrix H_m.

Then $p = n$.

Proof of Lemma 4.29. We prove this by induction on m.

Step 1. Let $m = 2^{j+1}$. In this case

$$H_m = \begin{bmatrix} H_{2^j} & H_{2^j} \\ H_{2^j} & -H_{2^j} \end{bmatrix} = \begin{bmatrix} H_k & H_k \\ H_k & -H_k \end{bmatrix}.$$

H_kH_k occurs once and $H_k - H_k$ occurs once. Therefore, $p = n = 1$.

Step 2. Assume that the result is true for $m = 2^{j+N}$ for some natural number N.

Step 3. Now suppose $m = 2^{j+N+1}$. Let $j + N = J$. So in this case $H_m = H_{J+1}$ is

$$\begin{bmatrix} H_J & H_J \\ H_J & -H_J \end{bmatrix}.$$

By our assumption in Step 2 the result is true in each H_J and $-H_J$. So here we need to pay attention to the H_ks forming the boundary between the two columns of H_Js in the matrix H_{J+1}. In the upper half if we have a_1 occurrences of H_kH_k or $-H_k \ - H_k$ in the boundary then in the lower half we have a_1 occurrences of $H_k \ - H_k$ or $-H_kH_k$. Similarly if there are b_1 occurrences of $H_k \ - H_k$ or $-H_kH_k$ in the boundary in the upper half then there are b_1 occurrences of H_kH_k or $-H_k - H_k$ in the boundary in the lower half.

In each H_J let $p = p_J$ and $n = n_J$. Note that due to our assumption $p_J = n_J$. Then in $H_{J+1} = H_{j+N+1}$,

$p = 4p_J + a_1 + b_1$ and

$n = 4n_J + b_1 + a_1$. Since $p_J = n_J$ we have $p = n$. □

Lemma 4.30. *Let,*

$$S = \sum_{m=1}^{N} x[m+k]x[m] \tag{4.31}$$

where N is such that counting the first N values of x will end at the last element of some Hadamard matrix in the sequence already described. Let $k > 0$ be given. Then there exists n such that $2^{n-1} < k \leqslant 2^n$. The contribution to the sum in (4.31) due to the function values ($x[m]s$) coming from all Hadamard matrices of size 2^{n+1} and bigger is 0.

Proof. Step 1. We will be looking at rows of the submatrix H_{2^n} in $H_{2^{n+1}}, H_{2^{n+2}}, H_{2^{n+3}}, \ldots$. Let us try to explain the idea with the case $k = 3$. Then $2^{2-1} < 3 \leqslant 2^2$ i.e., $n = 2$. So we would be looking at rows of H_4 in $H_8, H_{16}, H_{32} \ldots$. Here is the figure just for H_8.

$$H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc}
1 & \underbrace{1}_{x[m]} & \underbrace{1}_{x[m]} & \underbrace{1}_{x[m]} & \underbrace{1}_{x[m+3]} & \underbrace{1}_{x[m+3]} & \underbrace{1}_{x[m+3]} & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
-- & -- & -- & -- & -- & -- & -- & -- \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1
\end{array}\right].$$

The elements in the $k = 3$ columns (columns 2 to 4) in each occurrence of H_4 in $H_8, H_{16}, H_{32,} \ldots$ (except for the last H_4 in each row of H_4s) get multiplied to elements in the adjacent H_4. See the elements indicated in the matrix H_8. The elements from these columns will have zero contribution to the sum in (4.31). This is true because according to Lemma 4.29, H_4 occurs next to H_4 as often as it occurs next to $-H_4$ causing cancellations.

More generally, the elements in the k columns (columns $2^n - k + 1$ to 2^n) in each occurrence of H_{2^n} in $H_{2^{n+1}}, H_{2^{n+2}}, H_{2^{n+3}}, \ldots$ (except for the last H_{2^n} in each row of H_{2^n}s) get multiplied to elements in the adjacent H_{2^n}. The elements from these columns will have zero contribution to the sum in (4.31). This is true because according to Lemma 4.29, H_{2^n} occurs next to H_{2^n} as often as it occurs next to $-H_{2^n}$ in each of $H_{2^{n+1}}, H_{2^{n+2}}, H_{2^{n+3}}, \ldots$ causing cancellations.

Step 2. What is the contribution from the last $H_{2^n}(= H_4)$ submatrix in each row of the matrix $H_{2^{n+1}}(= H_8)$ or $H^{2^{n+2}}(= H_{16})$ etc. ? The question is also only for the elements from the k columns discussed in *Step 1*.

The answer is zero for the following reason. Due to the structure of the Hadamard matrices the last column of $\pm H_{2^n}$s in any higher order matrix like $H_{2^{n+1}}, H_{2^{n+2}}, \ldots$ has the same number of H_{2^n} as $-H_{2^n}$. In any Hadamard matrix of higher order the elements of the specified k columns of these $\pm H_{2^n}$s interact with H_{2^n}s occurring in the first column of $\pm H_{2^n}$s . But the first column of each higher order matrix just has $+H_{2^n}$ (no $-H_{2^n}$s). So enough cancellations result a contribution of zero.

Step 3. Now consider the contribution to the sum coming from elements in columns 1 to $2^n - k$ of the H_{2^n} submatrices in each row of H_{2^n} in $H_{2^{n+1}}, H_{2^{n+2}}, \ldots$. To analyze this part we think of the Hadamard matrices $H_{2^{n+1}}, H_{2^{n+2}}, \ldots$ as made up of rows and columns of $H_{2^{n-1}}(= H_2$ when $k = 3)$. Following is the situation in H_8 when $k = 3$.

$$H_8 = \left[\begin{array}{cccccccccc}
\underbrace{1}_{x[m]} & 1 & | & 1 & \underbrace{1}_{x[m+3]} & | & 1 & 1 & 1 & 1 \\
1 & -1 & | & 1 & -1 & | & 1 & -1 & 1 & -1 \\
-- & -- & | & -- & -- & -- & -- & -- & -- & -- \\
1 & 1 & | & -1 & -1 & | & 1 & 1 & -1 & -1 \\
1 & -1 & | & -1 & 1 & | & 1 & -1 & -1 & 1 \\
-- & -- & & -- & -- & -- & -- & -- & -- & -- \\
1 & 1 & & 1 & 1 & & -1 & -1 & -1 & -1 \\
1 & -1 & & 1 & -1 & & -1 & 1 & -1 & 1 \\
1 & 1 & & -1 & -1 & & -1 & -1 & 1 & 1 \\
1 & -1 & & -1 & 1 & & -1 & 1 & 1 & -1
\end{array}\right].$$

We can then use the argument used in *Step 1* by replacing $H_{2^n}(= H_4)$ by $H_{2^{n-1}}(= H_2)$ and conclude that the contribution to the sum in (4.31) due to these columns is also zero. $\square$

Theorem 4.32 (Zero autocorrelation). *The function x constructed from the Hadamard matrices in this section has a autocorrelation function which is*

$$A_x[k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Theorem 4.32. When $k = 0$,

$$A_x[0] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} x[m]\overline{x[m]} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} x[m]^2$$
$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} 1 = 1.$$

Let $k > 0$ be given and n be such that $2^n < k \leqslant 2^{n+1}$. It is enough to show that

$$AC[k] = \frac{1}{N} \sum_{m=1}^{N} x[m+k]x[m] \tag{4.33}$$

goes to zero as N goes to infinity. It is important to observe that N can be such that $x[N]$ occurs somewhere in the middle of a matrix. Each Hadamard matrix of size 2^n has 4^n elements. Recall that each such matrix will be repeated 2^n times. Thus,

$$N = \sum_{j=0}^{n+1} 2^j 4^j + \sum_{j=n+2}^{M} 2^j 4^j + P4^{M+1} + S = Q + R + \tilde{P} + S \tag{4.34}$$

where $0 \leqslant P < 2^{M+1}$, $0 \leqslant S < 4^{M+1}$ and $M \to \infty$ as $N \to \infty$. Therefore,

$$\begin{aligned} AC[k] \;&=\; \frac{1}{N}\Big[\sum_{m=1}^{Q} x[m+k]x[m] + \sum_{m=Q+1}^{Q+R} x[m+k]x[m] + \sum_{m=Q+R+1}^{Q+R+\tilde{P}} x[m+k]x[m] + \\ &\quad + \sum_{m=Q+R+\tilde{P}+1}^{Q+R+\tilde{P}+S} x[m+k]x[m]\Big]. \end{aligned} \tag{4.35}$$

Due to Lemma 4.30, the second and the third sums in (4.35) is equal to zero. So we have

$$AC[k] = \frac{1}{N}\Big[\sum_{m=1}^{Q} x[m+k]x[m] + \sum_{m=Q+R+\tilde{P}+1}^{Q+R+\tilde{P}+S} x[m+k]x[m]\Big].$$

Thus

$$\begin{aligned} |AC[k]| \;&\leqslant\; \frac{1}{N}\left(\sum_{m=1}^{Q} |x[m+k]x[m]| + \sum_{m=Q+R+\tilde{P}+1}^{Q+R+\tilde{P}+S} |x[m+k]x[m]|\right) \\ &=\; \frac{1}{N}\left(Q+S\right). \end{aligned} \tag{4.36}$$

Q depends on k and is finite. So $\lim_{N\to\infty} \frac{Q}{N} = 0$.

$$\begin{aligned} \frac{S}{N} &< \frac{4^{M+1}}{N} = \frac{4^{M+1}}{\sum_{j=0}^{n+1} 2^j 4^j + \sum_{j=n+2}^{M} 2^j 4^j + P4^{M+1} + S} \\ &< \frac{4^{M+1}}{\sum_{j=0}^{M} 8^j} = \frac{4^{M+1}}{\frac{8^{M+1}-1}{7}} \sim 7\frac{1}{2^{M+1}}. \end{aligned} \tag{4.37}$$

Since $M \to \infty$ as $N \to \infty$, $\lim_{N\to\infty} \frac{S}{N}$ goes to zero as N goes to infinity. From (4.36) this means that $AC[k] \to 0$ as $N \to \infty$ for $k > 0$. This proves that the autocorrelation is zero for all positive integers k but since the autocorrelation function is even it means that the autocorrelation is zero for all non-zero k. □

4.4.1 Error estimates

For practical purposes we would like to do the following: given $\epsilon > 0$ find $N \in \mathbb{N}$ such that for all non-zero $k \in \mathbb{Z}$

$$|AC[k]| = \left| \frac{1}{N} \sum_{m=1}^{N} x[m+k]x[m] \right| < \epsilon.$$

From (4.36) in the proof of Theorem 4.32 we have

$$\begin{aligned} |AC[k]| &\leqslant \frac{1}{N} \left(\sum_{m=1}^{Q} |x[m+k]x[m]| + \sum_{m=Q+R+\tilde{P}+1}^{Q+R+\tilde{P}+S} |x[m+k]x[m]| \right) \\ &= \frac{1}{N} (Q+S). \end{aligned}$$

From (4.37) we know that $\frac{S}{N} < 7\frac{1}{2^{M+1}}$, which is independent of k. Now $Q = \sum_{j=0}^{n+1} 8^j$ (see (4.34)) and n depends on k as $2^n < k \leqslant 2^{n+1}$. Consider the following table.

k	$2 < k \leqslant 4$	$5 \leqslant k \leqslant 8$	$9 \leqslant k \leqslant 16$	...
No. of values in Q	$\sum_{j=0}^{2} 8^j$	$\sum_{j=0}^{3} 8^j$	$\sum_{j=0}^{4} 8^j$	...
$\log_2(k)$	$1 < \log_2(k) \leqslant 2$	$2 < \log_2(k) \leqslant 3$	$3 < \log_2(k) \leqslant 4$	...

Therefore, $\lceil \log_2(k) \rceil$ is the upper limit of the summation in Q and

$$Q = \sum_{j=0}^{\lceil \log_2(k) \rceil} 8^j = \frac{8^{\lceil \log_2(k) \rceil + 1} - 1}{7}$$

Thus,

$$|AC[k]| \leqslant \frac{1}{N} \frac{8^{\lceil \log_2(k) \rceil + 1} - 1}{7} + 7 \frac{1}{2^{M+1}}. \tag{4.38}$$

The left hand side of (4.38) depends on k but the dependence is logarithmic and due to the slow rate of increase of the log function this means that AC is 'almost' independent of k..

Actually, due to (4.38) we have the following theorem.

Theorem 4.39. *Given $\epsilon > 0$ and K the smallest N such that*

$$\forall\ 0 < |k| \leqslant K, \quad \left| \frac{1}{N} \sum_{1}^{N} x[m+k]x[m] \right| < \epsilon$$

satisfies

$$\frac{1}{N} \frac{8^{\lceil \log_2(K) \rceil + 1} - 1}{7} + 7 \frac{1}{2^{M+1}} < \epsilon$$

where M is obtained from (4.34).

4.5 Multidimensional case

Let x be the function defined in section 4.2, 4.3 or 4.4. So for any given integer m, $x[m] = a_m$ where $a_m = \pm 1$ or some other root of unity, this being determined by the choice of x.

We define a function $h : \mathbb{Z}^d \to \mathbb{C}$ as follows

$$h[m] = h(m_1, m_2, \ldots, h_d) = x[m_1]. \tag{4.40}$$

We denote the $d-$dimensional square by $S(N)$ that is,

$$S(N) = \{(m_1, m_2, \ldots, m_d) : -N \leqslant m_i \leqslant N, i = 1, 2, \ldots, d\}.$$

For $k = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$,

$$\begin{aligned}
A_h[k] &= \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{m\in S(N)} h[m+k]\overline{h[m]} \\
&= \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{m_1=-N}^{N} \sum_{m_2=-N}^{N} \cdots \sum_{m_d=-N}^{N} h[m+k]\overline{h[m]} \\
&= \lim_{N\to\infty} \frac{1}{(2N+1)^d} \sum_{m_1=-N}^{N} \sum_{m_2=-N}^{N} \cdots \sum_{m_d=-N}^{N} x[m_1+k_1]\overline{x[m_1]} \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m_1=-N}^{N} x[m_1+k_1]\overline{x[m_1]} \\
&= \begin{cases} 0 & \text{if } k_1 \neq 0 \\ 1 & \text{if } k_1 = 0. \end{cases} \qquad (4.41)
\end{aligned}$$

So in this case the autocorrelation A_h is one on the hyperplane $(0, k_2, \ldots, k_d)$. If one defines $h[m] = x[m_i],\ i = 1, \cdots, d$ then the autocorrelation is 1 along the hyperplane defined by $k_i = 0$.

Chapter 5

Functions whose Autocorrelation is the Fourier Transform of the Fejér Kernel

Let $\triangle(t) = \max(1 - |t|, 0)$. On $[-1, 1]$, the graph of $\triangle$ consists of the equal legs of an isosceles triangle of height 1; $\triangle$ vanishes outside $[-1, 1]$. Let $\omega(\gamma) = \frac{1}{2\pi}\left(\frac{\sin\gamma/2}{\gamma/2}\right)^2$. We refer to ω as the Fejér function. The Fourier transform of $\triangle$ is $\omega_{2\pi}$. We begin this chapter with a discussion of functions defined on $\mathbb{Z}$ whose autocorrelation at integers is the triangle $\triangle$. For a given integer M, we construct a function whose autocorrelation is the triangle $\max(1 - \frac{|t|}{M}, 0)$. Given a positive, even function f that is decreasing and convex on $\mathbb{Z}^+$ we can approximate f by a sum of triangles of the form $\triangle$ and construct a function x on $\mathbb{Z}$ whose autocorrelation is f.

5.1 Background and preliminary results

It has been shown in [34] that if λ is a number in $(0, 1)$, with binary expression $0.\alpha_1\alpha_2\alpha_3\cdots$ and we define a function as

$$y[k] = \begin{cases} 2\alpha_{2n+1} - 1 & \text{if } k = n+1, n \in \mathbb{N} \cup \{0\} \\ 2\alpha_{2n} - 1 & \text{if } k = -n, n \in \mathbb{N} \end{cases} \tag{5.1}$$

then for *almost all* values of λ the autocorrelation of y, A_y, is

$$A_y[k] = \begin{cases} 0 & \forall k \neq 0 \\ 1 & \text{if } k = 0. \end{cases} \tag{5.2}$$

In other words, if y has the value $+1$ or the value -1 and if each choice of these values is independent of all others, then the probability is 0 that A_y will not have the value given by (5.2). *One should observe that* $A_y = \triangle$ *on integers.* Let us work out the details of this result following Wiener's work ([34]).

First let us discuss the necessary reduction of probabilities to Lebesgue measure. Let $\mathscr{B}(\mathbb{R})$ be the Borel algebra of subsets of the real line. We know from [23] that there is a one-to-one correspondence between probability measures P on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ and the distribution functions F on the real line $\mathbb{R}$. The measure P constructed from the function F by assigning

$$P(a, b] = F(b) - F(a)$$

for all $a, b, -\infty \leqslant a < b < \infty$ is usually called the Lebesgue-Stieltjes probability measure corresponding to the distribution function F. The case when

$$F(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \leqslant x \leqslant 1, \\ 1, & x > 1 \end{cases} \tag{5.3}$$

is particularly important. In this case, the corresponding probability measure (denoted by λ) is the Lebesgue measure on $[0, 1]$. Clearly $\lambda(a, b] = b - a$, which is the Lebesgue measure of $(a, b]$ (as well as any of the intervals (a, b), $[a, b]$ or $[a, b)$).

Recall that

$$A_y[k] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} y[m+k]y[m] \tag{5.4}$$

if the limit exists. We now establish that except for a set of values of λ of measure zero, $A_y[0] = 1$ and $A_y[k] = 0$ when k is an integer other than 0. That $A_y[0] = 1$

is an immediate consequence of the fact that $|y[n]| = 1$. If we can now show that for each particular non-zero integer k, $A_y[k] = 0$ except for a set of values of λ with zero measure, we may appeal to the theorem that the logical sum of a denumerable set of measure zero sets is of measure zero to complete the verification of (5.2).

Let us now consider $y[m+k]y[m]$ for a fixed k and variable m. For any m it assumes either the value $+1$ or the value -1, and if we take any finite consecutive set of numbers m, any sequence of signs is as probable as any other - that is, any sequence of signs corresponds to a region of λ of the same Lebesgue measure as that corresponding to any other. If we take $2N$ consecutive values of $y[m+k]y[m]$, there are 2^{2N} possible sequences and if we have $(N-j)$ $+1$s then we have $(N+j)$ -1s (where $0 \leqslant j \leqslant N$) and the absolute value of the sum of these values will be $N+j-(N-j) = 2j$. So the sum of these values exceeds $N\epsilon$ in absolute value if $\left[\frac{N\epsilon}{2}\right] \leqslant k \leqslant N$. The number of ways of having $(N-j)$ $+1$s or $(N+j)$ -1s is $\binom{2N}{N-j} = \binom{2N}{N+j}$. Therefore, the probability that the sum of $2N$ consecutive values exceeds $N\epsilon$ in absolute value is

$$\frac{\sum_{j=\left[\frac{N\epsilon}{2}\right]}^{N}\binom{2N}{N-j}}{2^{2N}} = \frac{\sum_{j=\left[\frac{N\epsilon}{2}\right]}^{N}\frac{(2N)!}{(N-j)!(N+j)!}}{2^{2N}}$$

(by Stirling's formula)

$$\begin{aligned}
&\sim \frac{1}{2N}\sum_{j=\left[\frac{N\epsilon}{2}\right]}^{N}\frac{\sqrt{2\pi}(2N)^{2N+\frac{1}{2}}e^{-2N}}{\sqrt{2\pi}\sqrt{2\pi}(N-j)^{N-j+\frac{1}{2}}(N+j)^{N+j+\frac{1}{2}}e^{-N+j}e^{-N-j}} \\
&= \frac{1}{\sqrt{\pi}}\sum_{j=\left[\frac{N\epsilon}{2}\right]}^{N}\frac{\sqrt{N}N^{2N}}{N^{N-j+\frac{1}{2}}\left(1-\frac{j}{N}\right)^{N-j+\frac{1}{2}}N^{N+j+\frac{1}{2}}\left(1+\frac{j}{N}\right)^{N+j+\frac{1}{2}}} \\
&= \frac{1}{\sqrt{\pi}}\sum_{j=\left[\frac{N\epsilon}{2}\right]}^{N}\frac{\sqrt{N}N^{2N}}{N^{2N+1}\left(1-\frac{j}{N}\right)^{N-j}\left(1+\frac{j}{N}\right)^{N+j}\sqrt{N-j}\sqrt{N+j}(\frac{1}{N})}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=\left[\frac{N\epsilon}{2}\right]}^{N} \frac{1}{\left(1-\frac{j}{N}\right)^{N-j}\left(1+\frac{j}{N}\right)^{N+j}} \left(\frac{N}{\pi\left(N^2-j^2\right)}\right)^{\frac{1}{2}} \\
&\sim \sum_{j=\left[\frac{N\epsilon}{2}\right]}^{N} e^{\frac{k}{N}(N-j-N-j)} \left(\frac{N}{\pi(N^2-j^2)}\right)^{\frac{1}{2}} \\
&= O\left(N^{\frac{1}{2}} e^{-\frac{N\epsilon^2}{2}}\right). \qquad (5.5)
\end{aligned}
$$

In as much as $\sum_{N=1}^{\infty} N^{\frac{1}{2}} e^{-\frac{N\epsilon^2}{2}}$ converges, the probability is zero that there should fail to be an integral value of N such that, from that value on,

$$
\left|\sum_{m=-N}^{N} y[k+m]y[m]\right| \leqslant N\epsilon + 1. \qquad (5.6)
$$

Thus, except for a set of values of λ of measure zero,

$$
\begin{aligned}
\overline{\lim_{N\to\infty}} \left|\frac{1}{2N+1} \sum_{m=-N}^{N} y[k+m]y[m]\right| \leqslant \overline{\lim_{N\to\infty}} \left|\frac{1}{2N} \sum_{m=-N}^{N} y[k+m]y[m]\right| &\leqslant \frac{\epsilon}{2} + \overline{\lim_{N\to\infty}} \frac{1}{2N} \\
&= \frac{\epsilon}{2}
\end{aligned}
$$

and since ϵ is arbitrary, and the sum of a denumerable set of measure zero sets has measure zero, we have

$$
\lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} y[k+m]y[m] = 0 \qquad (5.7)
$$

for almost every λ. This completes the proof of (5.2).

Remark. It was already discussed in Chapter 4 that Wiener has shown in [34] that it is possible to deterministically construct a function whose autocorrelation is the triangle (5.2).

5.2 Functions whose autocorrelation is a triangle

Keeping in mind the function y of Section 5.1 we are now in a position to state the following theorem.

Theorem 5.8. *Given $M \in \mathbb{N}$ there exists a constructible unimodular function $x : \mathbb{Z} \to \mathbb{C}$ such that its autocorrelation A_x is*

$$A_x[k] = \begin{cases} 1 - \frac{|k|}{M} & \textit{if } |k| \leqslant M \\ 0 & \textit{if } |k| > M. \end{cases} \tag{5.9}$$

Note that A_x is a triangle symmetric about the origin, has a base of length $2M$, where M is an integer, and height 1.

Proof. Given $M \in \mathbb{N}$ we now define a function $x : \mathbb{Z} \to \mathbb{C}$ as

$$x[k] = \begin{cases} 2\alpha_{2n+1} - 1 & \text{if } Mn < k \leqslant M(n+1), n \in \mathbb{N} \cup \{0\} \\ 2\alpha_{2n} - 1 & \text{if } -Mn < k \leqslant M(1-n), n \in \mathbb{N} \end{cases} \tag{5.10}$$

where the αs are as obtained in (5.1). Note that $x[k] = y[\lceil \frac{k}{M} \rceil]$. From Section 4.2 we know that such a y (and hence x) can be deterministically constructed. Also note that both x and y are unimodular.

We show that the autocorrelation of x is A_x. Since the autocorrelation function is an even function we will prove the result only for $k > 0$. Let $0 \leqslant Mp \leqslant k \leqslant M(p+1)$, $p \in \mathbb{N} \cup \{0\}$. Let N_n be the smallest integer such that $N < M(N_n + 1)$. Then we have

$$\begin{aligned} A_x[k] &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} x[m+k]\overline{x[m]} \\ &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} x[m+k]x[m] \end{aligned}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-MN_n}^{MN_n} x[k+m]x[m] + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{MN_n<|m|\leqslant N} x[m+k]x[m]$$

$$= S_1 + S_2. \tag{5.11}$$

First we calculate bounds on S_2.

$$\begin{aligned} |S_2| &= \left| \lim_{N\to\infty} \frac{1}{2N+1} \sum_{MN_n<|m|\leqslant N} x[m+k]x[m] \right| \\ &\leqslant \lim_{N\to\infty} \frac{1}{2N+1} \sum_{MN_n<|m|\leqslant N} |x[m+k]x[m]| \,. \end{aligned} \tag{5.12}$$

Now $x[m+k]x[m]$ is either 1 or -1, so,

$$|S_2| \leqslant \lim_{N\to\infty} \frac{1}{2N+1} \sum_{MN_n<|m|\leqslant N} 1 = \lim_{N\to\infty} \frac{2(N-MN_n)}{2N+1}.$$

We know from the property of n that $N - MN_n < M$. Therefore,

$$|S_2| \leqslant \lim_{N\to\infty} \frac{2M}{2N+1} = 0, \text{ or, } S_2 = 0. \tag{5.13}$$

Now we consider S_1.

$$\begin{aligned} S_1 &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-MN_n}^{MN_n} x[k+m]x[m] \\ &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N_n}^{N_n} \sum_{m=Mn+1}^{M(n+1)} x[k+m]x[m] + \lim_{N\to\infty} \frac{1}{2N+1} x[-N_n+k]x[-N_n]. \end{aligned} \tag{5.14}$$

By a property of the function x and the fact that the last term in (5.14) goes to 0 as N goes to infinity we have

$$\begin{aligned} S_1 &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N_n}^{N_n} \sum_{m=Mn+1}^{M(n+1)} x[m+k]x[M(n+1)] \\ &= \lim_{N\to\infty} \frac{1}{2N+1} \left(\sum_{m=Mn+1}^{Mn+M(p+1)-k} x[M(n+p+1)]x[M(n+1)] \right) + \\ &\quad + \sum_{Mn+M(p+1)-k+1}^{M(n+1)} x[M(n+p+2)]x[M(n+1)] \end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N_n}^{N_n} \left(\sum_{m=Mn+1}^{Mn+M(p+1)-k} y[n+p+1]y[n+1] + \right. \\
&\quad \left. + \sum_{m=Mn+M(p+1)-k+1}^{M(n+1)} y[n+p+2]y[n+1] \right) \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N_n}^{N_n} \big((M(p+1)-k)y[n+p+1]y[n+1] + \\
&\quad + (k-Mp)y[n+p+2]y[n+1]\big) \\
&= \lim_{N\to\infty} \frac{M(p+1)-k}{2N+1}(2N_n+1)\frac{1}{2N_n+1} \sum_{n=-N_n}^{N_n} y[n+p+1]y[n+1] + \\
&\quad + \lim_{N\to\infty} \frac{(k-Mp)}{2N+1}(2N_n+1)\frac{1}{2N_n+1} \sum_{n=-N_n}^{N_n} y[n+p+2]y[n+1].
\end{aligned}
$$

Since $N_n \to \infty$ as $N \to \infty$ we have

$$
\begin{aligned}
S_1 &= \lim_{N\to\infty} \frac{M(p+1)-k}{2N+1}(2N_n+1)A_y[p] + \lim_{N\to\infty} \frac{k-Mp}{2N+1}(2N_n+1)A_y[p+1] \\
&= \lim_{N\to\infty} \left(p+1-\frac{k}{M}\right) \frac{(2N_n+1)M}{2N+1} A_y[p] + \lim_{N\to\infty} (\frac{k}{M}-p) \frac{(2N_n+1)M}{2N+1} A_y[p+1].
\end{aligned}
\tag{5.15}
$$

Before we proceed further let us evaluate the following limit:

$$
\lim_{N\to\infty} \frac{(2N_n+1)M}{2N+1}. \tag{5.16}
$$

Note that the limit in (5.16) is the same as $\lim_{N\to\infty} \frac{2N_nM}{2N+1}$ since $\lim_{N\to\infty} \frac{M}{2N+1} = 0$.

From the choice of n, we have,

$$
\begin{aligned}
&MN_n \leqslant N < M(N_n+1) \\
&\text{or, } 2MN_n \leqslant 2N < 2M(N_n+1) \\
&\text{or, } 2MN_n+1 \leqslant 2N+1 < 2M(N_n+1)+1
\end{aligned}
$$

$$\text{or, } \frac{1}{2M(N_n+1)+1} < \frac{1}{2N+1} \leqslant \frac{1}{2MN_n+1}$$
$$\text{or, } \frac{2MN_n}{2M(N_n+1)+1} < \frac{2MN_n}{2N+1} \leqslant \frac{2MN_n}{2MN_n+1}. \tag{5.17}$$

N_n goes to infinity as N goes to infinity and so taking limits throughout as N goes to infinity we get

$$1 < \lim_{N\to\infty} \frac{2MN_n}{2N+1} \leqslant 1. \tag{5.18}$$

By the sandwiching principle

$$\lim_{N\to\infty} \frac{2MN_n}{2N+1} = 1$$

which means

$$\lim_{N\to\infty} \frac{M(2N_n+1)}{2N+1} = 1 \tag{5.19}$$

and we have evaluated (5.16). Using (5.19) in (5.15) gives

$$S_1 = \left(p+1-\frac{k}{M}\right) A_y[p] + \left(\frac{k}{M}-p\right) A_y[p+1].$$

Since we evaluated S_2 to be zero we have from (5.11) that

$$A_x[k] = S_1 = \left(p+1-\frac{k}{M}\right) A_y[p] + \left(\frac{k}{M}-p\right) A_y[p+1] \tag{5.20}$$

If $0 \leqslant k \leqslant M$ then $p = 0$. For every other range of k, p is non-zero. Using the values of $A_y[p]$ as given by (5.2) and the fact that A_x is an even function one gets the desired result of (5.9). $\square$

Remark. The triangle considered in Theorem 5.8 can have any arbitrary height. However, the corresponding code x will not be unimodular. In this case, letting

$\lambda \in (0,1)$ have binary expansion $.\alpha_1\alpha_2\alpha_3\ldots$ the code $x : \mathbb{Z} \to \mathbb{C}$ defined, for a given positive number K and a positive integer M, as

$$x[k] = \begin{cases} \sqrt{K}(2\alpha_{2n+1} - 1) & \text{if } Mn < k \leqslant M(n+1), n \in \mathbb{N} \cup \{0\} \\ \sqrt{K}(2\alpha_{2n} - 1) & \text{if } -Mn < k \leqslant M(1-n), n \in \mathbb{N} \end{cases} \tag{5.21}$$

will have autocorrelation

$$A_x[p] = \begin{cases} K(1 - \frac{|p|}{M}), & 0 \leqslant |p| \leqslant M \\ 0 & \text{otherwise.} \end{cases}$$

A_x is a triangle symmetric about the origin having base of length $2M$ and height K. The code x is such that $|x| = \sqrt{K}$. In this case the code x is bounded.

5.3 Functions whose autocorrelation is the sum of triangles

Theorem 5.22. *Suppose we are given two distinct positive integers M_1 and M_2. We know from Section 5.2 that one can construct functions $x : \mathbb{Z} \to \mathbb{C}$ and $y : \mathbb{Z} \to \mathbb{C}$ such that*

$$A_x[k] = \begin{cases} 1 - \frac{|k|}{M_1}, & 0 \leqslant |k| \leqslant M_1 \\ 0 & \textit{otherwise} \end{cases}$$

and

$$A_y[k] = \begin{cases} 1 - \frac{|k|}{M_2}, & 0 \leqslant |k| \leqslant M_2 \\ 0 & \textit{otherwise.} \end{cases}$$

The function z such that $A_z = A_x + A_y$ is given by $z = x + y$.

Note that for convenience we have considered codes x and y whose autocorrelation functions A_x and A_y are triangles symmetric about the origin with height 1 and bases of length $2M_1$ and $2M_2$ respectively. Referring to the remark at the end of Theorem 5.8 and based on the proof of Theorem 5.22 that will follow, it is worth noting that if instead A_x and A_y were triangles of height K_1 and K_2 respectively, the result of Theorem 5.22 would still hold i.e., $z = x+y$ would have autocorrelation $A_z = A_x + A_y$.

Proof of Theorem 5.22.

$$\begin{aligned}
A_z[k] &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} z[m+k]\overline{z[m]} \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} (x[m+k]+y[m+k])(\overline{x[m]}+\overline{y[m]}) \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} x[m+k]\overline{x[m]} + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} y[m+k]\overline{y[m]} + \\
&\quad + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} x[m+k]\overline{y[m]} + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} y[m+k]\overline{x[m]} \\
&= A_x(k) + A_y(k) + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} x[m+k]\overline{y[m]} + \\
&\quad + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} y[m+k]\overline{x[m]}. \qquad (5.23)
\end{aligned}$$

Let us denote the last two terms in the right side of (5.23) by S_1 and S_2 respectively. We want to show that $S_1 = 0$ and $S_2 = 0$.

$$S_1 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} x[m+k]\overline{y[m]}. \qquad (5.24)$$

Since y is real valued, (5.24) becomes

$$S_1 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} x[m+k]y[m]. \qquad (5.25)$$

Let P_N be the largest integer so that

$$M_2 P_N \leqslant N \leqslant M_2(P_N+1). \qquad (5.26)$$

Then S_1 can be written as

$$\begin{aligned} S_1 &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{-M_2P_N-1} x[m+k]y[m] + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=M_2P_N+1}^{N} x[m+k]y[m] + \\ &\quad + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-M_2P_N}^{M_2P_N} x[m+]y[m]. \end{aligned} \tag{5.27}$$

Let us denote the first two terms of (5.27) by s_1 and s_2 respectively. Now,

$$|s_1| \leqslant \sum_{m=-N}^{-M_2P_N-1} 1 = N - M_2P_N$$

and

$$|s_2| \leqslant \sum_{m=M_2P_N+1}^{N} 1 = N - M_2P_N.$$

From (5.26),

$$N - M_2P_N \leqslant M_2(P_N+1) - M_2P_N = M_2$$

which means $|s_1| \leqslant M_2$ and $|s_2| \leqslant M_2$. Therefore,

$$\lim_{N\to\infty} \frac{|s_1|}{2N+1} \leqslant \lim_{N\to\infty} \frac{M_2}{2N+1} = 0$$

and also

$$\lim_{N\to\infty} \frac{|s_2|}{2N+1} \leqslant \lim_{N\to\infty} \frac{M_2}{2N+1} = 0.$$

Thus,

$$\begin{aligned} S_1 &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-M_2P_N}^{M_2P_N} x[m+k]y[m] \\ &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-P_N}^{P_N-1} \sum_{m=M_2n+1}^{M_2(n+1)} x[m+k]y[m] + \\ &\quad + \lim_{N\to\infty} \frac{1}{2N+1} x[-M_2P_N+k]y[-M_2P_N] \\ &\quad \text{(the second term goes to 0 as } N \to \infty\text{)} \end{aligned}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-P_N}^{P_N-1} \sum_{m=M_2n+1}^{M_2(n+1)} x[m+k]y[M_2(n+1)]. \tag{5.28}$$

$y[M_2(n+1)]$ is either $+1$ or -1. Between (M_2n+1) and $M_2(n+1)$ there are M_2 terms. So there are M_2 values of x. Suppose that of these M_2 values there are j that have the value $+1$ and $(M_2 - j)$ that have the value -1. Upon multiplication by $y(M_2(n+1))$ we have either j values that are -1 and $(M_2 - j)$ values that are $+1$ or vice versa.

In the sum on the right side of (5.28) there are $2P_N$ blocks of length M_2. Let us say that the first block has j_1 terms equal $+1$ and $(M_2 - j_1)$ terms equal to -1, the second block has j_2 terms equal to $+1$ and $(M_2 - j_2)$ terms equal to -1 and so on. Together, there are $(j_1 + j_2 + \cdots + j_{2P_N})$ terms equal to $+1$ and $(M_2 - j_1 + M_2 - j_2 + \cdots + M_2 - j_{2P_N}) = 2P_NM_2 - (j_1 + j_2 + \cdots + j_{2P_N})$ terms equal to -1. Let $P_NM_2 = M$ and $j_1 + j_2 + \cdots + j_{2P_N} = M - j$ where $0 \leqslant j \leqslant M$. Then $2P_NM_2 - (j_1 + j_2 + \cdots + j_{2P_N}) = 2M - (M - j) = M + j$. So out of $2M$ consecutive values of $x[m+k]y[m]$ there are $(M-j)$ values that are $+1$ and $(M+j)$ values that are -1. So the absolute value of the sum of $2P_NM_2$ consecutive values of $x[m+k]y[m]$ would be $M+j-(M-j) = 2j$. So the sum of these values exceeds $M\epsilon$ in absolute value if $[M\epsilon] \leqslant 2j \leqslant 2M$. The number of ways of having $(M-j)$ $+1$s and $(M+j)$ -1s is $\binom{2M}{M-j} = \binom{2M}{M+j}$. Therefore, for a given ϵ, the probability that the sum of $2M$ consecutive values of $x[m+k]y[m]$ exceeds $M\epsilon$ in absolute value is $\frac{\sum_{j=[\frac{M\epsilon}{2}]}^{M}}{2^{2M}}\binom{2M}{M-j}$. It can be shown in a manner identical to that in (5.5) that this goes to zero as M goes to infinity. Thus, the probability is zero that there should fail to be an integral value of $M = P_N M_2$ such that from that value on (see (5.28))

$$\Big|\sum_{m=-M}^{M} x[m+k]y[m]\Big| \leqslant M\epsilon + 1.$$

Therefore,

$$\overline{\lim}_{N\to\infty} = \Big|\frac{1}{2N+1}\sum_{m=-M}^{M} x[m+k]y[m]\Big| \leqslant \frac{M\epsilon+1}{2N+1} = \frac{P_N M_2 \epsilon}{2N+1} + \frac{1}{2N+1}. \quad (5.29)$$

From (5.26),

$$\frac{P_N M_2}{2N+1} \leqslant \frac{N}{2N+1} \to \frac{1}{2}$$

as N goes to infinity. So, the left side of (5.29) is less than $\frac{\epsilon}{2}$ and for almost all λ,

$$\lim_{N\to\infty} \sum_{m=-N}^{N} x[m+k]y[m] = 0.$$

In a similar way one can show that

$$\lim_{N\to\infty} \sum_{m=-N}^{N} x[m]y[m+k] = 0$$

for almost every λ. This concludes showing that if $z[k] = x[k] + y[k]$ then $A_z = A_x + A_y$. □

Remark. One should note that even though the argument in the proof of Theorem 5.22 appears to be probabilistic, the deterministic construction of such a z from some suitable x and y can be done (see proof of Theorem 5.8).

Theorem 5.30. *An even, non-negative function on $\mathbb{Z}$ that is convex and decreasing to zero on $\mathbb{Z}^+$ and has finite support can be written as a finite sum of triangles. Each triangle in the sum is symmetric about the origin, has base length $2M$ for some integer M and height K where K is a positive number.*

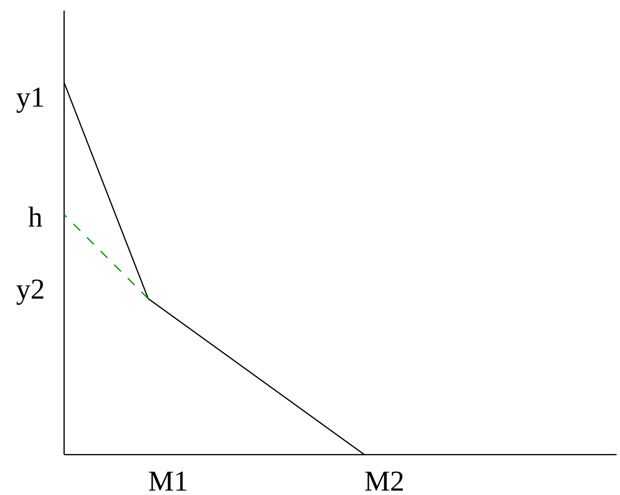

Figure 5.1: Sum of two triangles.

Proof. Since the function is even we will restrict ourselves only on the positive integers, $\mathbb{Z}^+$. We will determine the height and base of the defining triangles. If one were to imagine the function defined on $\mathbb{R}$ instead of $\mathbb{Z}$ then the function satisfying the properties stated in the theorem would be piecewise linear. In this case let the points of discontinuity be the set $\{M_1, M_2, \ldots, M_n\}$ such that $M_1 > M_2 > \cdots > M_n$. Suppose that $f[0] = y_1, f[M_1] = y_2, f[M_2] = y_3, \ldots, f[M_{n-1}] = y_n, f[M_n] = 0$. So on $\mathbb{Z}^+$ the support of f is contained in $[0, M_n]$. The function f will be the sum of triangles with base lengths $M_1, M_2, \ldots, M_n$. The problem is to find the heights of these triangles.

We first demonstrate this for the simple function shown in Figure 5.1. We are given y_1 and y_2. Thus we know h. The function is made up of the two triangles shown in Figure 5.2. We have to find their heights r_1 and r_2. Now

$$r_1 = h.$$

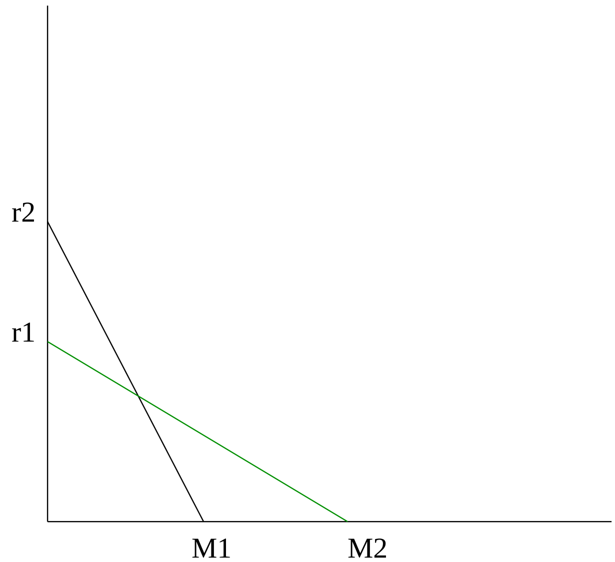

Figure 5.2:

Besides,

$$r_1 + r_2 = h_1, \text{ or, } r_2 = h_1 - r_1 = h_1 - h.$$

Moving on to a function that could be the sum of three triangles let us look at Figure 5.3.

We have to find the heights r_1, r_2 and r_3 of these triangles. Since we are given y_1, y_2 and y_3 we know P_1 and P_2.

$$P_1 = r_3,$$

$$P_2 = r_1 + r_3 \implies r_1 = -r_3 + P_2 = -P_1 + P_2,$$

$$h_1 = r_1 + r_2 + r_3 \implies r_2 = h_1 - r_1 - r_3 = h_1 - P_2.$$

Following are the necessary steps to show this for a function that is the sum of n triangles. See Figure 5.4.

(i) We are given the points $(0, y_1), (M_1, y_2), (M_2, y_3), (M_3, y_4), \cdots, (M_{n-1}, y_n), (M_n, 0)$.

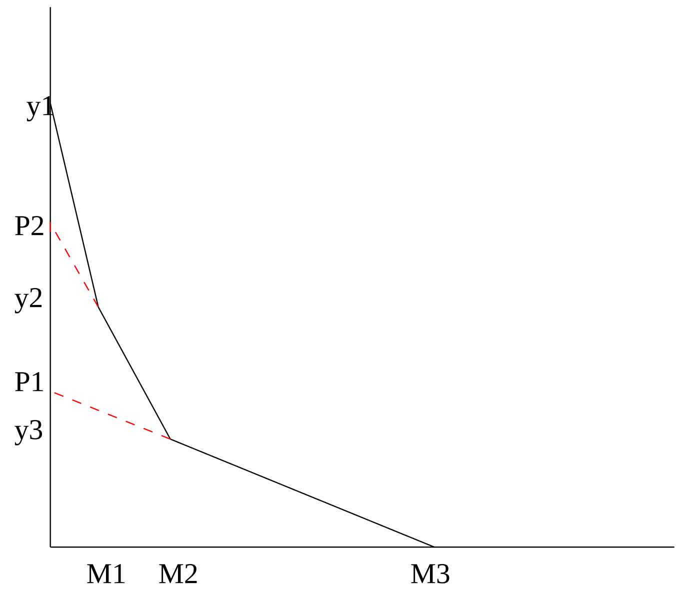

Figure 5.3: Sum of three triangles.

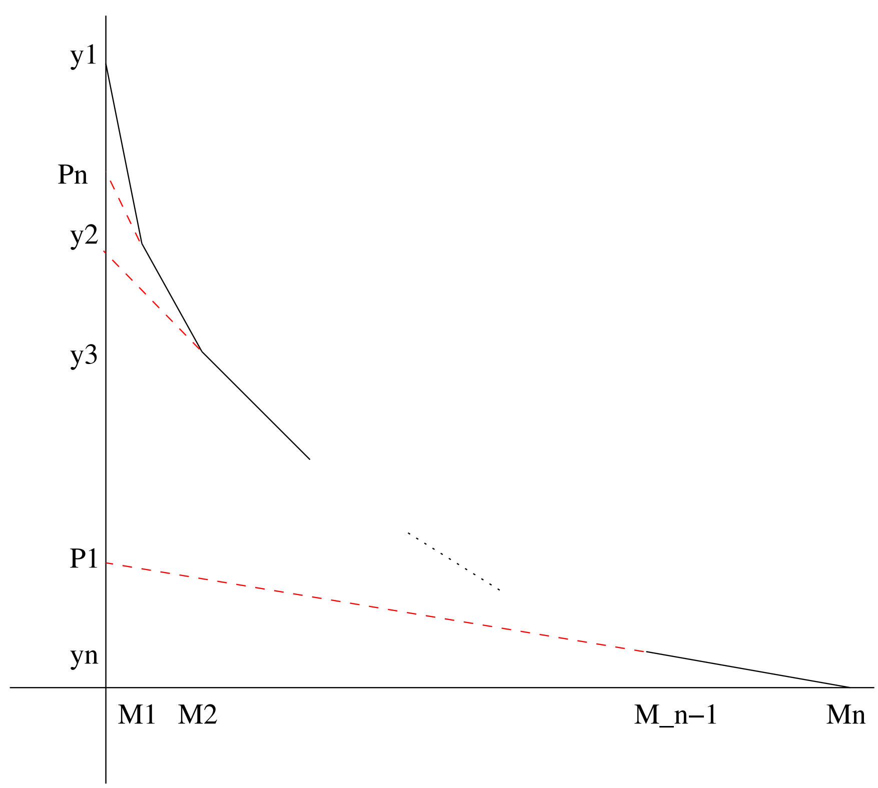

Figure 5.4: Sum of n triangles.

(ii) We extend the line joining (M_{n-1}, y_n) and $(M_n, 0)$ till it joins the y axis. Let is intersect the y axis at P_1. See Figure 5.4. Then the triangle with base M_n has height P_1.

(iii) Next we extend the line joining (M_{n-1}, y_n) and (M_{n-2}, y_{n-1}) to make it intersect the y axis. Let this line intersect the y axis at P_2.

(iv) Suppose the heights of the n triangles whose sum is the given function is $r_1, r_2, \ldots, r_n$ with base lengths $M_1, M_2, \ldots, M_n$ respectively. We already found out that $r_n = P_1$. Now,

$$P_2 = r_n + r_{n-1} \implies r_{n-1} = P_2 - r_n = P_2 - P_1.$$

(v) Just as we did in (iii) we extend the line joining (M_{n-2}, y_{n-1}) and (M_{n-3}, y_{n-2}) and let its points of intersection with the y axis be P_3.

$$P_3 = r_n + r_{n-1} + r_{n-2}.$$

Since we already know r_n and r_{n-1} we would get

$$r_{n-2} = P_3 - r_n - r_{n-1}.$$

(vi) If we continue as above then we would finally get $r_1 + r_2 + \cdots r_n = y_1$ where $r_2, \ldots, r_n$ are known by now and we can find r_1.

$\square$

Corollary 5.31. *One can approximate the class of even and positive functions on $\mathbb{Z}$ that decrease to zero and are convex on $\mathbb{Z}^+$ by a finite sum of triangles.*

Proof. For any given $\epsilon > 0$, let M_ϵ be the integer such that for all $m \in \mathbb{N}$ with $|m| \geqslant M_\epsilon$, $f[m] \leqslant \epsilon$. This is possible due to the decreasing property of our even function.

Let us now define the function f_ϵ as

$$f_\epsilon[m] = \begin{cases} f[m] & \text{if } |m| \leqslant M_\epsilon \\ 0 & \text{if } |m| > M_\epsilon. \end{cases}$$

One can observe that for all $m \in \mathbb{Z}$, $|f[m] - f_\epsilon[m]| \leqslant \epsilon$.

f_ϵ has support contained in $[-M_\epsilon, M_\epsilon]$ i.e., it has compact support and so f_ϵ can be approximated by a sum of triangles as shown in Theorem 5.30. □

5.4 Remarks

We know from [4] that

$$\max(1 - |t|, 0) \longleftrightarrow \left(\frac{\sin \pi\gamma}{\pi\gamma}\right)^2.$$

Letting $\omega(\gamma) = \frac{1}{2\pi}\left(\frac{\sin \gamma/2}{\gamma/2}\right)^2$ one has for $\lambda > 0$,

$$\max(1 - \frac{2\pi|t|}{\lambda}, 0) \longleftrightarrow \omega_\lambda(\lambda)$$

where ω_λ is the dilation of ω i.e., $\omega_\lambda(\gamma) = \lambda(\omega(\lambda\gamma)) = \frac{\lambda}{2\pi}\left(\frac{\sin(\frac{\lambda\gamma}{2})}{\frac{\lambda\gamma}{2}}\right)^2$.

Let $\frac{|t|}{M} = \frac{2\pi|t|}{\lambda}$. Then $\lambda = 2\pi M$. So

$$\max(1 - \frac{|t|}{M}, 0) \longleftrightarrow \omega_{2\pi M}(\gamma).$$

Note that $\omega_{2\pi M}(\gamma) = M\left(\frac{\sin \pi M\gamma}{\pi M\gamma}\right)^2$. If we wish to consider triangles of arbitrary height K i.e., $K\max\left(1-\frac{|t|}{M},0\right)$ then

$$K\max\left(1-\frac{|t|}{M},0\right)\longleftrightarrow K\omega_{2\pi M}(\gamma)=KM\Big(\frac{\sin\pi M\gamma}{\pi M\gamma}\Big)^2.$$

It is rather obvious that the function $\sqrt{K}x$ (where x is as defined in (5.10)) has its autocorrelation at some non-zero integer k to be $K\max\left(1-\frac{|k|}{M},0\right)$.

As something pertinent to section 5.3 one should also note that using properties of the Fourier transform one has, for given $K_1,K_2,\dots,K_n$ and $M_1,M_2,\dots,M_n$,

$$\sum_{i=1}^n K_i\max(1-\frac{|t|}{M_i},0)\longleftrightarrow\sum_{i=1}^n K_iM_i\Big(\frac{\sin\pi M_i\gamma}{\pi M_i\gamma}\Big)^2.$$

Thus the function f described in Theorem 5.30 is the inverse Fourier transform of a positive function of the form $K_1M_1\Big(\frac{\sin\pi M_1\gamma}{\pi M_1\gamma}\Big)^2+K_2M_2\Big(\frac{\sin\pi M_2\gamma}{\pi M_2\gamma}\Big)^2+K_nM_n\Big(\frac{\sin\pi M_n\gamma}{\pi M_n\gamma}\Big)^2$. It should be noted that due to Theorem 5.22 the waveform x whose autocorrelation is f can be constructed deterministically as $x=x_1+\cdots x_n$ where each $x_i (i=1,\dots,n)$ has autocorrelation equal to $K_i\max(1-\frac{|t|}{M_i},0)$.

Chapter 6

Conclusion

6.1 Summary of results

Here is a summary of the main results of the thesis.

(i) The Wiener-Wintner Theorem is proved in the setting of $\mathbb{Z}^d$. This gives the construction of a *locally bounded* function x whose autocorrelation is the inverse Fourier transform of a given positive function.

(ii) Since the codes constructed by the Wiener Wintner Theorem is only locally bounded and we are aiming at getting codes with *constant amplitude* we next give an approach using u.d. mod - 1 sequences that could give *bounded* codes that has autocorrelation equal to the inverse Fourier transform of a given positive function. We however, show that it is impossible to use our technique of u.d. mod - 1 sequences for the purpose.

(iii) We construct a number of *unimodular* codes whose autocorrelation is zero everywhere except at the origin where it is one. This solves the problem when the given positive function is $F \equiv 1$ on $\mathbb{T}$. The following codes are constructed:

  (a) $x[n] = e^{2\pi i n^\alpha \theta}$ where $\alpha \in \mathbb{N}$ and $\theta \notin \mathbb{Q}$.

  (b) For a given integer $n > 2$ let the n roots of unity be $\{w_1, \ldots, w_n\}$. Let x take values on the positive integers in the order

$[w_0, w_1, \ldots, w_{n-1}]$ repeated $n^0 = 1$ times.

$[w_0, w_0; w_0, w_1; \ldots; w_0, w_{n-1}; w_1, w_0; w_1, w_1; \ldots; w_1, w_{n-1}; \ldots;$

$w_{n-1}, w_0, w_{n-1}, w_1, \ldots, w_{n-1}, w_{n-1}]$ repeated $n^1 = n$ times.

$[w_0, w_0, w_0; w_0, w_0, w_1; \ldots w_0, w_0, w_{n-1}; \ldots; w_{n-1}, w_{n-1}, w_0; \ldots;$

$w_{n-1}, w_{n-1}, w_{n-1}]$ repeated n^2 times.

$\vdots$

and for $k \in \mathbb{N}$, let $x[-k] = x[k]$.

(c) Let H_{2^n} be the real Hadamard matrix of size $2^n \times 2^n$. Let x take values on the positive integers from elements of the Hadamard matrices arranged in the following order.

H_1 (H_1 is repeated once (2^0)).

H_2 H_2 (H_2 is repeated twice (2^1)).

H_4 H_4 H_4 H_4 (H_4 is repeated $2^2 = 4$ times).

$\vdots$

and for $k \in \mathbb{N}$, let $x[-k] = x[k]$.

(iv) For a given positive integer M and a positive number K the triangle identified by $K \max\left(1 - \frac{|t|}{M}, 0\right)$ is the Fourier transform of the function $KM\left(\frac{\sin \pi M\gamma}{\pi M\gamma}\right)^2$. We construct codes x whose autocorrelation is the triangle $K \max\left(1 - \frac{|t|}{M}, 0\right)$ and also codes whose autocorrelation is the sum $\sum_{i=1}^{n} K_i \max\left(1 - \frac{|t|}{M_i}, 0\right)$. Based on this we construct codes x whose autocorrelation is a function that is positive and even on $\mathbb{Z}$, convex and decreasing to zero on $\mathbb{Z}^+$.

6.2 Future research

- It seems that it might be possible to try other techniques to construct *unimodular* codes whose autocorrelation is zero everywhere except at the origin where it is one.

  (i) The classical Rudin-Shapiro construction produces a sequence of polynomials with coefficients that are ± 1. One could use these coefficients to construct a sequence with the zero autocorrelation property.

  (ii) The $N \times N$ Discrete Fourier Transform (DFT) matrix $\mathcal{D}_N$ is defined as $(\frac{1}{\sqrt{N}} W_N^{mn}), m, n = 0, \ldots, N-1$ where $W_N = e^{-2\pi i/N}$, i.e.,

$$\mathcal{D}_N = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-2\pi i/N} & e^{-2\pi i 2/N} & \cdots & e^{-2\pi i(N-1)/N} \\ 1 & e^{-2\pi i 2/N} & e^{-2\pi i 4/N} & \cdots & e^{-2\pi i 2(N-1)/N} \\ \vdots & & & & \\ 1 & e^{-2\pi i(N-1)/N} & e^{-2\pi i 2(N-1)/N} & \cdots & e^{-2\pi i(N-1)(N-1)/N} \end{pmatrix}.$$

  One observes that the elements of this matrix are unimodular and so might be used to construct sequences having zero autocorrelation. Also, the rows of the DFT matrix are mutually orthogonal and this is actually an example of a complex Hadamard matrix.

- It would be interesting to delve into the case of the unimodular codes defined by $x[n] = e^{2\pi i n^\alpha \theta}$ where θ is irrational but α *is not an integer.*

  When α is an integer we have discussed several multidimensional cases when

α is equal to 2. Calculations get very complicated when α is greater than 2 but it would be nice to resolve this.

- For the codes constructed from Hadamard matrices or roots of unity we have addressed one way of extending such codes to higher dimensions but there can be other interesting constructions and it would be nice to see what the autocorrelation looks like. For example, in $\mathbb{Z}^2$, one could define a code as shown in Figure 6.1, where $x[0], x[1], \ldots$ are the values of the corresponding function (from Hadamard or others) in $\mathbb{Z}$.

- An approximation problem: Given a positive measure μ and $\epsilon > 0$. Suppose μ has Fourier transform A_μ. Take a positive definite compactly supported version of A_μ, A_{μ_ϵ} (ϵ away from A_μ in some norm, i.e., $||\mu - \mu_\epsilon||_1 < \epsilon$). A_{μ_ϵ} can be approximated by triangles. A_{μ_ϵ} is the autocorrelation of some f_ϵ. Does f_ϵ converge to a bounded function?

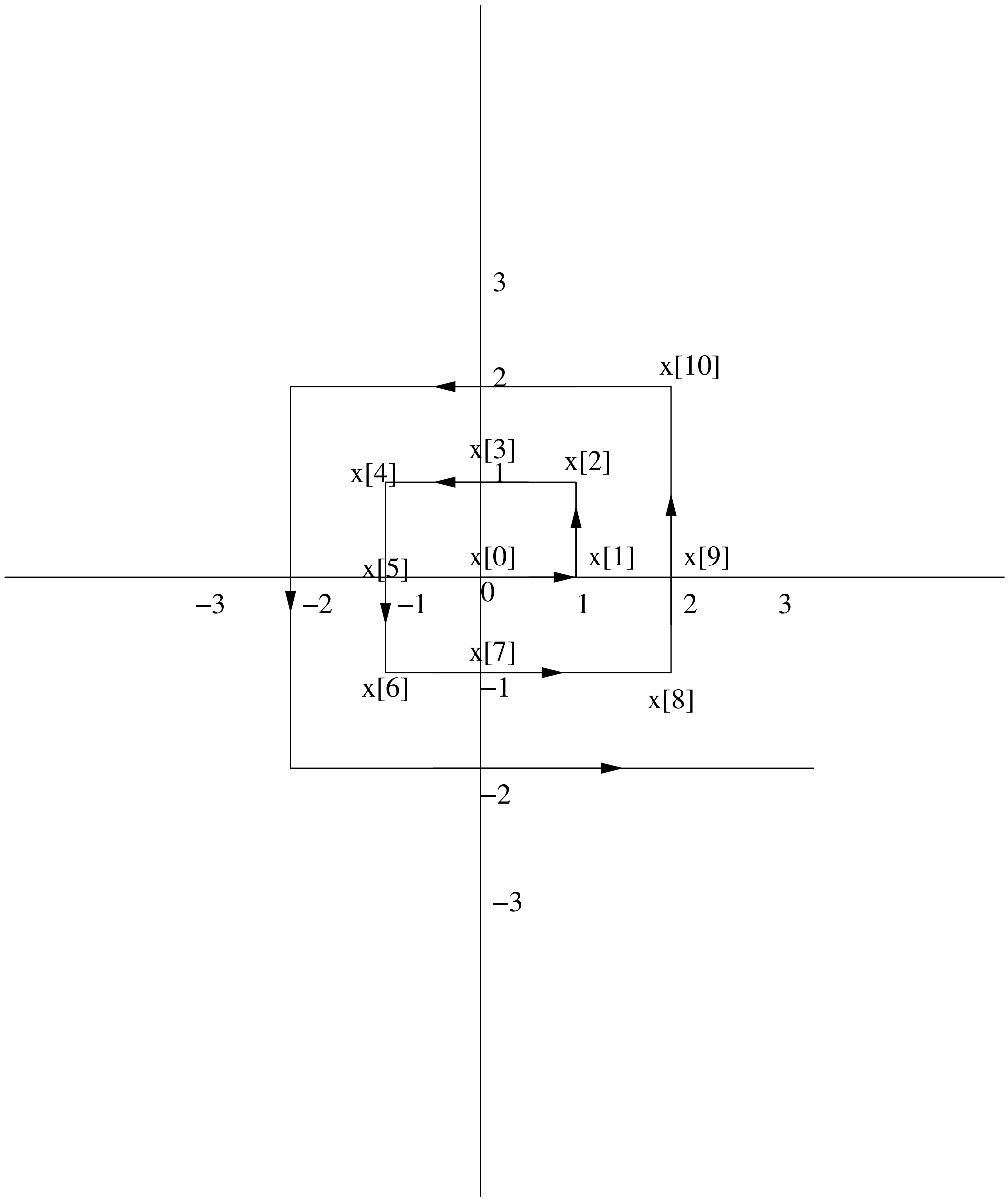

Figure 6.1: Unimodular codes in $\mathbb{Z}^2$.

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